Joint Distributions

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Announcements

Midterm:

- Tuesday the 23rd 7-9PM
- Covers through today*
- Unlimited notes/textbook, no calculator or computer.
- More review sheets coming today.

PS4:

- Out today! Problems above the line recommended for before midterm.

Review

And here we are

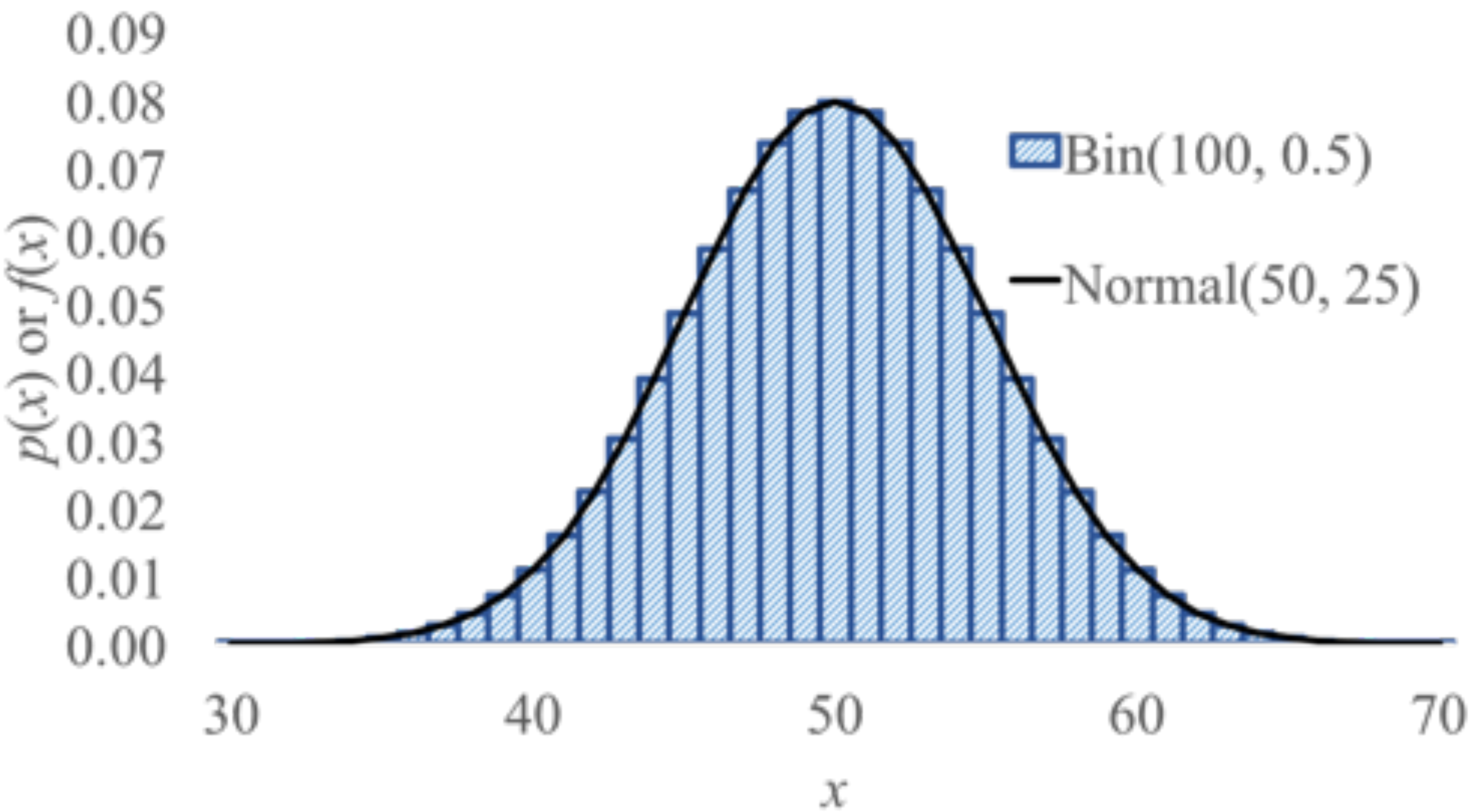
$$\mathcal{N}(\mu, \sigma^2)$$

CDF of Standard Normal: A function that has been solved for numerically

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

The cumulative density function (CDF) of any normal

Normal Approximates Binomial



Continuity Correction

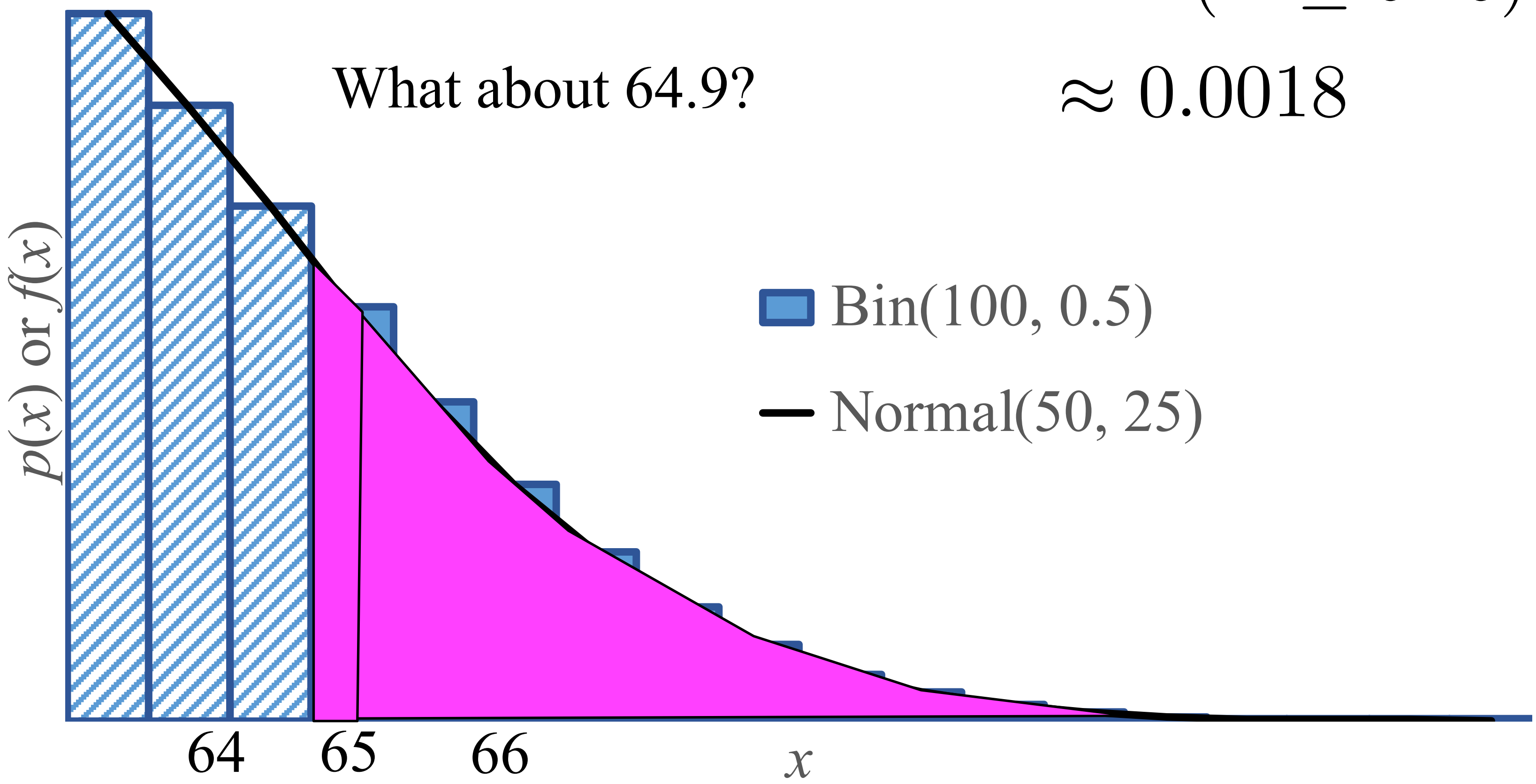
If Y (normal) approximates X (binomial)

$$P(X \geq 65)$$

$$\approx P(Y \geq 64.5)$$

$$\approx 0.0018$$

What about 64.9?

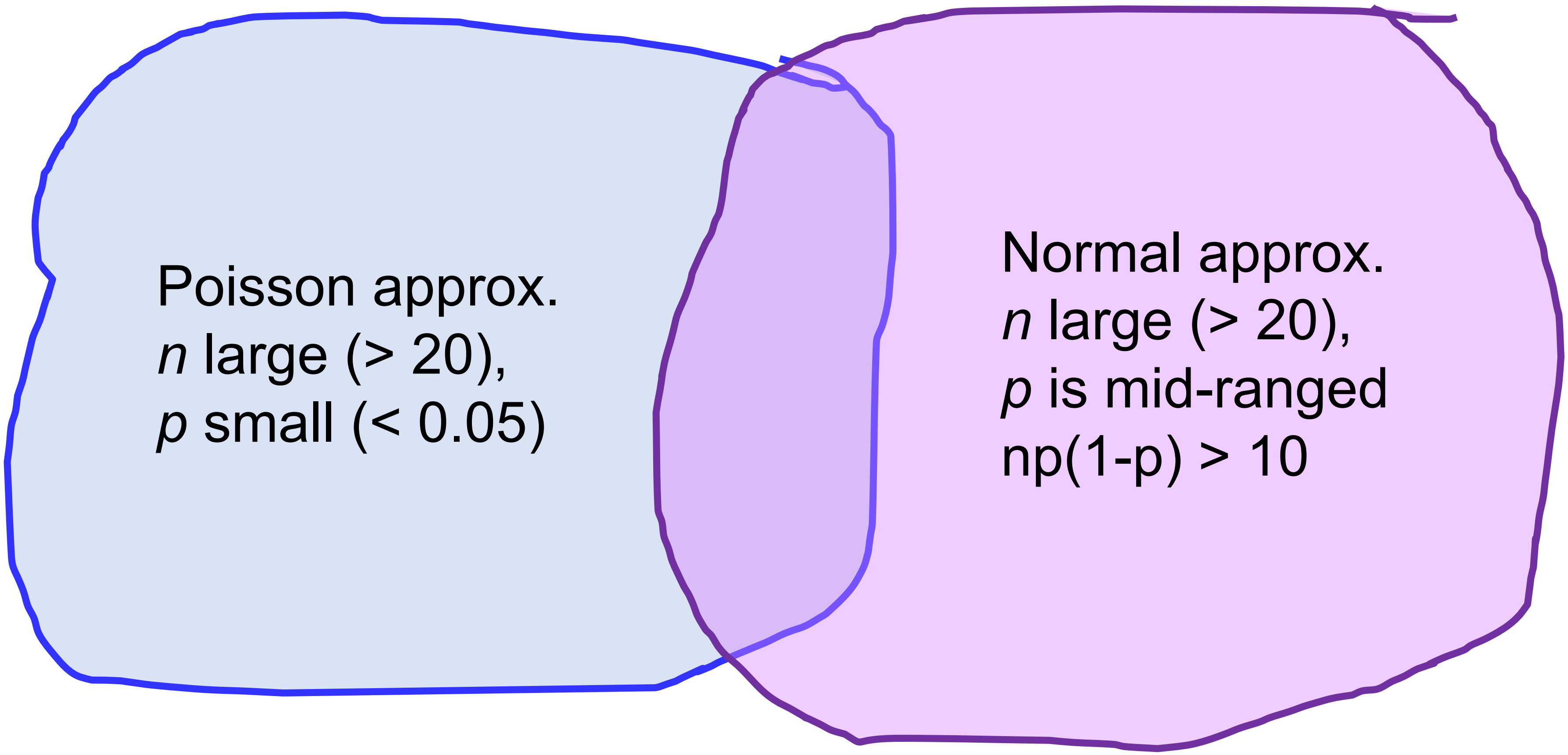


Continuity Correction

Use the continuity correction when approximating a **discrete value** with a **continuous distribution**

Who Gets to Approximate?

