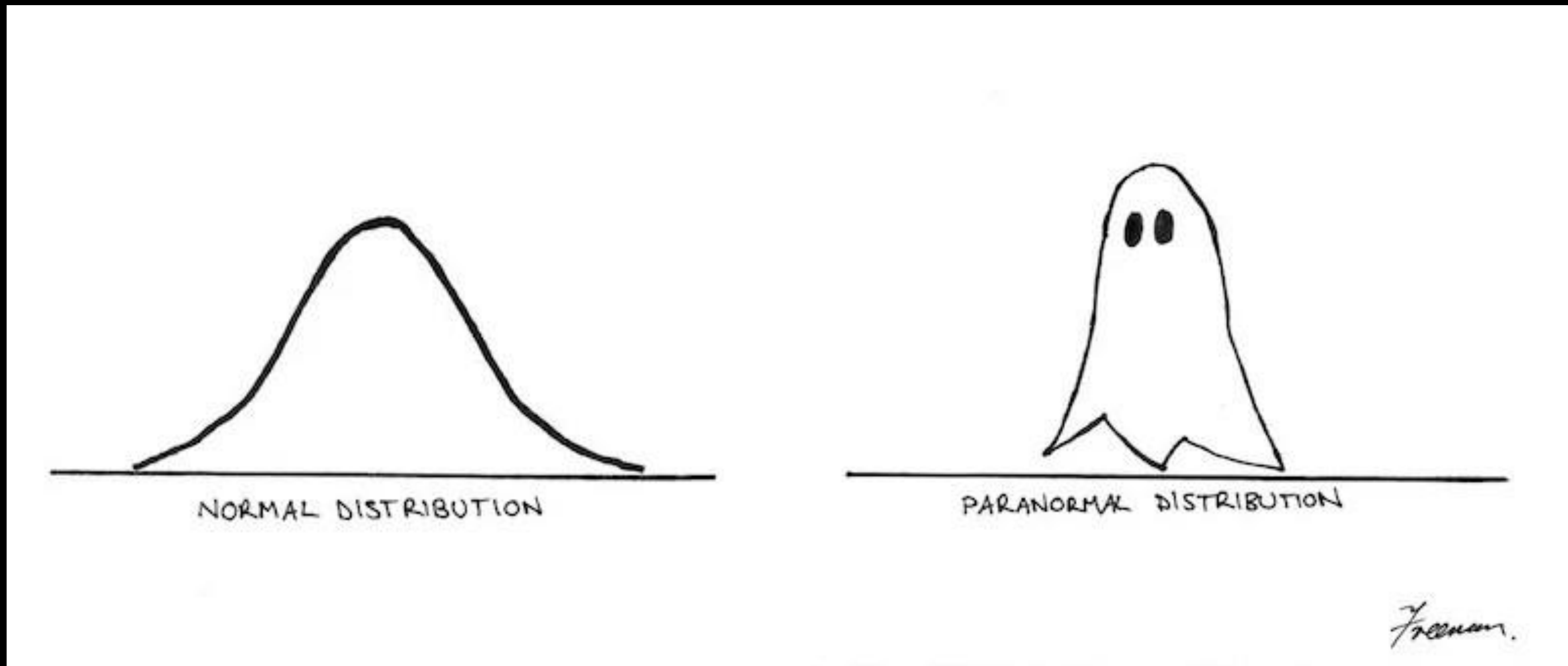

CS 109 Midterm Review

Julie Wang, 10/26/2019

Slides from Julia Daniel's Fall 2018 review and Lisa Yan's 2019 slides

Outline

- **Exam Logistics and Coverage**
- **General Strategies**
- **Counting and Events**
- **Probability Rules**
- **Random Variables**
- **Practice Problems!**

Logistics

- Midterm will be held on Tuesday October 29 from 7pm – 9pm in Hewlett 200
- Closed book, closed calculator, closed computer
- You may bring three 8.5” x 11” pages, front and back of notes to the exam.
  - Check out the midterm section on the website for some great notes and practice material
- Midterm is 20% of your final grade in the course
  - But what’s most important is your own understanding of the material

