Combinatorics

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Review

Counting

We are counting:

of **events**,

of **outcomes**,

of **objects**

Two Key Rules

Counting outcomes with or:

Inclusion Exclusion:

If outcomes can come from set A **or** set B, then the total number of outcomes is $|A| + |B| - |A \cap B|$.

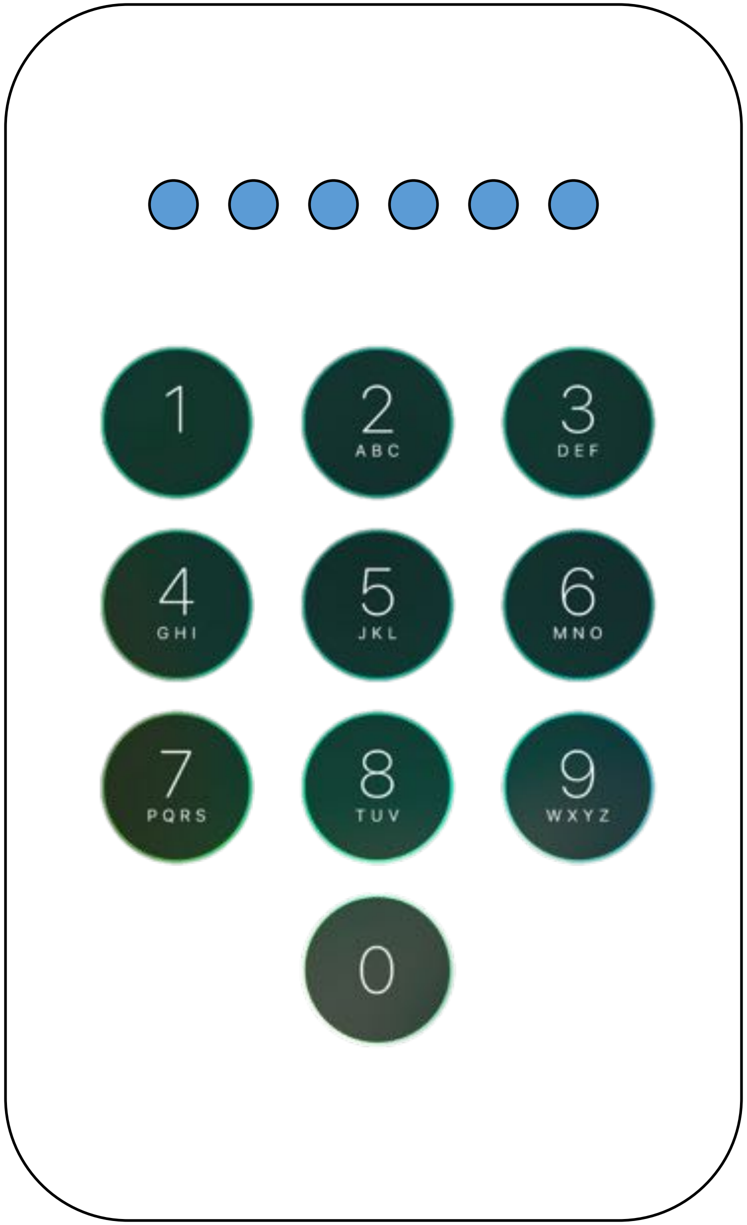
Counting outcomes with steps:

Product Rule of Counting:

If outcomes are generated via a process with r **steps**, where step i has n_i outcomes, then the total number of outcomes is:

$$n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i$$

How Many Unique 6 digit passcodes?



How many unique 6 digit passcodes are there?

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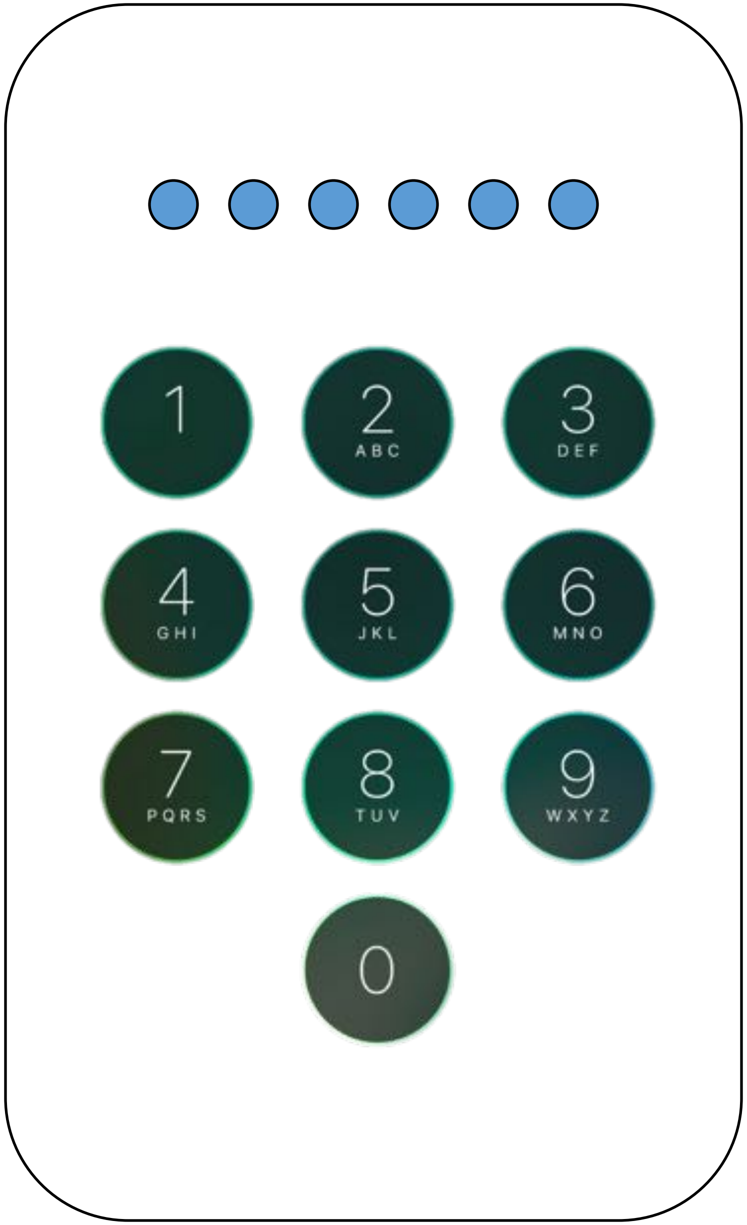