X ~ Bin(*n*, *p*)



If there is a choice, go with the normal approximation

Probability Table for Discrete

- States all possible outcomes with several discrete variables
- A probability table is not "parametric"
- If #variables is > 2, you can have a probability table, but you can't draw it on a slide

All values of A

All values of B

	0	1	2
0			
1		$P(A = 1, B = 1)$	
2			

Every outcome falls into a bucket

Here "," means "and"

Discrete Joint Mass Function

- For two discrete random variables *X* and *Y*, the **Joint Probability Mass Function** is:

$$p_{X,Y}(a,b) = P(X = a, Y = b)$$

- Marginal distributions:

$$p_X(a) = P(X = a) = \sum_y p_{X,Y}(a, y)$$

$$p_Y(b) = P(Y = b) = \sum_x p_{X,Y}(x, b)$$

- Example: X = value of die D_1, Y = value of die D_2

$$P(X = 1) = \sum_{y=1}^{6} p_{X,Y}(1, y) = \sum_{y=1}^{6} \frac{1}{36} = \frac{1}{6}$$

A Computer (or Three) In Every House

- Consider households in Silicon Valley
  - A household has X Macs and Y PCs
  - Can't have more than 3 Macs or 3 PCs

Y \ X	0	1	2	3	$p_Y(y)$
0	0.16	0.12	?	0.04	
1	0.12	0.14	0.12	0	
2	0.07	0.12	0	0	
3	0.04	0	0	0	
$p_X(x)$					

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Y \ X	0	1	2	3	$p_Y(y)$
0	0.16	0.12	0.07	0.04	0.39
1	0.12	0.14	0.12	0	0.38
2	0.07	0.12	0	0	0.19
3	0.04	0	0	0	0.04
$p_X(x)$	0.39	0.38	0.19	0.04	1.00

Marginal distributions

End Review

Permutations

How many ways are there to order n distinct objects?

$$n!$$

Binomial

How many ways are there to make an unordered selection of r objects from n objects?

How many ways are there to order n objects such that:
r are the same (indistinguishable)
$(n-r)$ are the same (indistinguishable)?

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Called the "binomial" because of something from Algebra

Binomial Distribution

- Consider *n* independent trials of Ber(p) rand. var.
  - X is number of successes in *n* trials
  - X is a **Binomial** Random Variable: X ~ Bin(n, p)

Binomial # ways of ordering the successes

$$P(X=i) = p(i) = \binom{n}{i} p^i (1-p)^{n-i} \quad i = 0,1,\ldots,n$$

Probability of exactly *i* successes

Probability of each ordering of *i* successes is equal + mutually exclusive

Multinomial

How many ways are there to order n objects such that:

n_1 are the same (indistinguishable)

n_2 are the same (indistinguishable)

$\dots$

n_r are the same (indistinguishable)?

$$\frac{n!}{n_1! n_2! \dots n_r!} = \binom{n}{n_1, n_2, \dots, n_r}$$

Note: Multinomial > Binomial

The Multinomial

- Multinomial distribution
  - n independent trials of experiment performed
  - Each trial results in one of m outcomes, with respective probabilities: $p_1, p_2, \ldots, p_m$ where $\sum_{i=1}^{m} p_i = 1$
  - X_i = number of trials with outcome i

$$P(X_1 = c_1, X_2 = c_2, \ldots, X_m = c_m) = \binom{n}{c_1, c_2, \ldots, c_m} p_1^{c_1} p_2^{c_2} \ldots p_m^{c_m}$$

Joint distribution

Multinomial # ways of ordering the successes

Probabilities of each ordering are equal and mutually exclusive

where $\sum_{i=1}^{m} c_i = n$ and $\binom{n}{c_1, c_2, \ldots, c_m} = \frac{n!}{c_1! c_2! \cdots c_m!}$

Hello Die Rolls, My Old Friends

- 6-sided die is rolled 7 times
  - Roll results: 1 one, 1 two, 0 three, 2 four, 0 five, 3 six

$$P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, X_6 = 3)$$

$$= \frac{7!}{1!1!0!2!0!3!}\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^3 = 420\left(\frac{1}{6}\right)^7$$

- This is generalization of Binomial distribution
  - Binomial: each trial had 2 possible outcomes
  - Multinomial: each trial has *m* possible outcomes

Probabilistic Text Analysis

- Ignoring order of words, what is probability of any given word you write in English?
  - P(word = “the”) > P(word = “transatlantic”)
  - P(word = “Stanford”) > P(word = “Cal”)
  - Probability of each word is just multinomial distribution
- What about probability of those same words in someone else’s writing?
  - P(word = “probability” | writer = you) >

    P(word = “probability” | writer = non-CS109 student)
  - After estimating P(word | writer) from known writings, use Bayes’ Theorem to determine P(writer | word) for new writings!

A Document is a Large *Multinomial*

According to the Global Language Monitor there are 988,968 words in the english language used on the internet.

Text is a Multinomial

Example document:
"Pay for Viagra with a credit-card. Viagra is great. So are credit-cards. Risk free Viagra. Click for free."
n = 18

It's a Multinomial!

$$P\left(\begin{array}{l}\text{Viagra} = 2\\ \text{Free} = 2\\ \text{Risk} = 1\\ \text{Credit-card: } 2\\ \dots\\ \text{For} = 2\end{array} \middle| \text{spam}\right) = \frac{n!}{2!2!\dots2!} p_{\text{viagra}}^2 p_{\text{free}}^2 \dots p_{\text{for}}^2$$

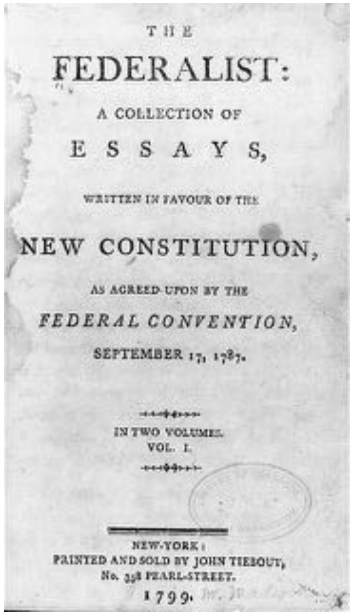
Probability of seeing this document | spam

The probability of a word in spam email being viagra

Who wrote the federalist papers?

Old and New Analysis

- Authorship of "Federalist Papers"
  - 85 essays advocating ratification of US constitution
  - Written under pseudonym "Publius"
    - Really, Alexander Hamilton, James Madison and John Jay
  - Who wrote which essays?
    - Analyzed probability of words in each essay versus word distributions from known writings of three authors

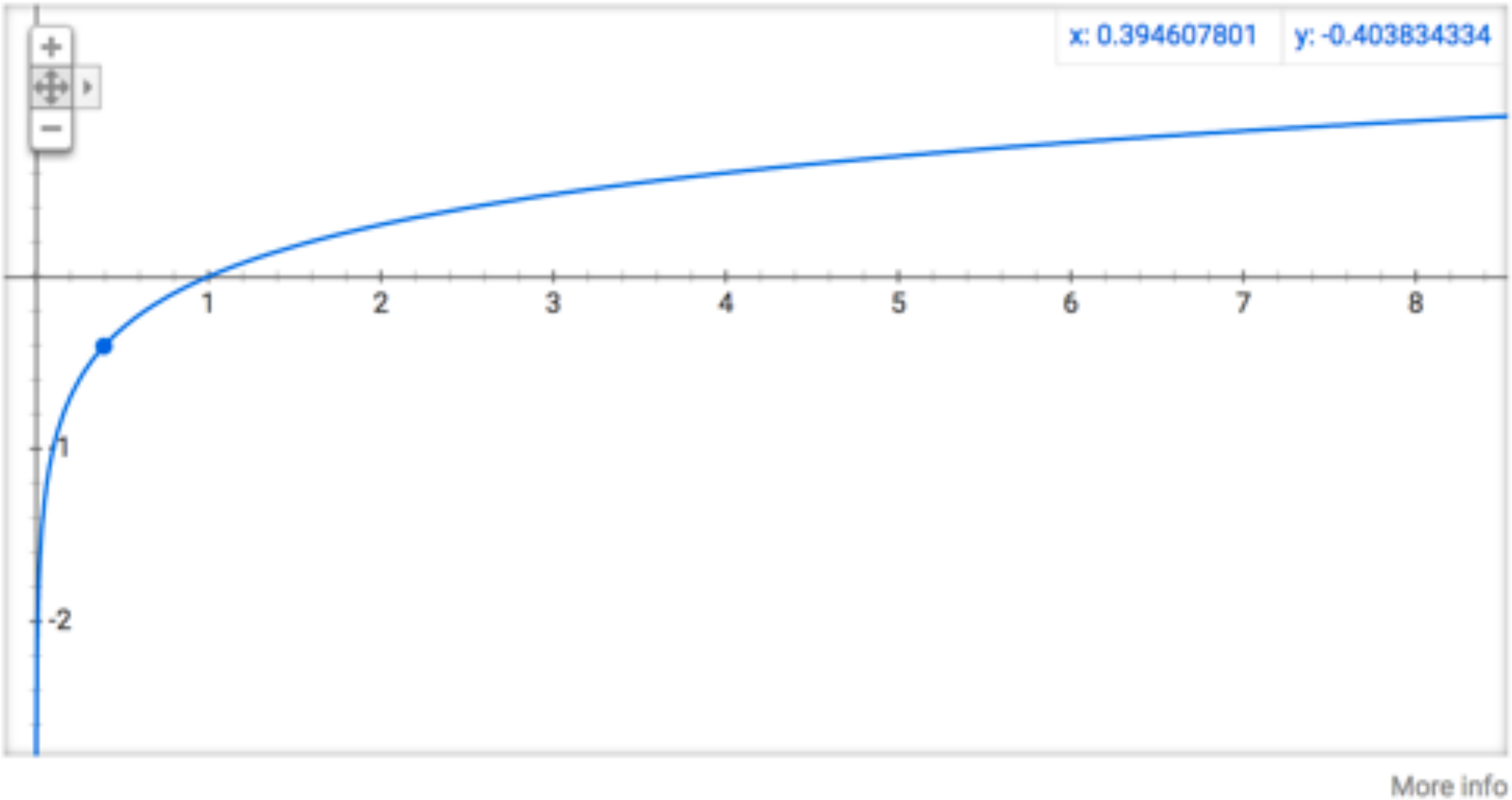

THE
FEDERALIST:
A COLLECTION OF
ESSAYS,
WRITTEN IN FAVOUR OF THE
NEW CONSTITUTION,
AS AGREED UPON BY THE
FEDERAL CONVENTION,
SEPTEMBER 17, 1787.
IN TWO VOLUMES.
VOL. I.
NEW-YORK:
PRINTED AND SOLD BY JOHN TIEBOUT,
No. 358 PEARL-STREET.
1799.

Let’s write a program!

Log Review

$$e^y = x \qquad \log(x) = y$$

Graph for log(x)



Log Identities

$$\log(a \cdot b) = \log(a) + \log(b)$$

$$\log(a/b) = \log(a) - \log(b)$$

$$\log(a^n) = n \cdot \log(a)$$

Products become Sums!

$$\log(a \cdot b) = \log(a) + \log(b)$$

$$\log(\prod_i a_i) = \sum_i \log(a_i)$$

This is important because the product of many small numbers gets hard for computers to represent.

Stretch!

Continuous Joint

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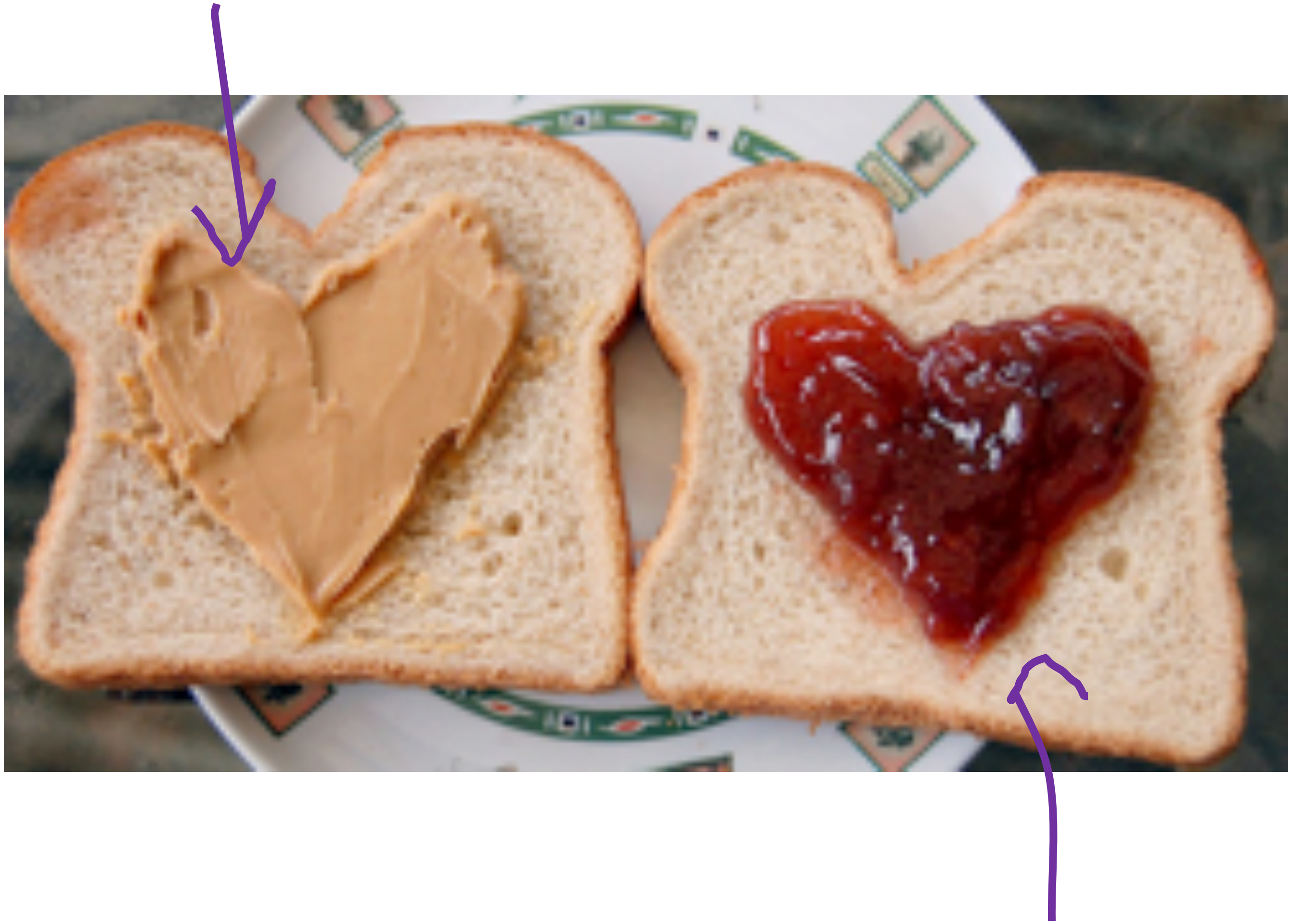
Continuous Random Variables



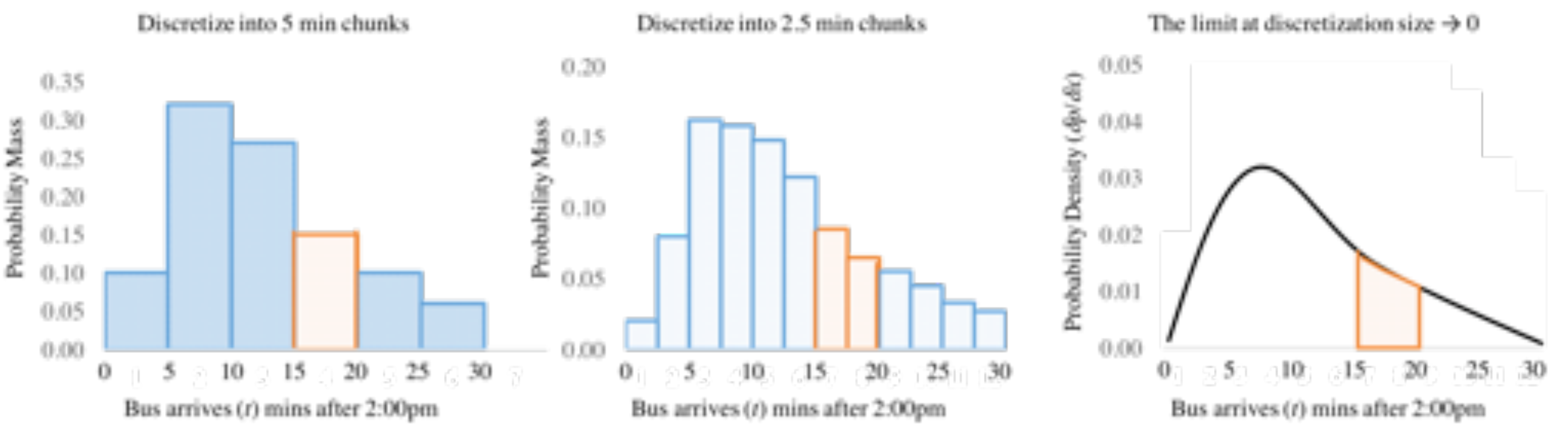
Joint Distributions

Continuous Joint Distribution

Riding the Marguerite

You are running to the bus stop. You don't know exactly when the bus arrives. You arrive at 2:20pm.

What is P(wait < 5 min)?



Joint Dart Distribution

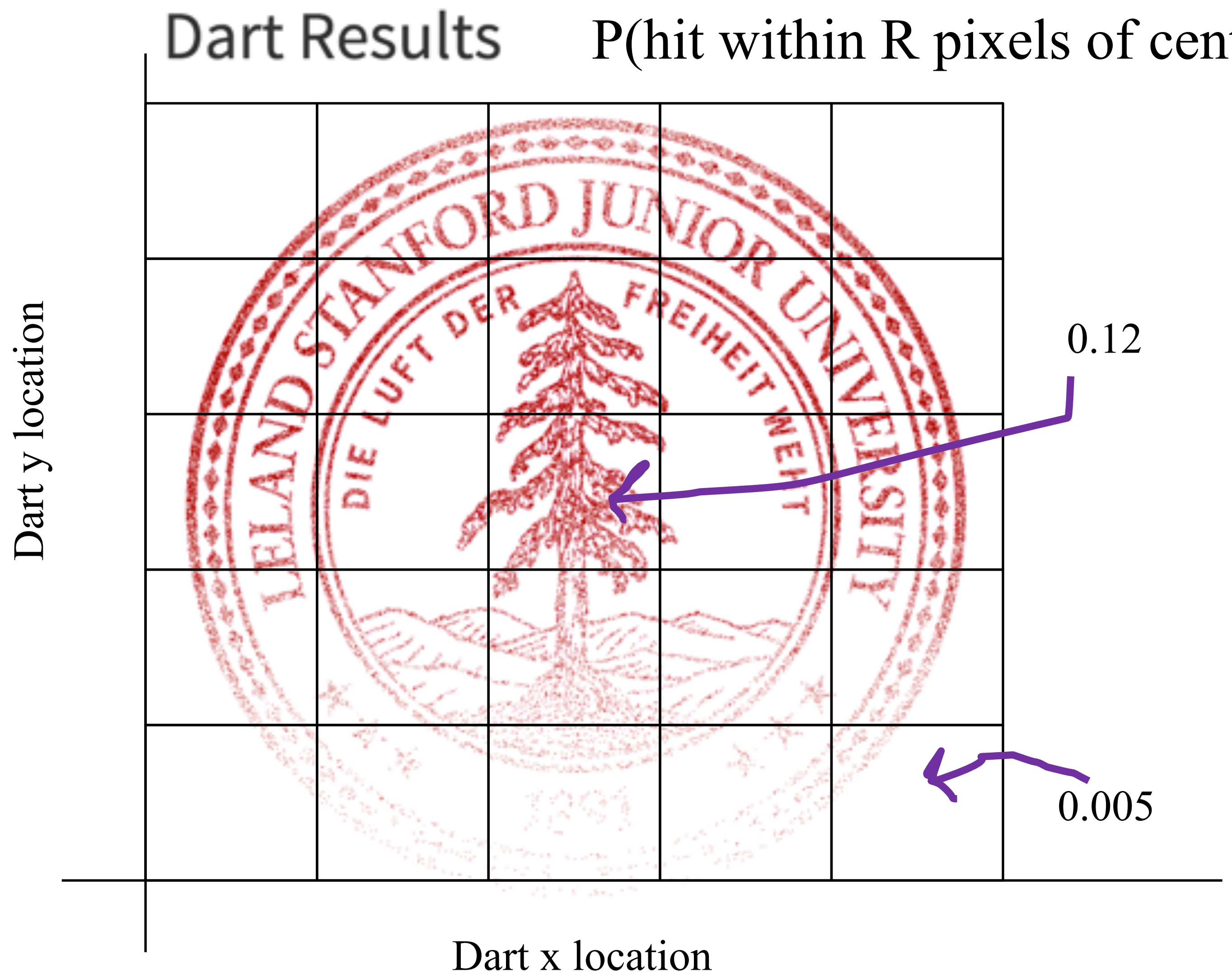
Dart Results

P(hit within R pixels of center)?

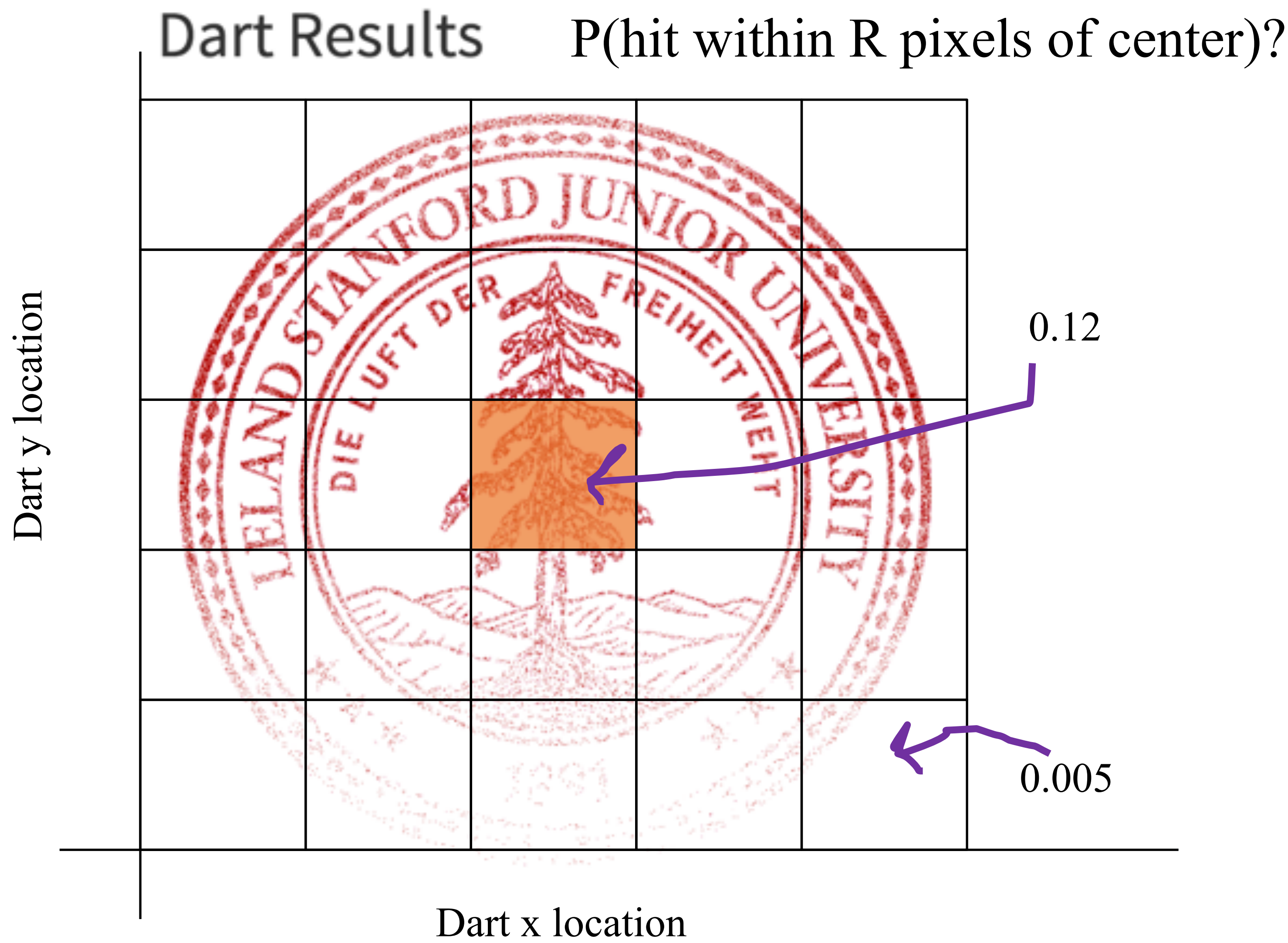
What is the probability that a dart hits at (456.234231234122355, 532.12344123456)?

Joint Dart Distribution

P(hit within R pixels of center)?