How many unique 6 digit passcodes are there?

$$10^6 = 1,000,000$$

Combinatorics

Counting tasks on *n* objects

Sort objects (permutations)

Choose *k* objects (combinations)

Put objects in *r* buckets

Combinatorics

Counting tasks on *n* objects

- Sort objects (permutations)
  - Distinct
- Choose *k* objects (combinations)
- Put objects in *r* buckets

Sort *n* Distinct Objects

Ayesha Tim Irina Joey Waddie

Sort *n* Distinct Objects

Sort 5 distinct cans:

Step 1: Chose first can (5 options)

Irina

Sort *n* Distinct Objects

Sort 5 distinct cans:

Step 1: Chose first can (5 options)

Step 2: Chose second can (4 options)

Irina

Waddie

$$5 \times 4 \times 3 \times 2 \times 1 =$$

$\cdots$ **120 unique sorts**

Sort *n* Distinct Objects

Def Permutations:

A permutation is an ordered arrangement of distinct object.

n objects can be permuted in:

$$n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 = n!$$

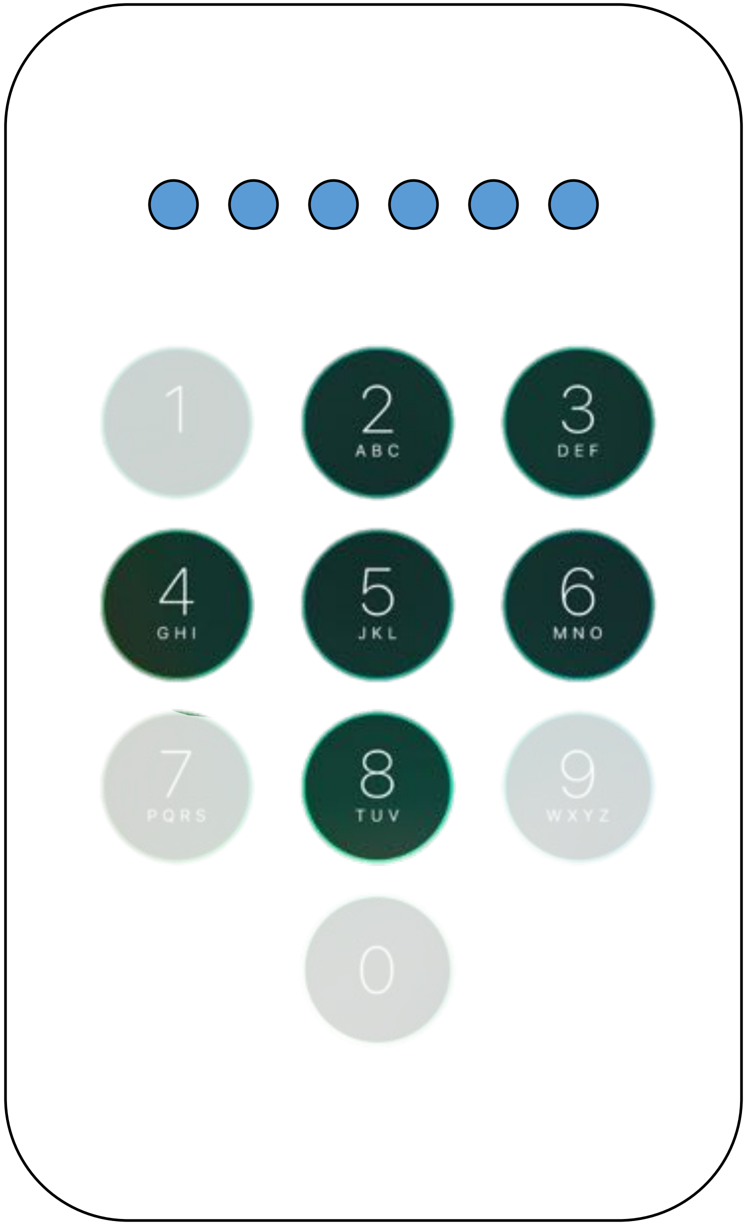
(Select 1st object out of n, then 2nd object out of n – 1, etc.)

Sort Distinct Objects

Ayesha Tim Irina Joey Waddie

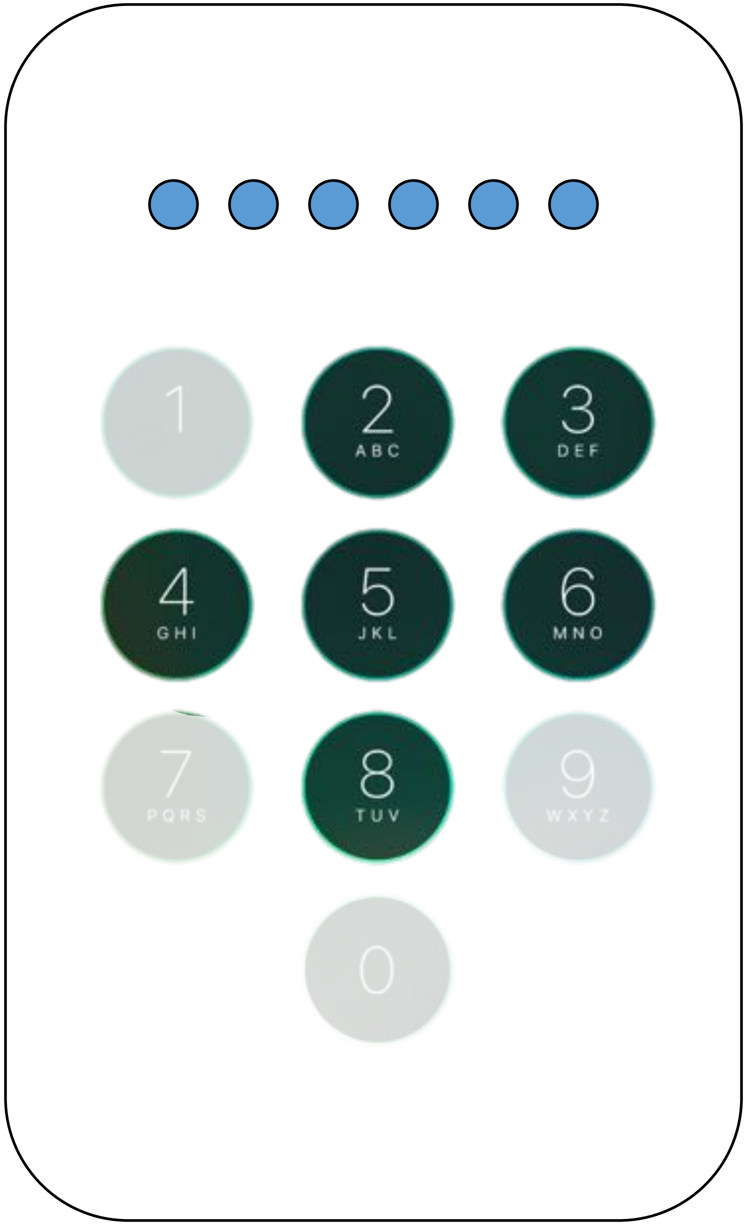
$$= 120$$

How many possible codes 6 smudges?