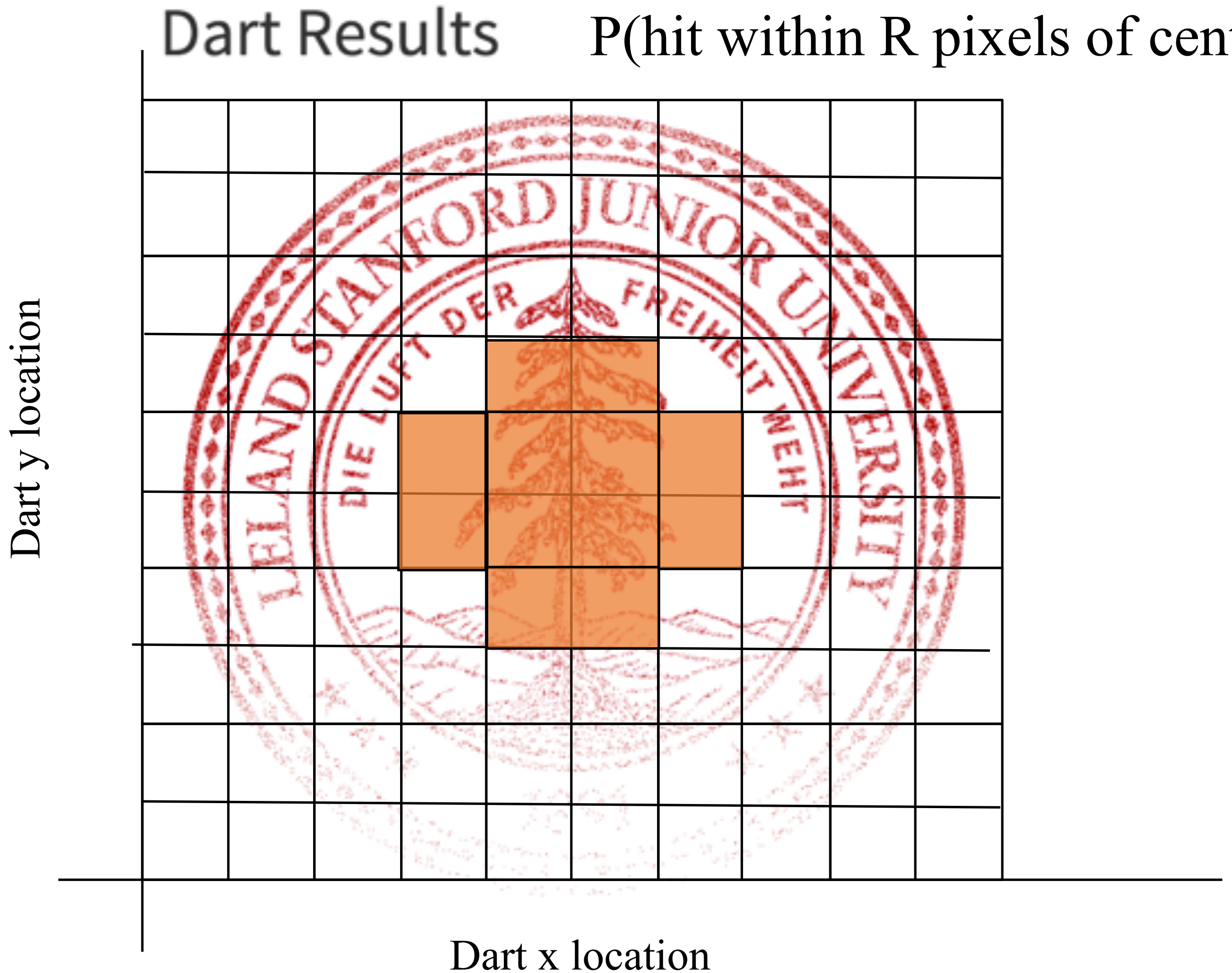


Joint Dart Distribution

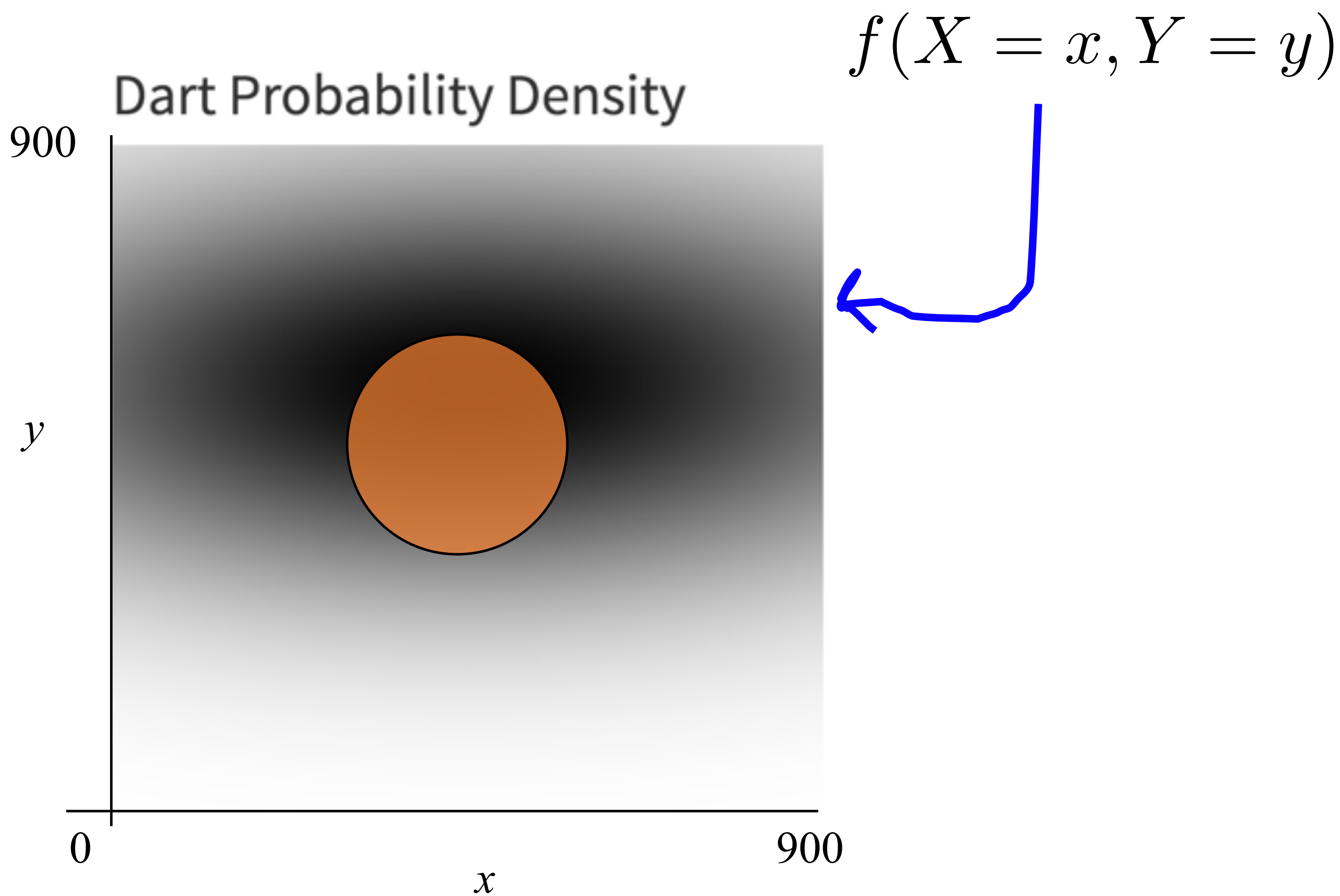


Joint Dart Distribution

P(hit within R pixels of center)?



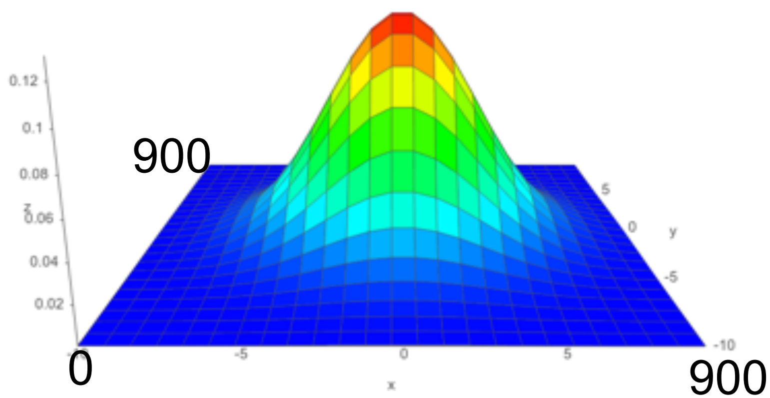
Joint Dart Distribution

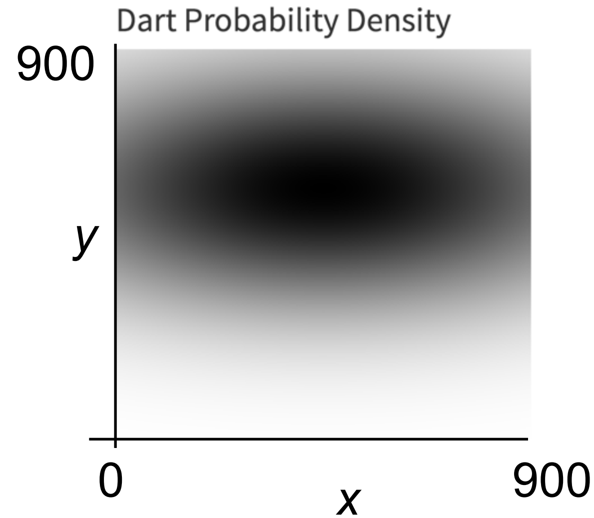


In the limit, as you break down continuous values into intestinally small buckets, you end up with multidimensional probability density

Joint Probability Density Funciton

A **joint probability density function** gives the relative likelihood of **more than one** continuous random variable **each** taking on a specific value.





$$P(a_1 < X < a_2, b_1 < Y < b_2) = \int_{x=a_1}^{a_2} \int_{y=b_1}^{b_2} f(X = x, Y = y)\ \partial y \partial x$$

Joint Probability Density Funciton

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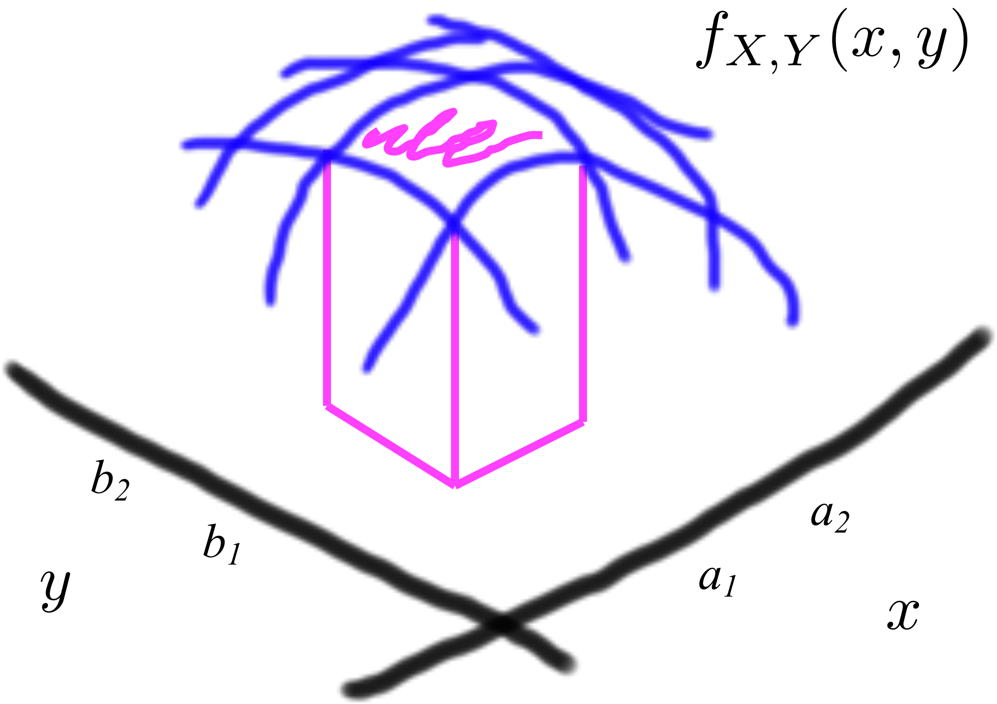

Joint Probability Density Function

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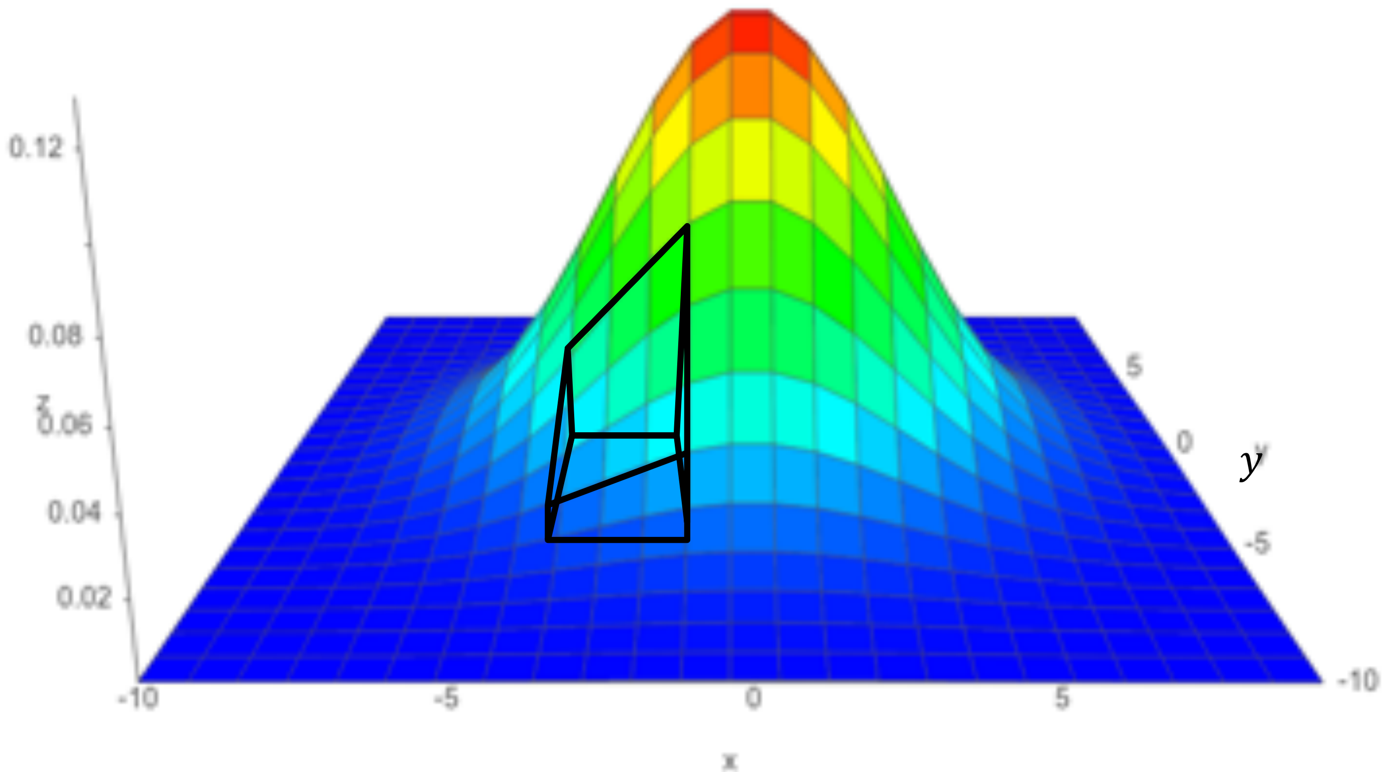


plot by Academo

Multiple Integrals Without Tears

- Let *X* and *Y* be two continuous random variables
  - where $0 \le X \le 1$ and $0 \le Y \le 2$
- We want to integrate $g(x,y) = xy$ w.r.t. *X* and *Y*:
  - First, do "innermost" integral (treat *y* as a constant):

$$\int_{y=0}^{2}\int_{x=0}^{1} xy\,dx\,dy = \int_{y=0}^{2}\left(\int_{x=0}^{1} xy\,dx\right)dy = \int_{y=0}^{2} y\left[\frac{x^2}{2}\right]_0^1 dy = \int_{y=0}^{2} y\,\frac{1}{2}\,dy$$

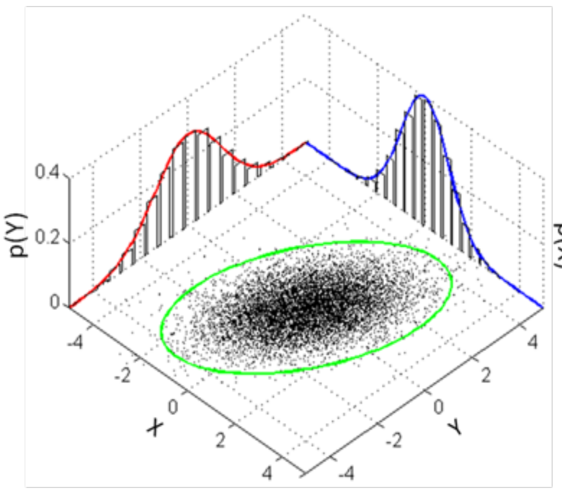
  - Then, evaluate remaining (single) integral:

$$\int_{y=0}^{2} y\,\frac{1}{2}\,dy = \left[\frac{y^2}{4}\right]_0^2 = 1 - 0 = 1$$

Marginalization

Marginal probabilities give the distribution of **a subset of the variables** (often, just one) of a joint distribution.

Sum/integrate over the variables you don't care about.

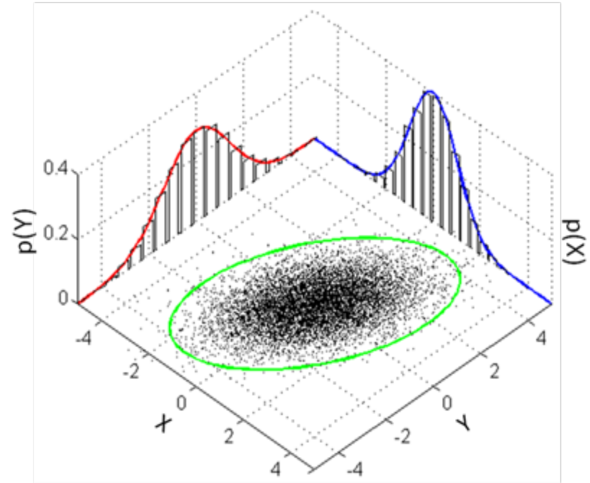


$$p_X(a) = \sum_y p_{X,Y}(a, y)$$

$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y)\ dy$$

$$f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x, b)\ dx$$

Marginalization



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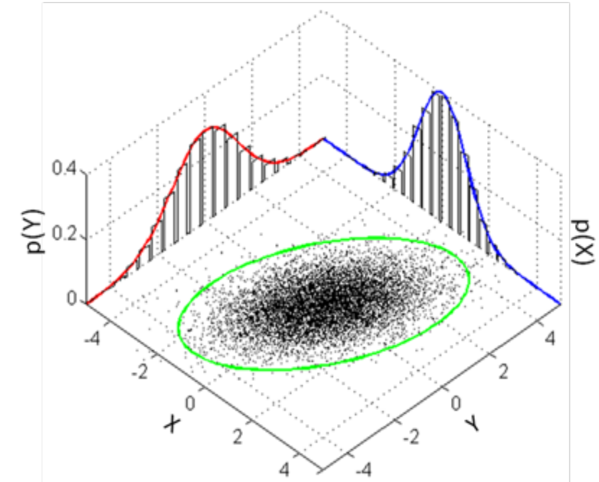

$$p_X(a) = \sum_y P(X = a, Y = y)$$

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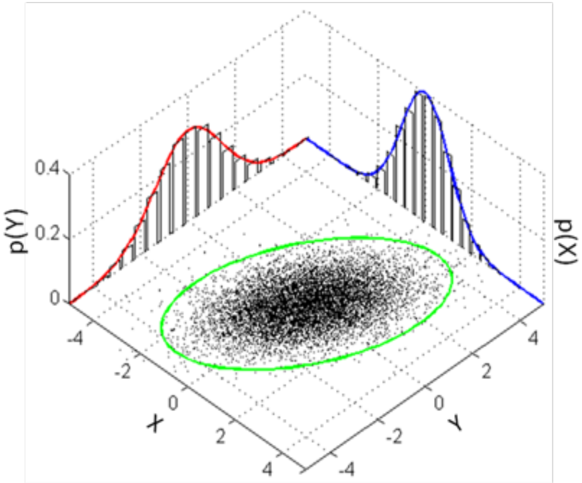

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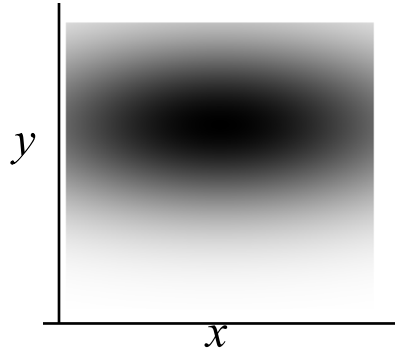
$$f(X = a) = \int_{-\infty}^{\infty} f(X = a, Y = y)$$

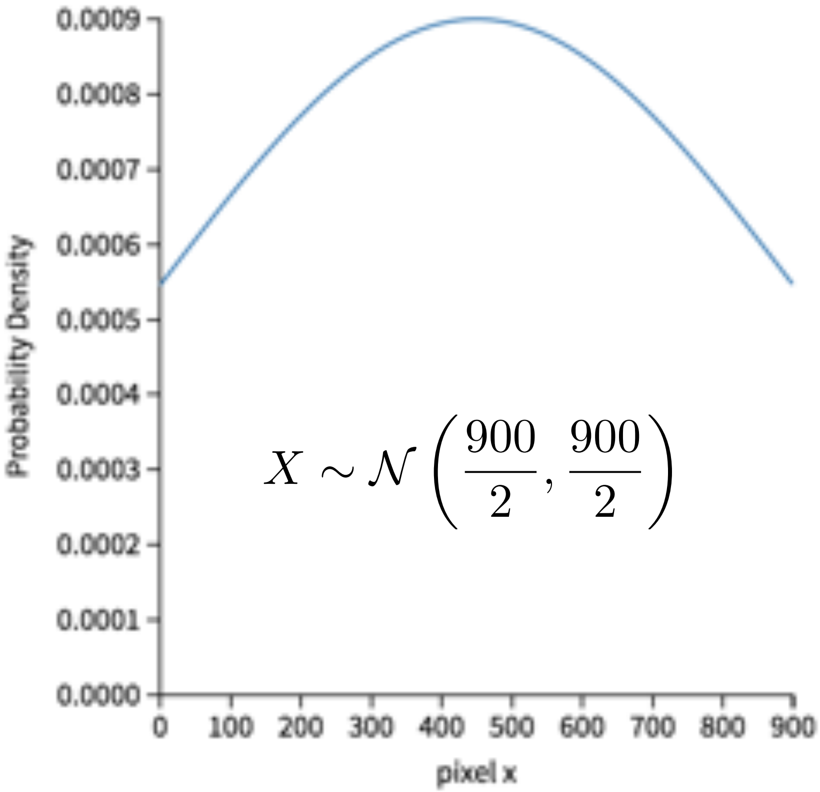
$$f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x, b)\ dx$$

Darts!

Dart PDF

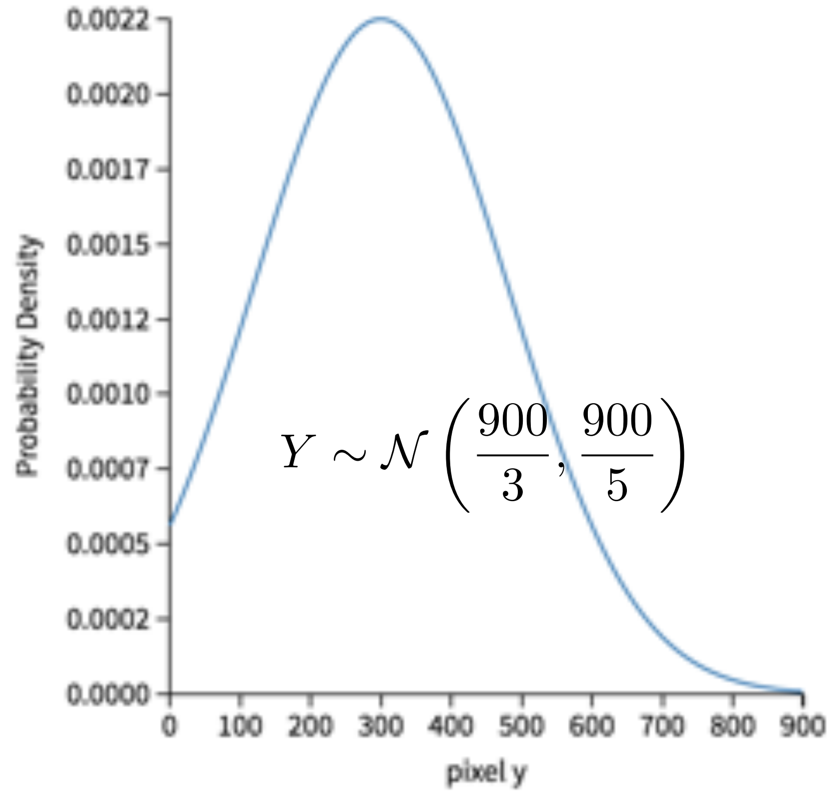


X-Pixel Marginal



$$X \sim \mathcal{N}\left(\frac{900}{2}, \frac{900}{2}\right)$$

Y-Pixel Marginal



$$Y \sim \mathcal{N}\left(\frac{900}{3}, \frac{900}{5}\right)$$

Joint Cumulative Density Function

Cumulative Density Function (CDF):