If a phone password uses each of **six** distinct numbers, how many unique six digit passcodes are there?

How many possible codes 6 smudges?



If a phone password uses each of **six** distinct numbers, how many unique six digit passcodes are there?

$$6! = 720$$

Sort Distinct Objects

Ayesha Tim Irina Joey Waddie

$= 120$

Sort Semi-Distinct Objects

Coke Tim Coke Joey Waddie

Sort Semi-Distinct Objects

Coke Tim Coke Joey Waddie

$$= 120/2$$

Sort Semi-Distinct Objects

Making perms of distinct objects is a two step process

Step 1

Step 2

perms of distinct objects = perms considering some objects are indistinct × perms of just the indistinct objects

Sort Semi-Distinct Objects

perms of distinct objects = perms considering some objects are indistinct × perms of just the indistinct objects

Sort Semi-Distinct Objects

$$\frac{\text{perms of distinct objects}}{\text{perms of just the indistinct objects}} = \text{perms considering some objects are indistinct}$$

General Way to Count Permutations

Def: General Permutations:

When there are n objects
n_1 are the same (indistinguishable) and
n_2 are the same and

...

n_r are the same,

There are:

$$\frac{n!}{n_1!n_2!\ldots n_r!}$$

Unique orderings ("permutations")

How many orderings?

Coke

Coke0

Coke

Coke0

Coke0

How many orderings?

Coke Coke0 Coke Coke0 Coke0

$$= 120/(3! \times 2!) = 10$$

How many orderings of letters?

MOO

MISSISSIPPI

How many orderings of letters?

MOO $= \frac{3!}{2!} = 3$

MISSISSIPPI

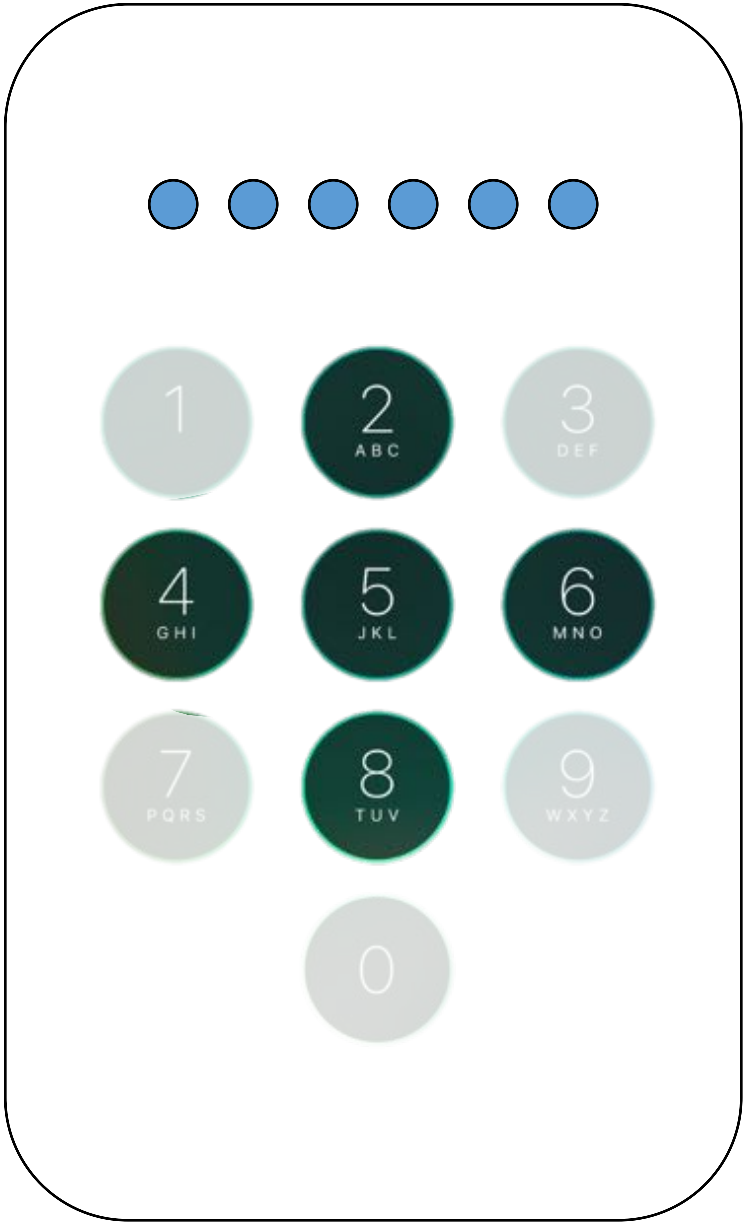
How many orderings of letters?