$$F_{X,Y}(a, b) = P(X < a, Y < b)$$

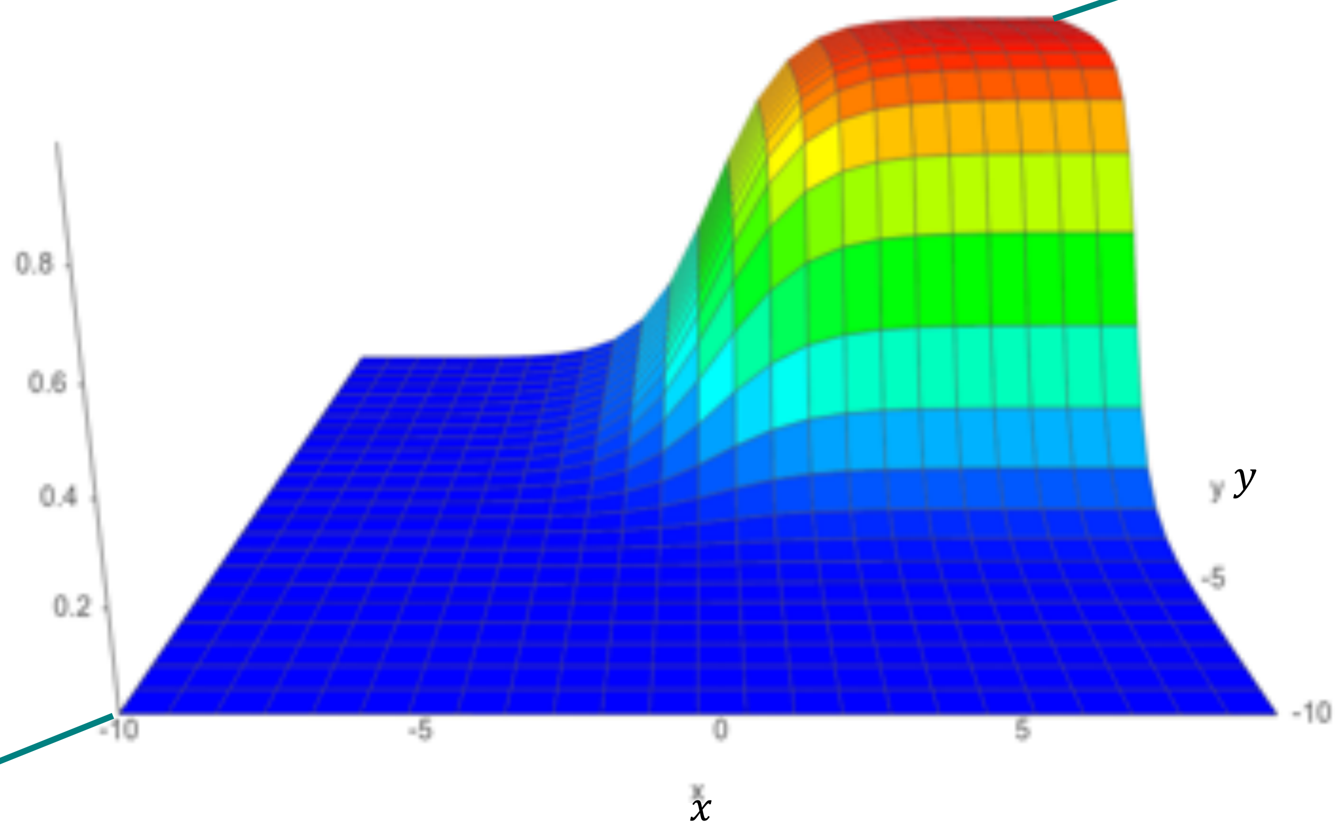
$$F_{X,Y}(a,b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x, y)\, dy\, dx$$

$$f_{X,Y}(a,b) = \frac{\partial^2}{\partial a\, \partial b} F_{X,Y}(a,b)$$

Joint CDF

$$F_{X,Y}(a,b) = P(X < a, Y < b)$$

to 1 as
x → +∞,
y → +∞



to 0 as
x → -∞,
y → -∞

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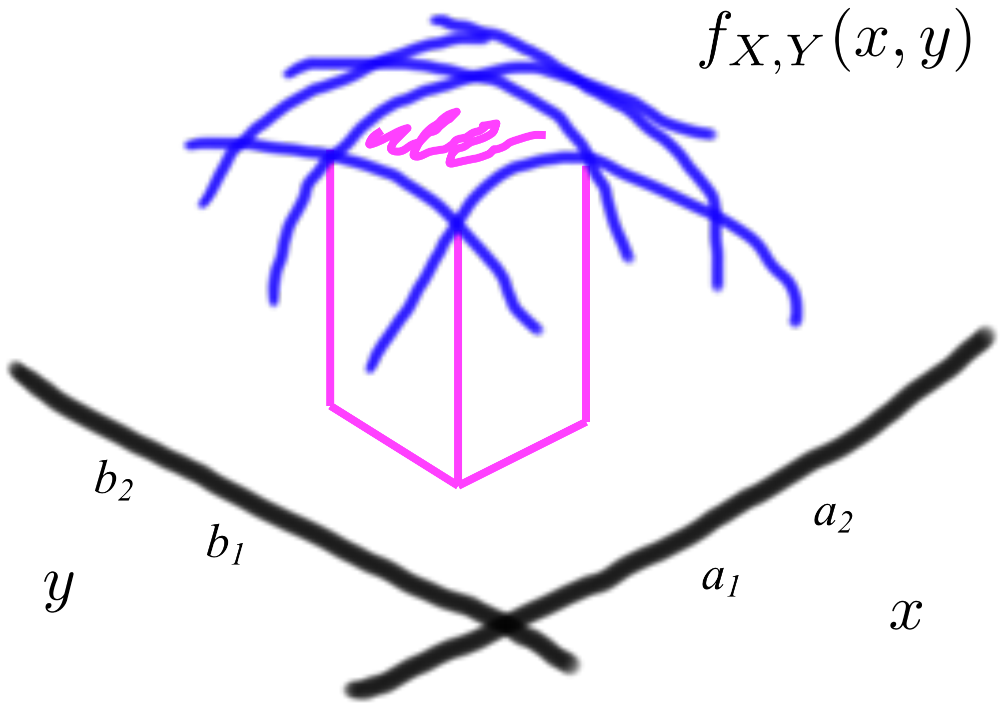
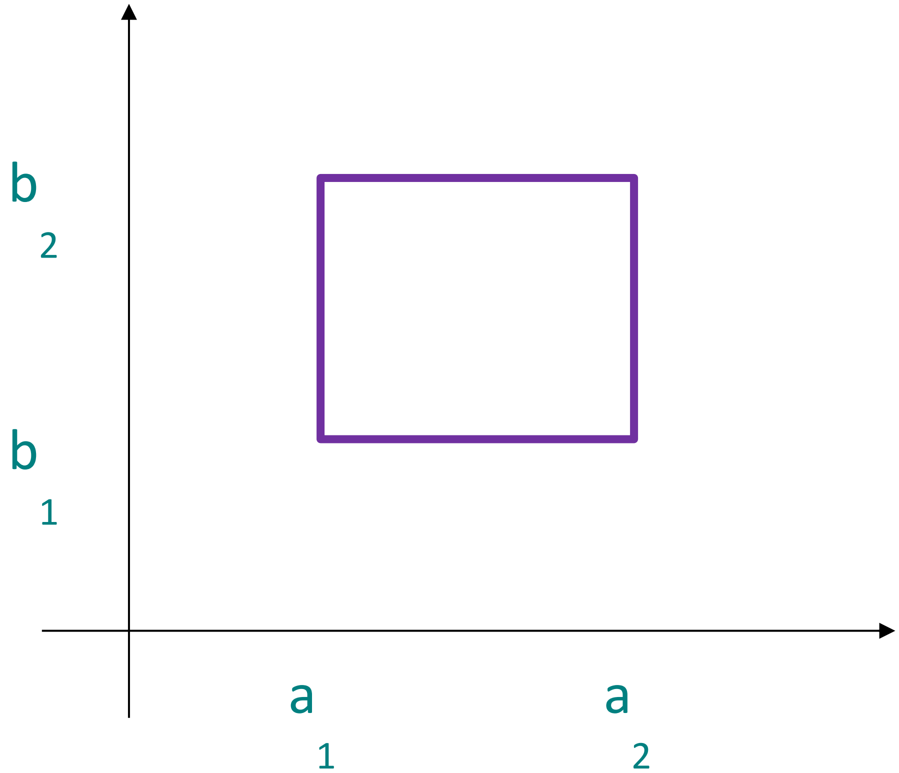
Jointly Continuous

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x,y)\, dy\, dx$$



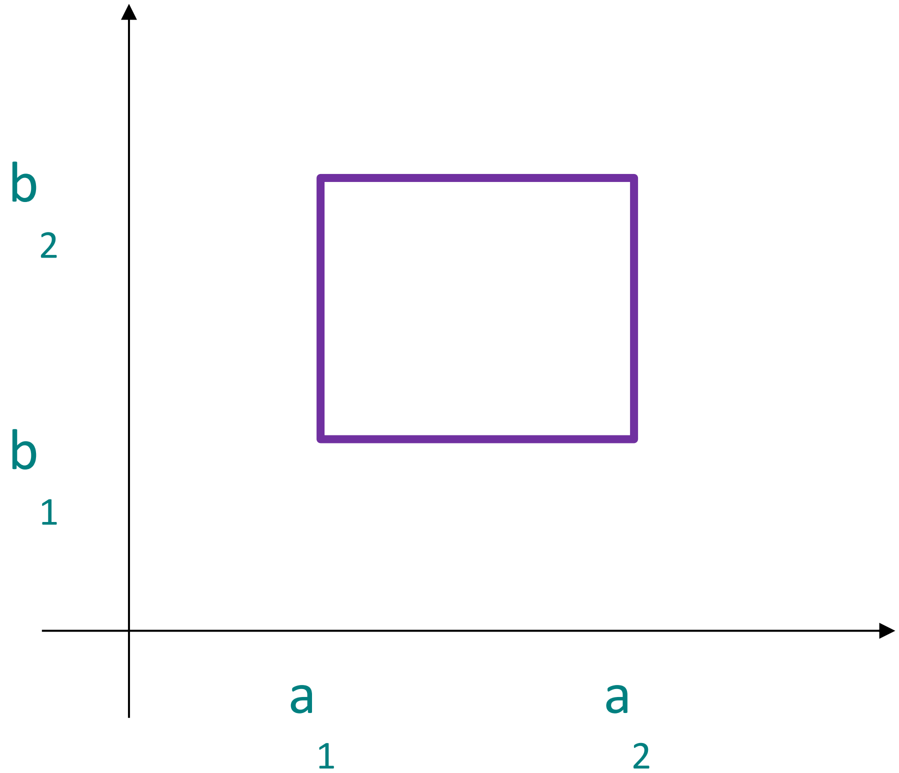
Probabilities from Joint CDF

$$P(a_1 < X \le a_{2,} b_1 < Y \le b_2)$$



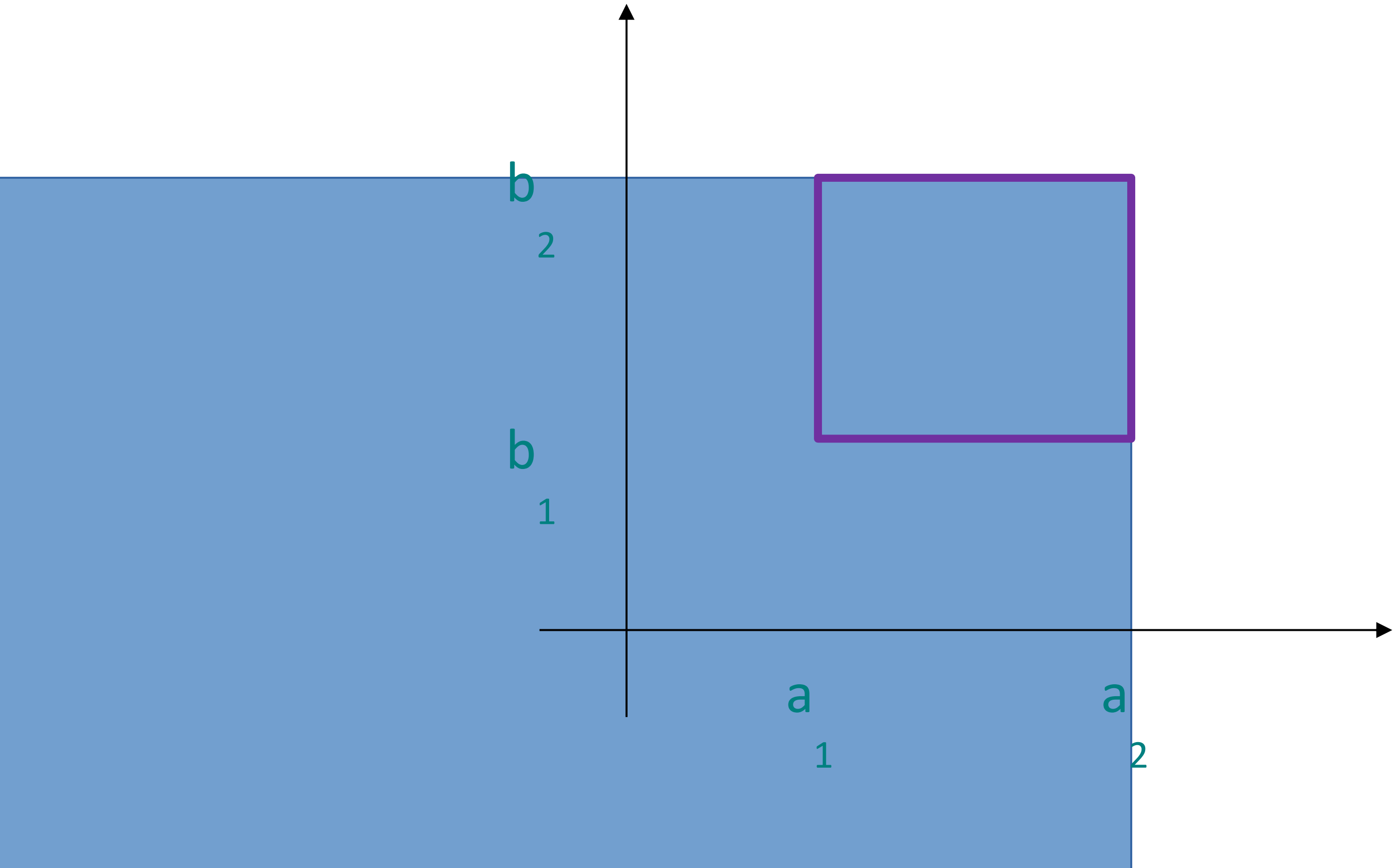
Probabilities from Joint CDF

$$P(a_1 < X \le a_{2,} b_1 < Y \le b_2) = F_{X,Y}(a_{2,} b_2)$$



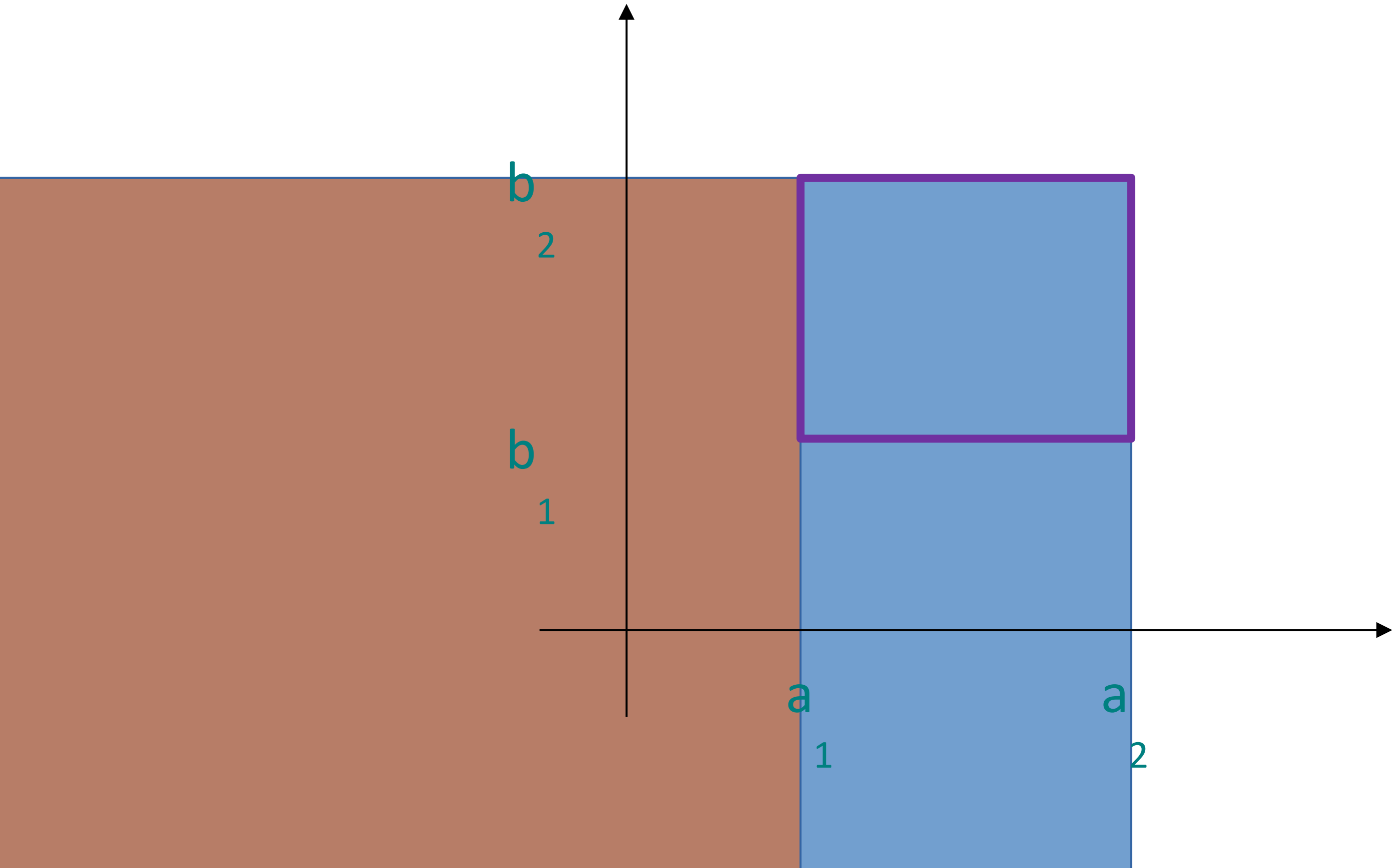
Probabilities from Joint CDF

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2)$$



Probabilities from Joint CDF

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2)$$
$$-F_{X,Y}(a_1, b_2)$$



Probabilities from Joint CDF

$$P(a_1 < X \le a_2, b_1 < Y \le b_2) = F_{X,Y}(a_2, b_2)$$
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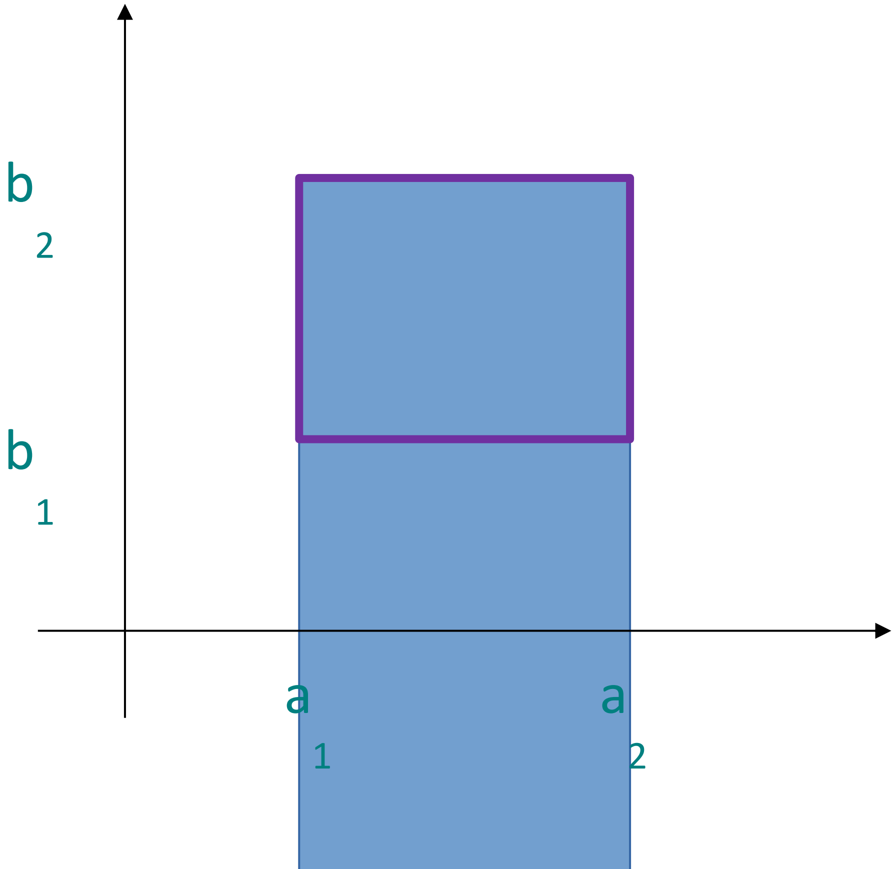

Probabilities from Joint CDF

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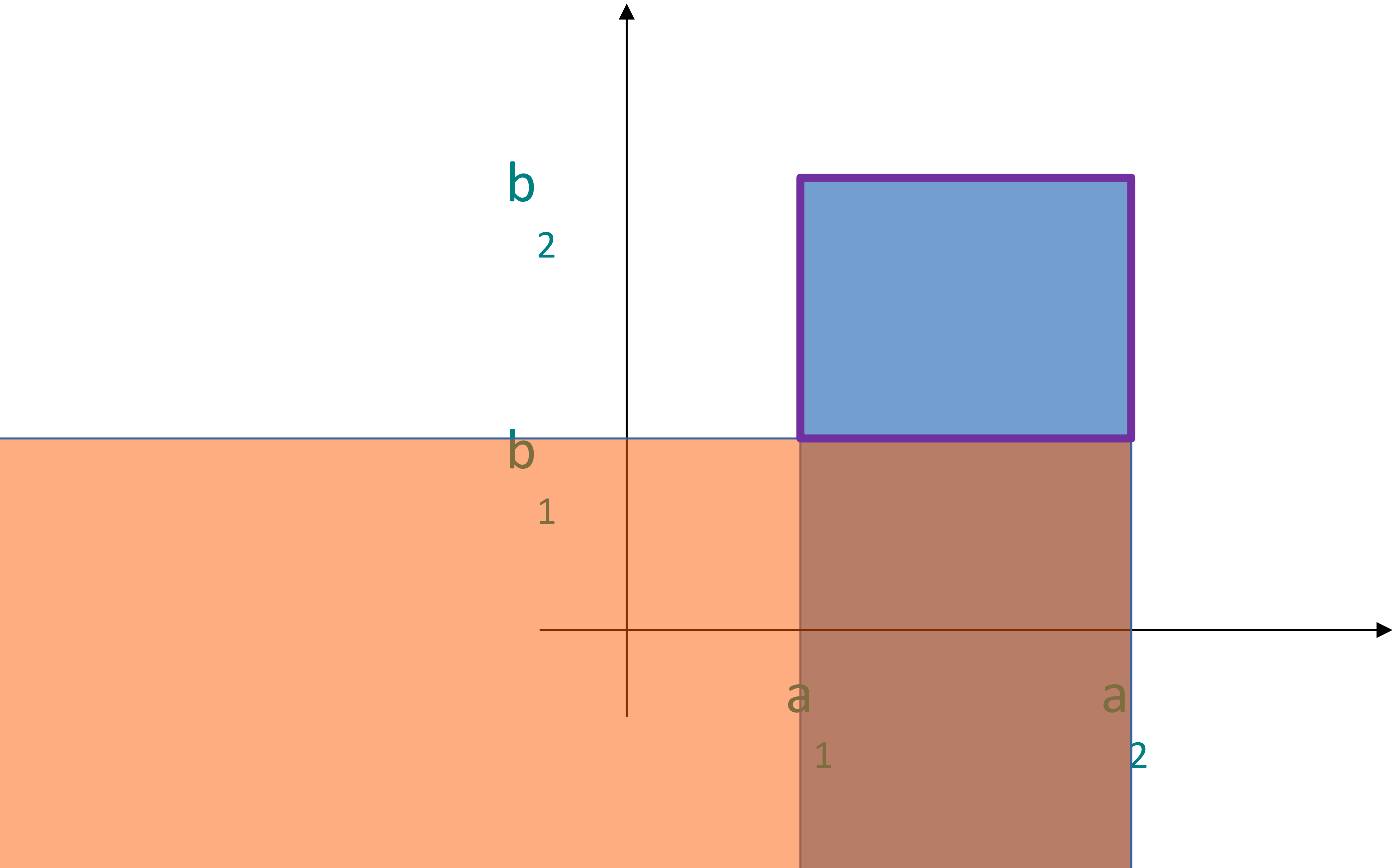
$$- F_{X,Y}(a_1, b_2)$$
$$- F_{X,Y}(a_2, b_1)$$