MOO $= \frac{3!}{2!} = 3$

MISSISSIPPI

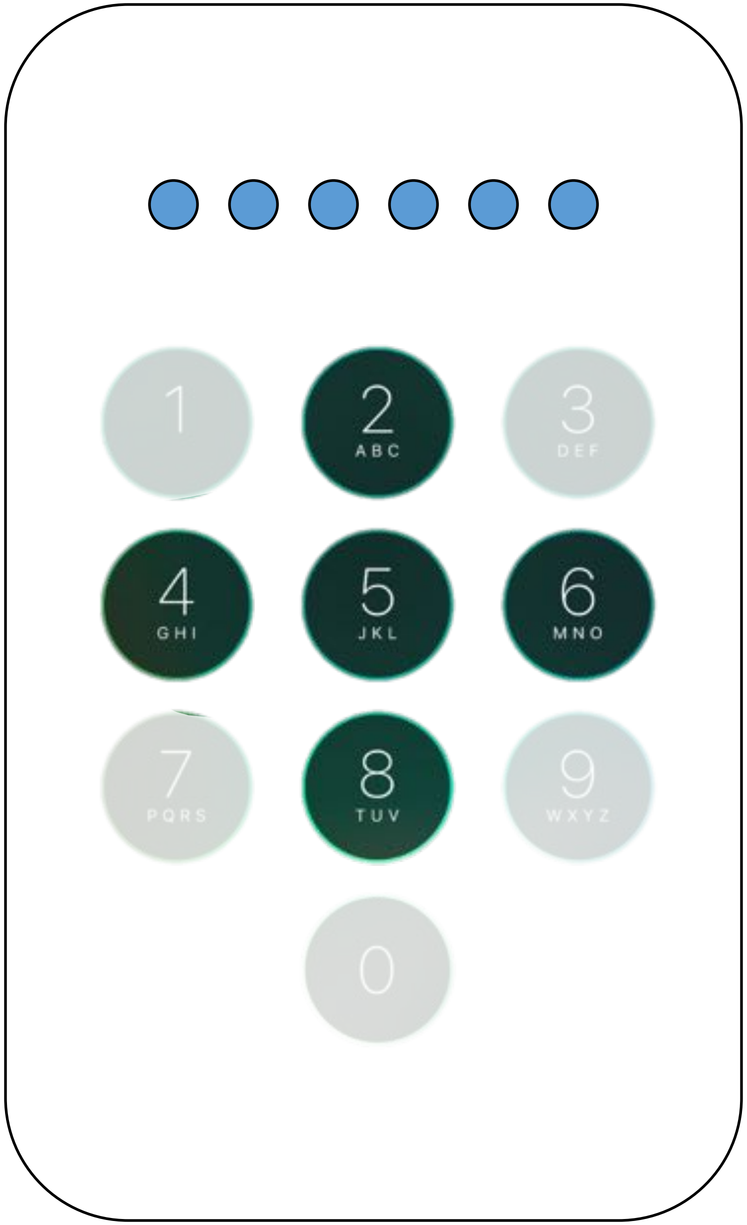
$$= \frac{11!}{1!4!4!2!} = 34,650$$

How many possible codes 5 smudges?



If a phone password uses each of **five** distinct numbers, how many unique six digit passcodes are there?

How many possible codes 5 smudges?



If a phone password uses each of **five** distinct numbers, how many unique six digit passcodes are there?

Five mutually exclusive cases:
2 was repeated
4 was repeated
5 was repeated
6 was repeated
8 was repeated

$$= 5 \times \frac{6!}{2!} = 1,800$$

Combinatorics

Counting tasks on *n* objects

Sort objects (permutations)

Choose *k* objects (combinations)

Put objects in *r* buckets

Distinct

$$n!$$

Some Distinct

$$\frac{n!}{n_1!n_2!\dots}$$

Combinatorics

Counting tasks on *n* objects

- Sort objects (permutations)
  - Distinct: $n!$
  - Some Distinct: $$\frac{n!}{n_1!n_2!\ldots}$$
- Choose *k* objects (combinations)
  - Distinct
- Put objects in *r* buckets

Combinatorics

There are $n = 20$ people

How many ways can we **Choose** $k = 5$ people to get cake?

Consider this
generative process

Step 1: Randomly order people

There are $n = 20$ people

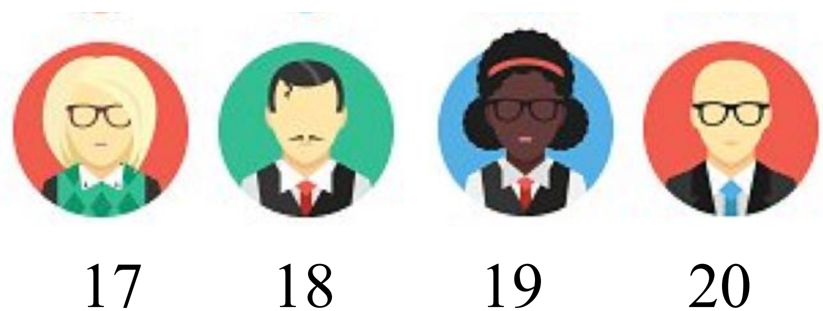
How many ways can we **Choose** $k = 5$ people to get cake?

$$\text{step 1 ways} = n!$$

Step 2: Draw a line at pos k

There are $n = 20$ people

How many ways can we **Choose** $k = 5$ people to get cake?



$$\text{step 2 ways} = 1$$

Step 3: Allow Cake Group to Mingle

There are $n = 20$ people

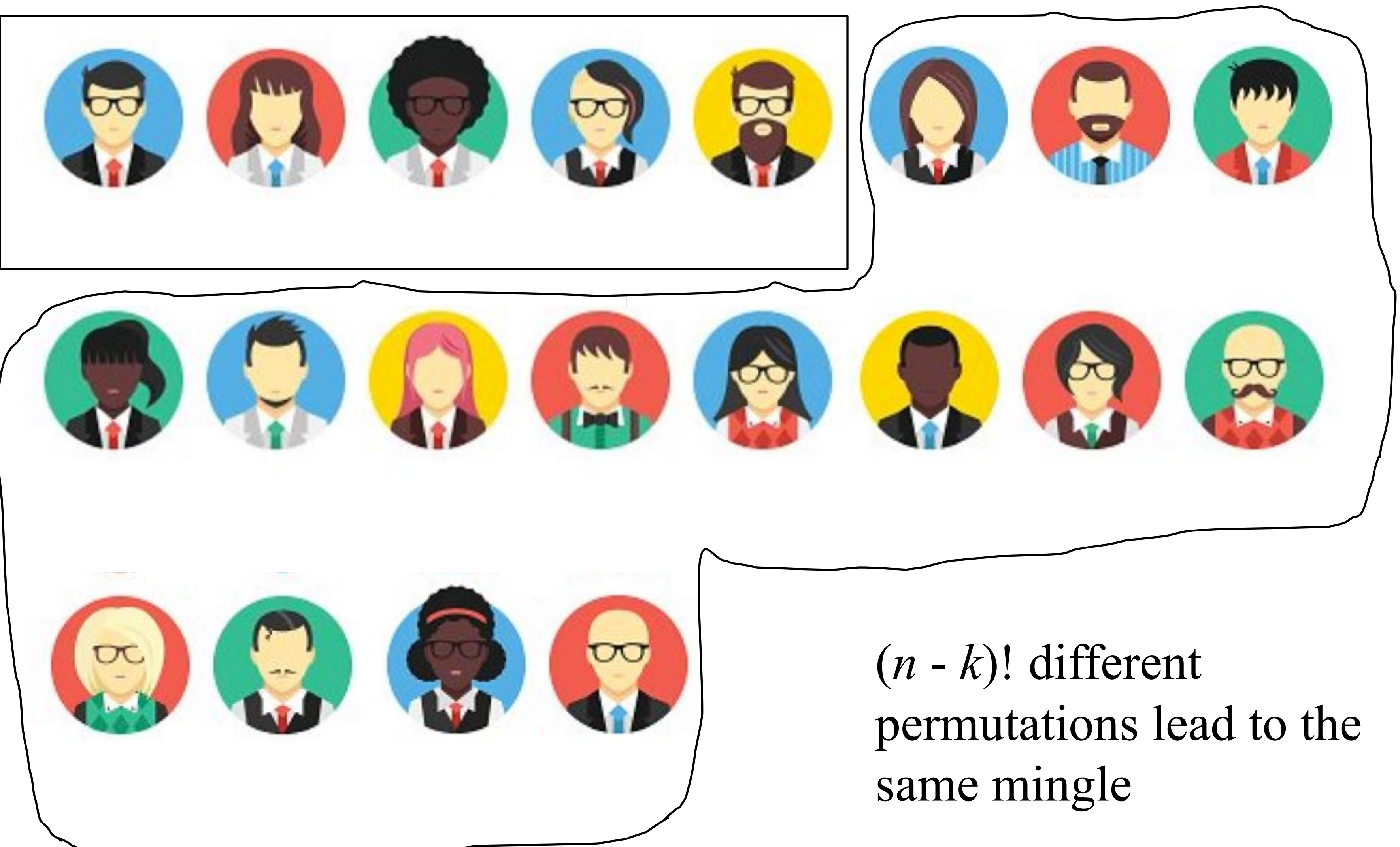
How many ways can we **Choose** $k = 5$ people to get cake?

$k!$ different permutations lead to the same mingle

Step 4: Allow nonCake Group to Mingle

There are n = 20 people

How many ways can we **Choose** k = 5 people to get cake?



(n - k)! different permutations lead to the same mingle

Step 4: Allow nonCake Group to Mingle

There are $n = 20$ people

How many ways can we **Choose** $k = 5$ people to get cake?

Randomly order n objects

Designate the first k as chosen

$$\text{num ways} = n! \times 1 \times \frac{1}{k!(n-k)!}$$

Any ordering of chosen group is the same choice

Any ordering of non-chosen group is the same choice

Step 4: Allow nonCake Group to Mingle

There are $n = 20$ people

How many ways can we **Choose** $k = 5$ people to get cake?

$$\text{num ways} = \binom{n}{k}$$

* Also called binomial coefficients

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k}$$

Combinatorics

Counting tasks on *n* objects

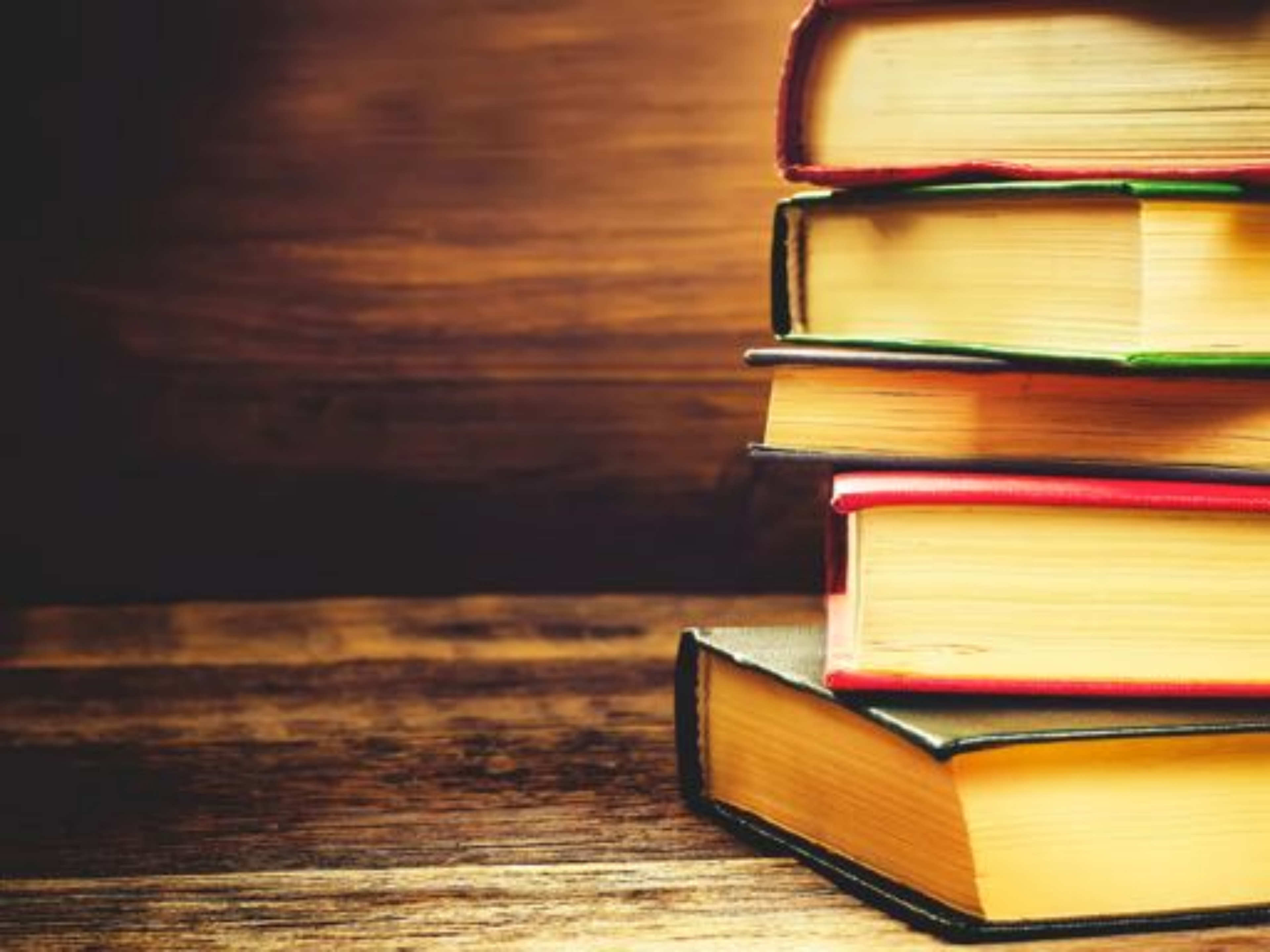
- Sort objects (permutations)
  - Distinct: $n!$
  - Some Distinct: $\frac{n!}{n_1!n_2!\ldots}$
- Choose *k* objects (combinations)
  - Distinct: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Put objects in *r* buckets