Probabilities from Joint CDF

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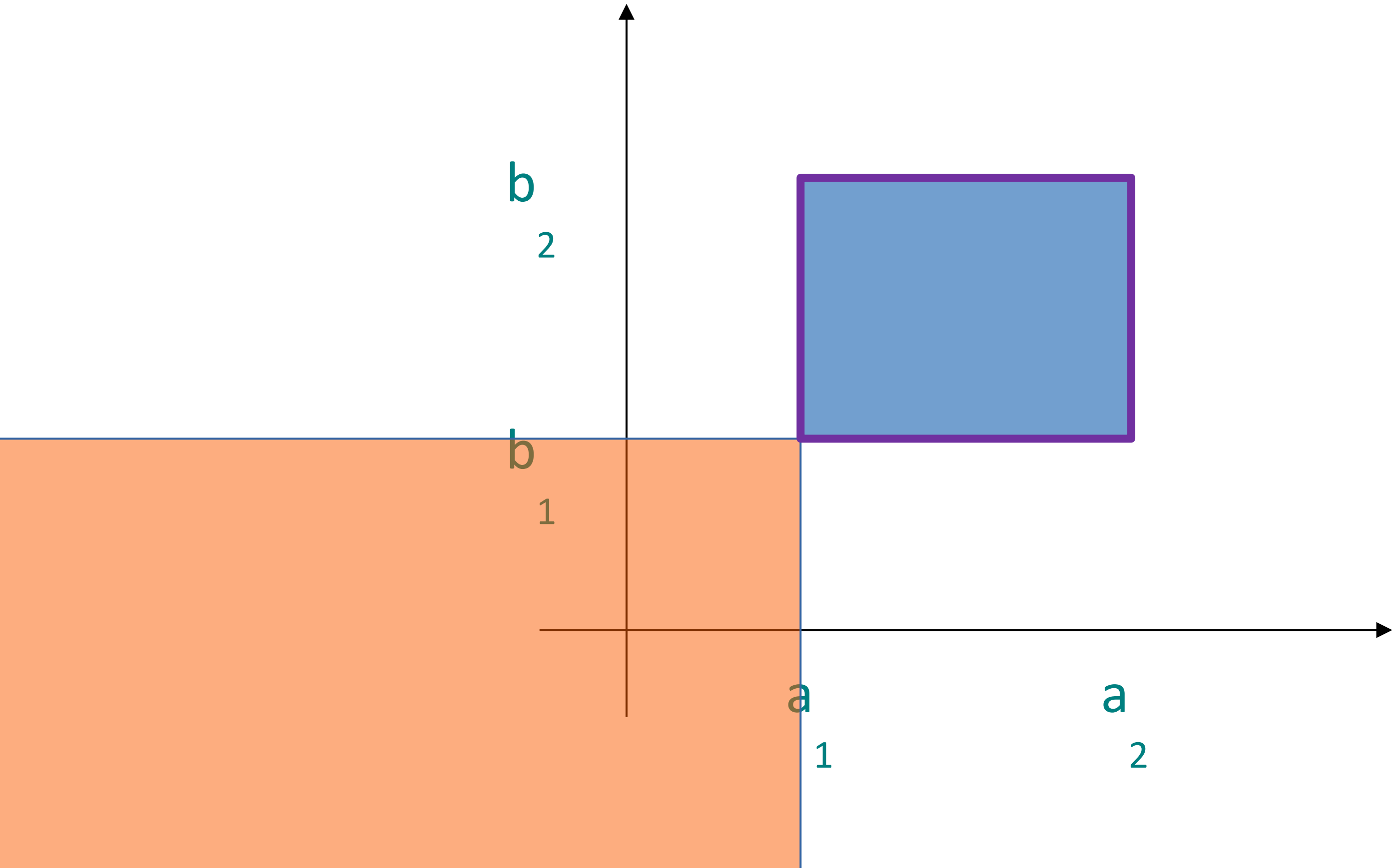
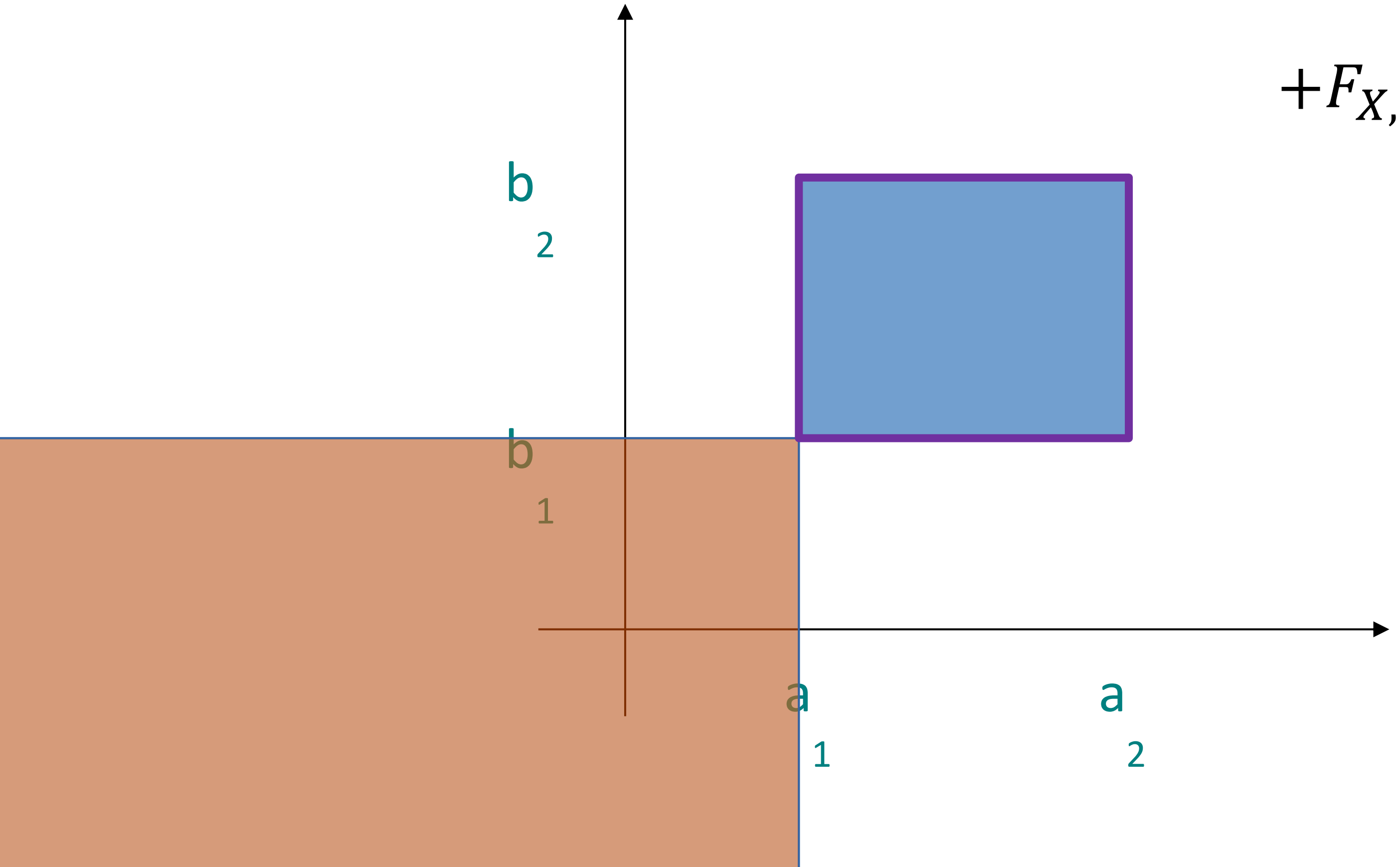


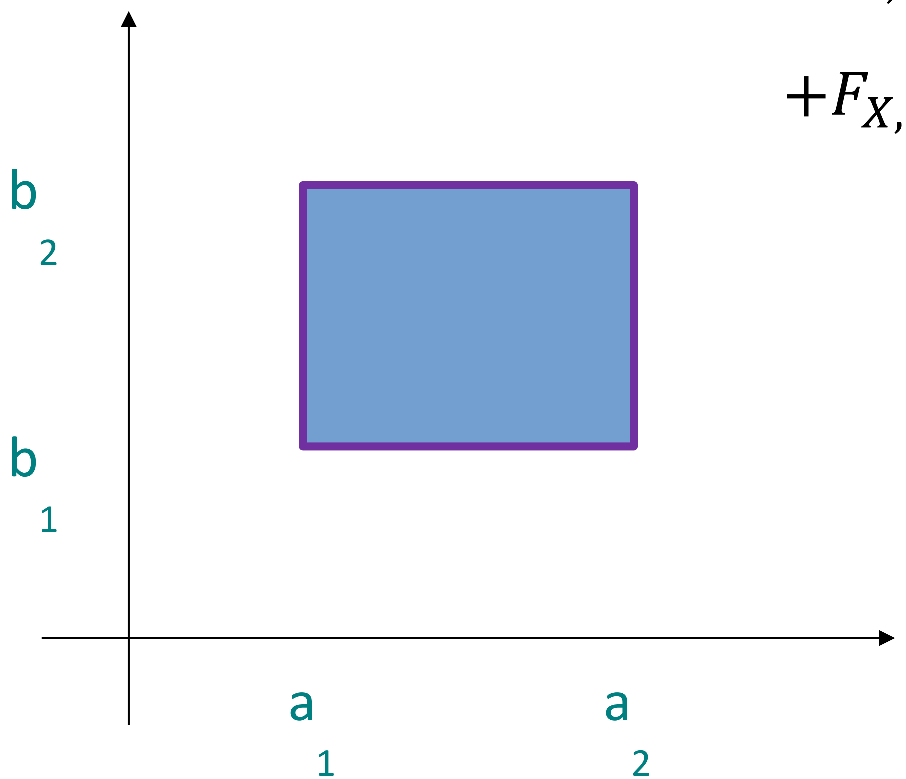
Probabilities from Joint CDF

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2)$$
$$-F_{X,Y}(a_1, b_2)$$
$$-F_{X,Y}(a_2, b_1)$$
$$+F_{X,Y}(a_1, b_1)$$



Probabilities from Joint CDF

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2)$$
$$-F_{X,Y}(a_1, b_2)$$
$$-F_{X,Y}(a_2, b_1)$$
$$+F_{X,Y}(a_1, b_1)$$



Probability for Instagram!

Gaussian Blur

In image processing, a Gaussian blur is the result of blurring an image by a Gaussian function. It is a widely used effect in graphics software, typically to reduce image noise.

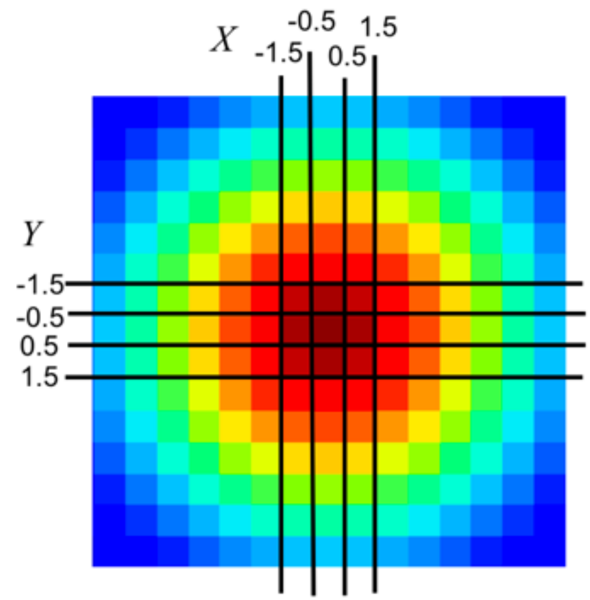
Gaussian blurring with StDev = 3, is based on a joint probability distribution:

Joint PDF

$$f_{X,Y}(x,y) = \frac{1}{2\pi \cdot 3^2} e^{-\frac{x^2+y^2}{2\cdot 3^2}}$$

Joint CDF

$$F_{X,Y}(x,y) = \Phi\left(\frac{x}{3}\right) \cdot \Phi\left(\frac{y}{3}\right)$$



Used to generate this weight matrix

Gaussian Blur

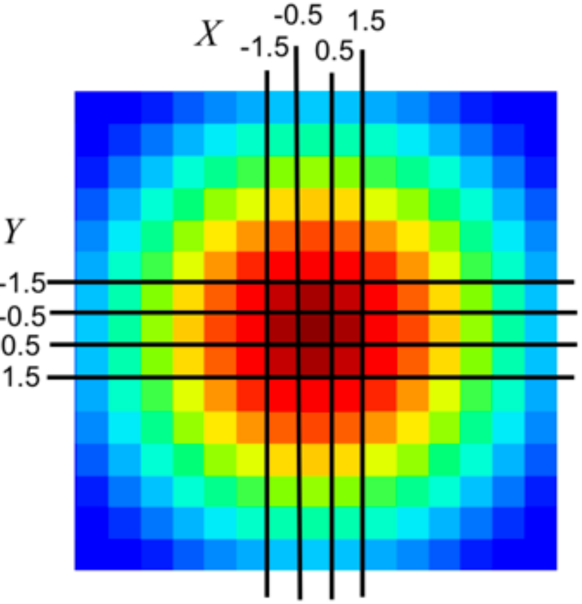
Joint PDF

$$f_{X,Y}(x,y) = \frac{1}{2\pi \cdot 3^2} e^{-\frac{x^2+y^2}{2 \cdot 3^2}}$$

Joint CDF

$$F_{X,Y}(x,y) = \Phi\left(\frac{x}{3}\right) \cdot \Phi\left(\frac{y}{3}\right)$$

Weight Matrix



Each pixel is given a weight equal to the probability that X and Y are both within the pixel bounds. The center pixel covers the area where

$-0.5 \le x \le 0.5$ and $-0.5 \le y \le 0.5$

What is the weight of the center pixel?

$$\begin{aligned}
&P(-0.5 < X < 0.5, -0.5 < Y < 0.5) \\
=&P(X < 0.5, Y < 0.5) - P(X < 0.5, Y < -0.5) \\
&- P(X < -0.5, Y < 0.5) + P(X < -0.5, Y < -0.5) \\
=&\phi\left(\frac{0.5}{3}\right) \cdot \phi\left(\frac{0.5}{3}\right) - 2\phi\left(\frac{0.5}{3}\right) \cdot \phi\left(\frac{-0.5}{3}\right) \\
&+ \phi\left(\frac{-0.5}{3}\right) \cdot \phi\left(\frac{-0.5}{3}\right) \\
=&0.5662^2 - 2 \cdot 0.5662 \cdot 0.4338 + 0.4338^2 = 0.206
\end{aligned}$$

How do you integrate under a circle?

$$f(X = x, Y = y)$$