8,000 villagers.

How many distinct ways can you **Choose** 2 to play a game?

8,000 villagers.
How many distinct ways can you
Choose 2 to play a game?

$$= \frac{8000!}{7998!2!} = 31,996,000$$



Combinatorics

Counting tasks on *n* objects

Sort objects (permutations)

- Distinct: $n!$
- Some Distinct: $\frac{n!}{n_1!n_2!\ldots}$

Choose *k* objects (combinations)

- Distinct: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Put objects in *r* buckets

- Distinct
- None Distinct

Combinatorics

Counting tasks on *n* objects

- Sort objects (permutations)
  - Distinct: $n!$
  - Some Distinct: $\frac{n!}{n_1!n_2!\dots}$
- Choose *k* objects (combinations)
  - Distinct: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Put objects in *r* buckets
  - Distinct: r^n
  - None Distinct

Combinatorics

Counting tasks on *n* objects

- **Sort objects (permutations)**
  - Distinct: $n!$
  - Some Distinct: $\frac{n!}{n_1!n_2!\ldots}$
- **Choose *k* objects (combinations)**
  - Distinct: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- **Put objects in *r* buckets**
  - Distinct: r^n
  - None Distinct: $\frac{(n+r-1)!}{n!(r-1)!}$

Stretch!

Probability

Sample Space

- **Sample space**, S, is set of all possible outcomes of an experiment
  - Coin flip: S = {Head, Tails}
  - Flipping two coins: S = {(H, H), (H, T), (T, H), (T, T)}
  - Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}
  - # emails in a day: $S = \{x \mid x \in \mathbf{Z}, x \geq 0\}$ (non-neg. ints)
  - YouTube hrs. in day: $S = \{x \mid x \in \mathbf{R}, 0 \leq x \leq 24\}$

Events

- **Event**, E, is some subset of S   ($E \subseteq S$)
  - Coin flip is heads: E = {Head}
  - ≥ 1 head on 2 coin flips: E = {(H, H), (H, T), (T, H)}
  - Roll of die is 3 or less: E = {1, 2, 3}
  - # emails in a day ≤ 20: $E = \{x \mid x \in \mathbf{Z}, 0 \leq x \leq 20\}$
  - Wasted day (≥ 5 YT hrs.): $E = \{x \mid x \in \mathbf{R}, 5 \leq x \leq 24\}$

What is a probability?

Number between 0 and 1

Ascribe Meaning

$$P(E)$$

* Our belief that an event E occurs

What is a Probability

What is a probability?

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

What is a Probability

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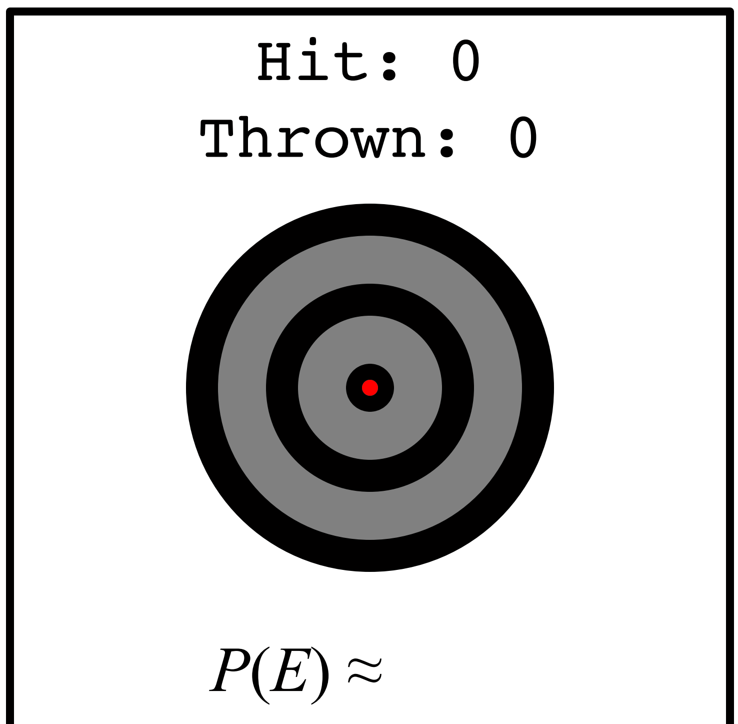
n is the number of trails

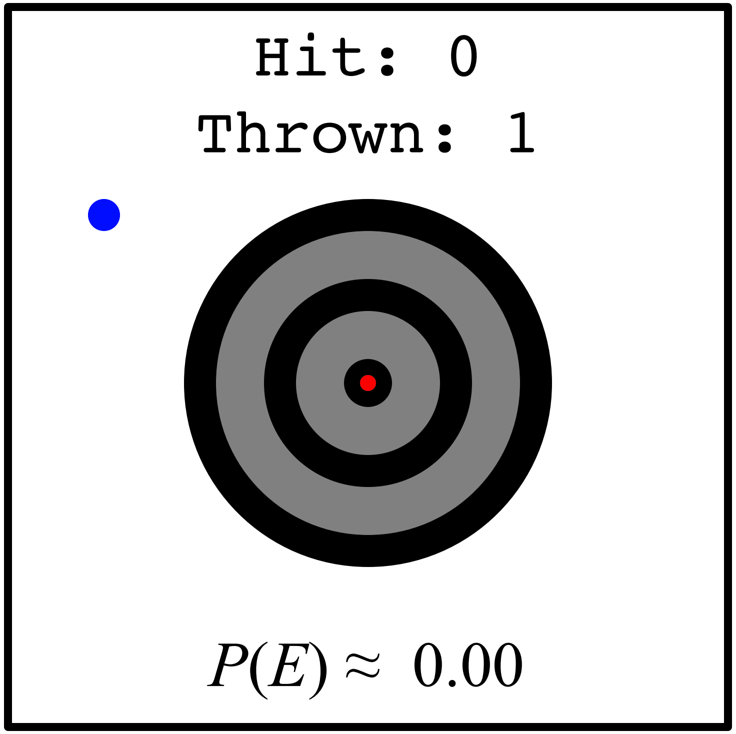


The "event" *E* is that you hit the target

What is a Probability

What is a probability?

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$



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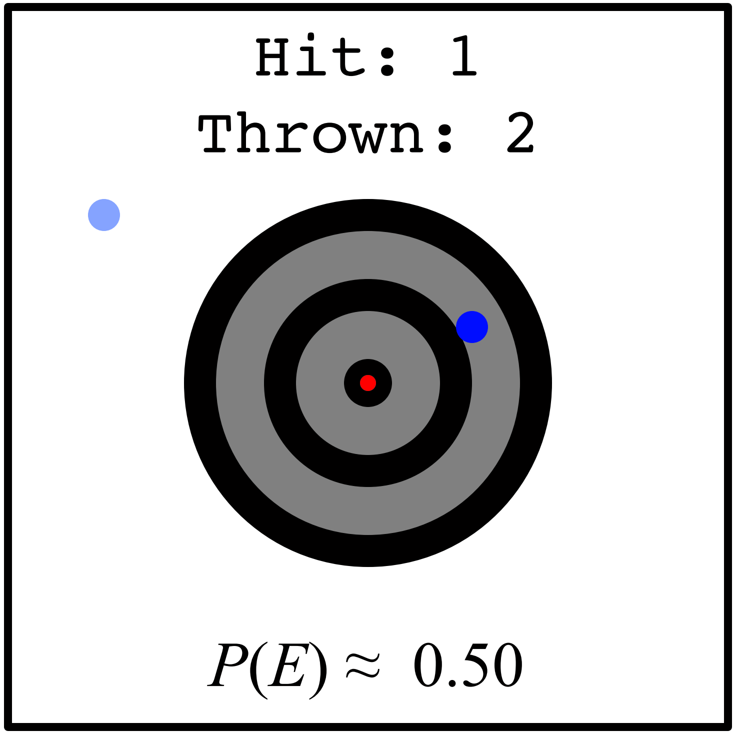

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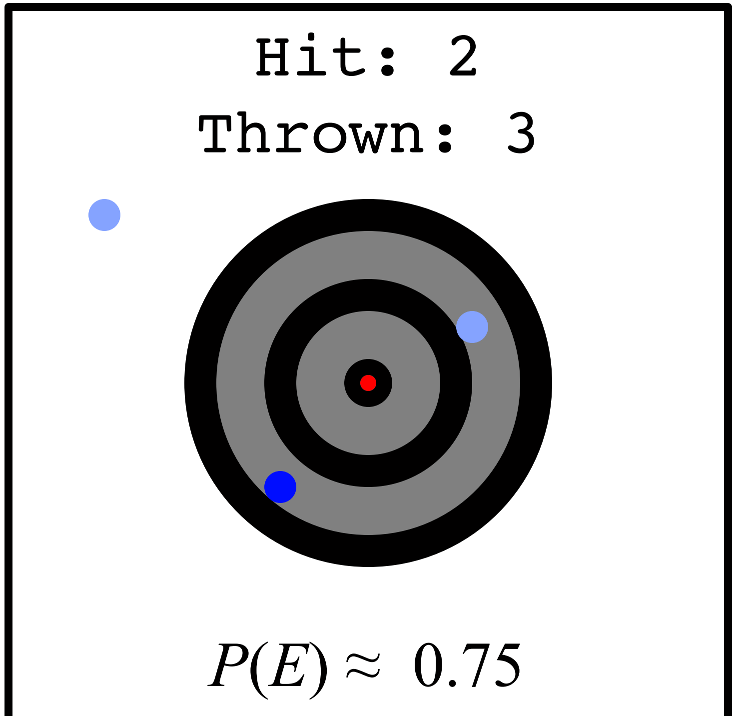

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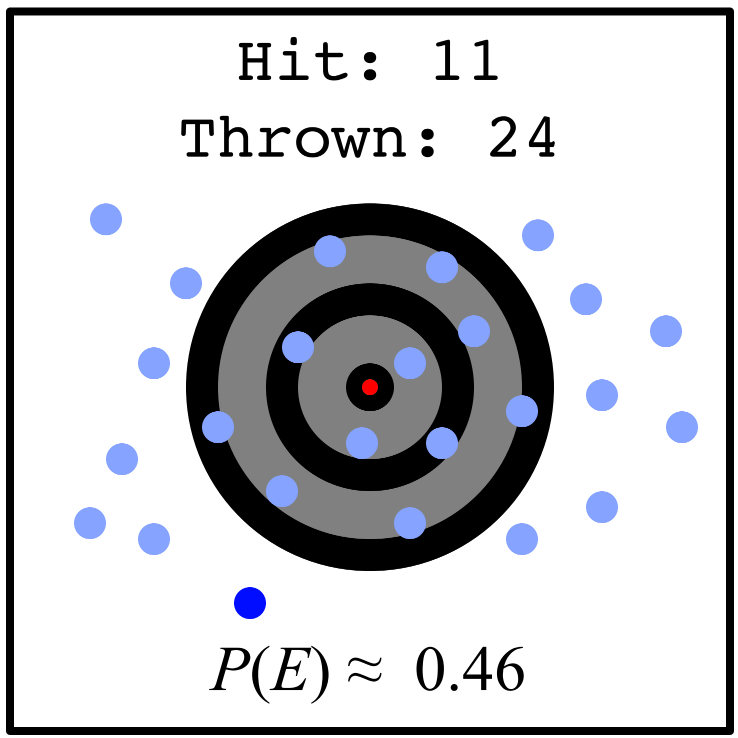
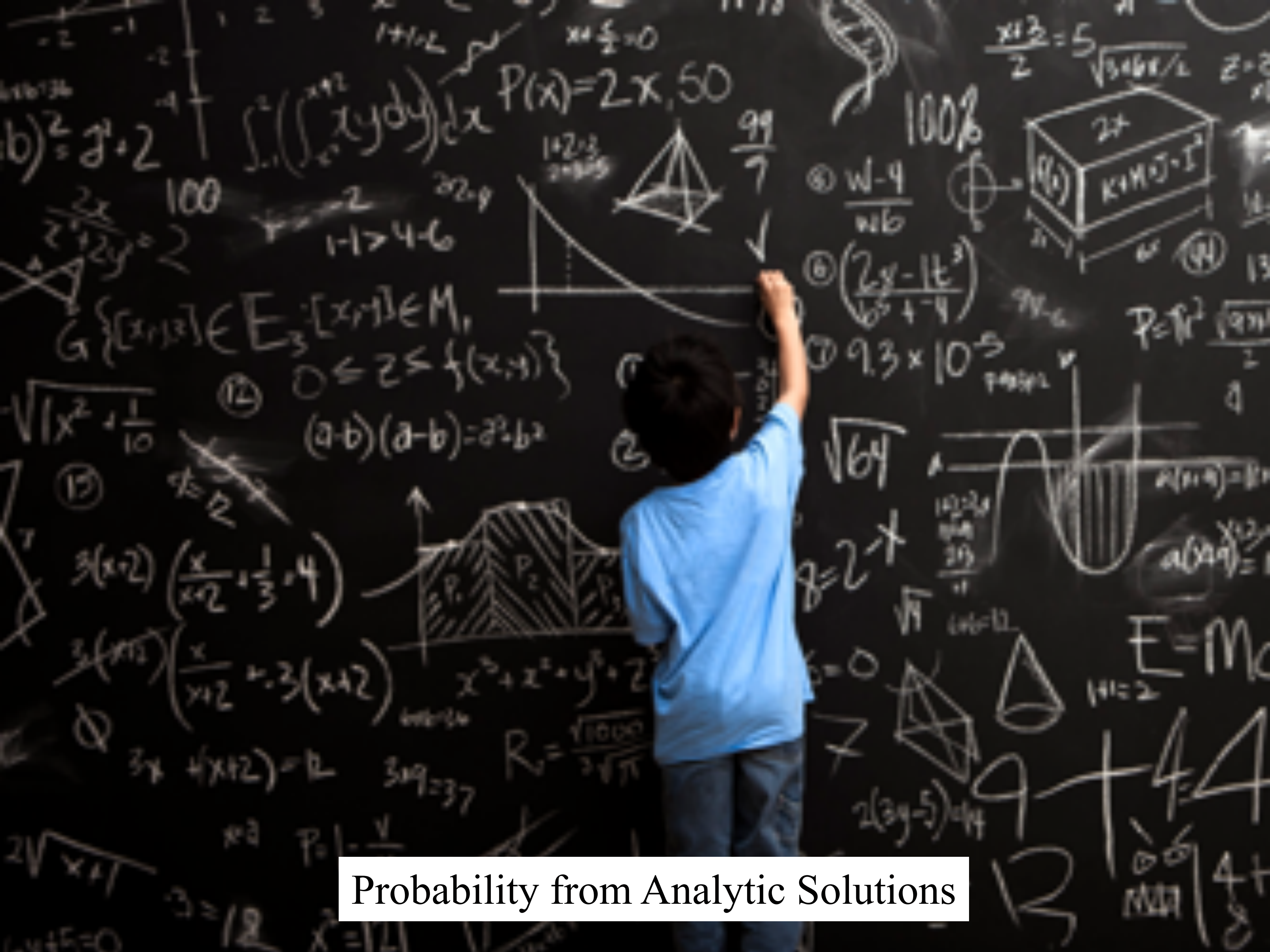


The "event" *E* is that you hit the target


90



Probability from Analytic Solutions

Axioms of Probability

Recall: S = all possible outcomes. E = the event.

- Axiom 1: $0 \leq P(E) \leq 1$
- Axiom 2: $P(S) = 1$
- Axiom 3: $P(E^c) = 1 - P(E)$

Aside: axiom 3 is often stated as the probability of mutually exclusive events. We'll come back to that later in the lecture…

Special Case of
Analytic Probability

Equally Likely Outcomes

- Some sample spaces have equally likely outcomes
  - Coin flip: S = {Head, Tails}
  - Flipping two coins: S = {(H, H), (H, T), (T, H), (T, T)}
  - Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}
- P(Each outcome) = $\frac{1}{|S|}$
- In that case, P(E) = $\frac{\text{number of outcomes in E}}{\text{number of outcomes in S}} = \frac{|E|}{|S|}$

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1891

Rolling Two Dice

- Roll two 6-sided dice.
  - What is P(sum = 7)?

- S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
  (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
  (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
  (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
  (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
  (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}
- E = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

- P(sum = 7) = |E|/|S| = 6/36 = 1/6

Not Equally Likely

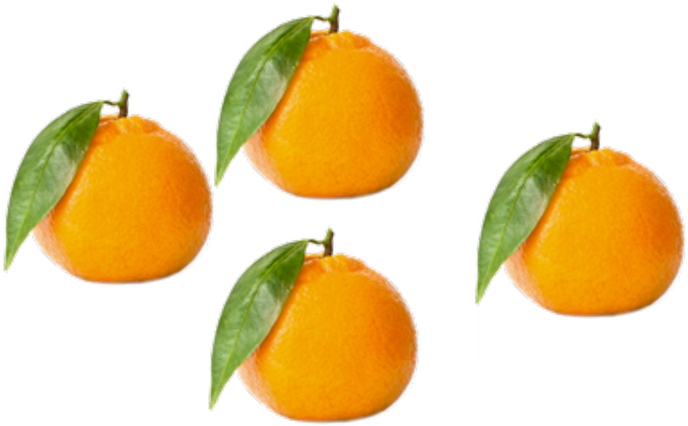
- Play lottery.
  - What is P(Win)?

- S = {Lose, Win}
- E = {Win}
- P(Win) = |E|/|S| = 1/2 = 50%

Mandarins and Bananas

- 4 Mandarins and 3 Bananas in a Bag. 3 drawn.
  - What is P(1 Mandarin and 2 Bananas drawn)?

Equally likely sample space? Thought experiment



When approaching an "**equally likely probability**" problem, start by defining **sample spaces** and **event spaces**.