Coverage

- Lecture Notes 1 – 13
- Counting
    - Sum Rule, Product Rule
    - Inclusion-Exclusion
    - Pigeonhole Principle
    - Permutations, Combinations, and Buckets
- Probability
    - Events Spaces and Sample spaces
    - Probability axioms
    - Conditional Probability
    - Bayes Theorem
    - Independence
- Random Variables
    - Discrete and Continuous
    - PMF’s, PDF’s, CDF’s
    - Expectation and Variance
    - Bernoulli, Binomial, Poisson , Geometric, Negative Binomial, Exponential Normal
- Multiple Random Variables
    - Joint Distributions
    - Continuous and Discrete
    - Independence of Joint Distributions
    - Product and sum of distributions

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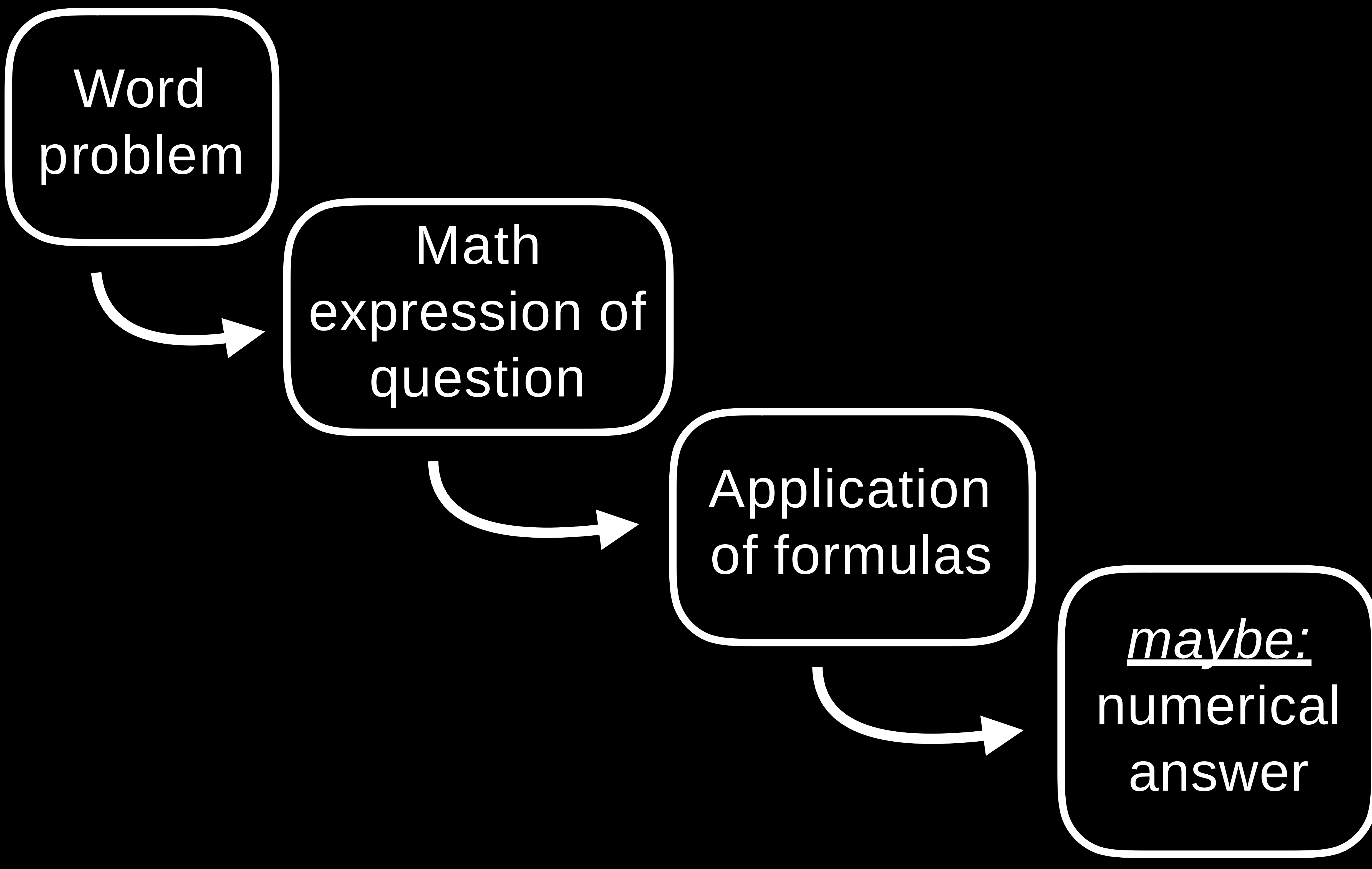
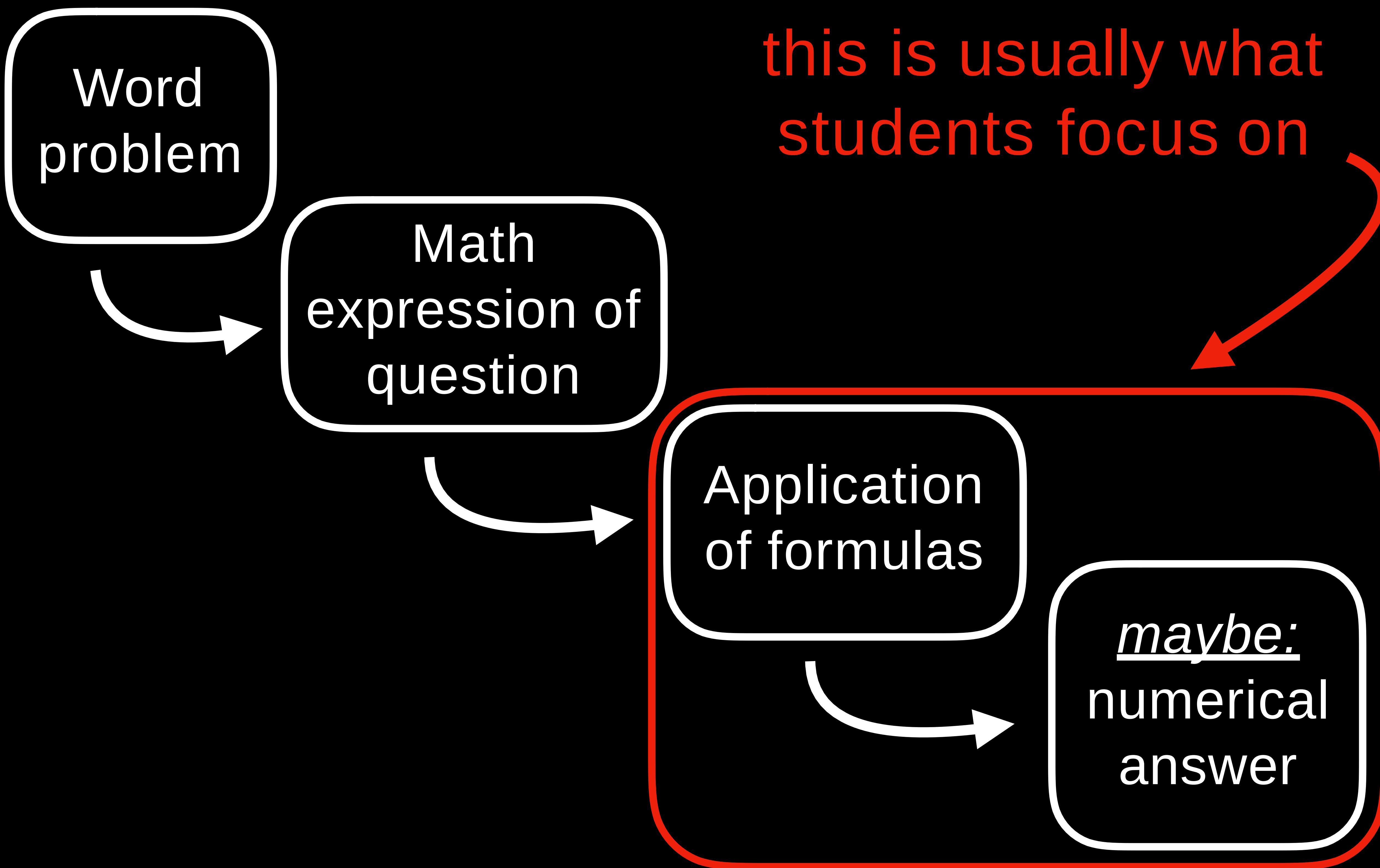
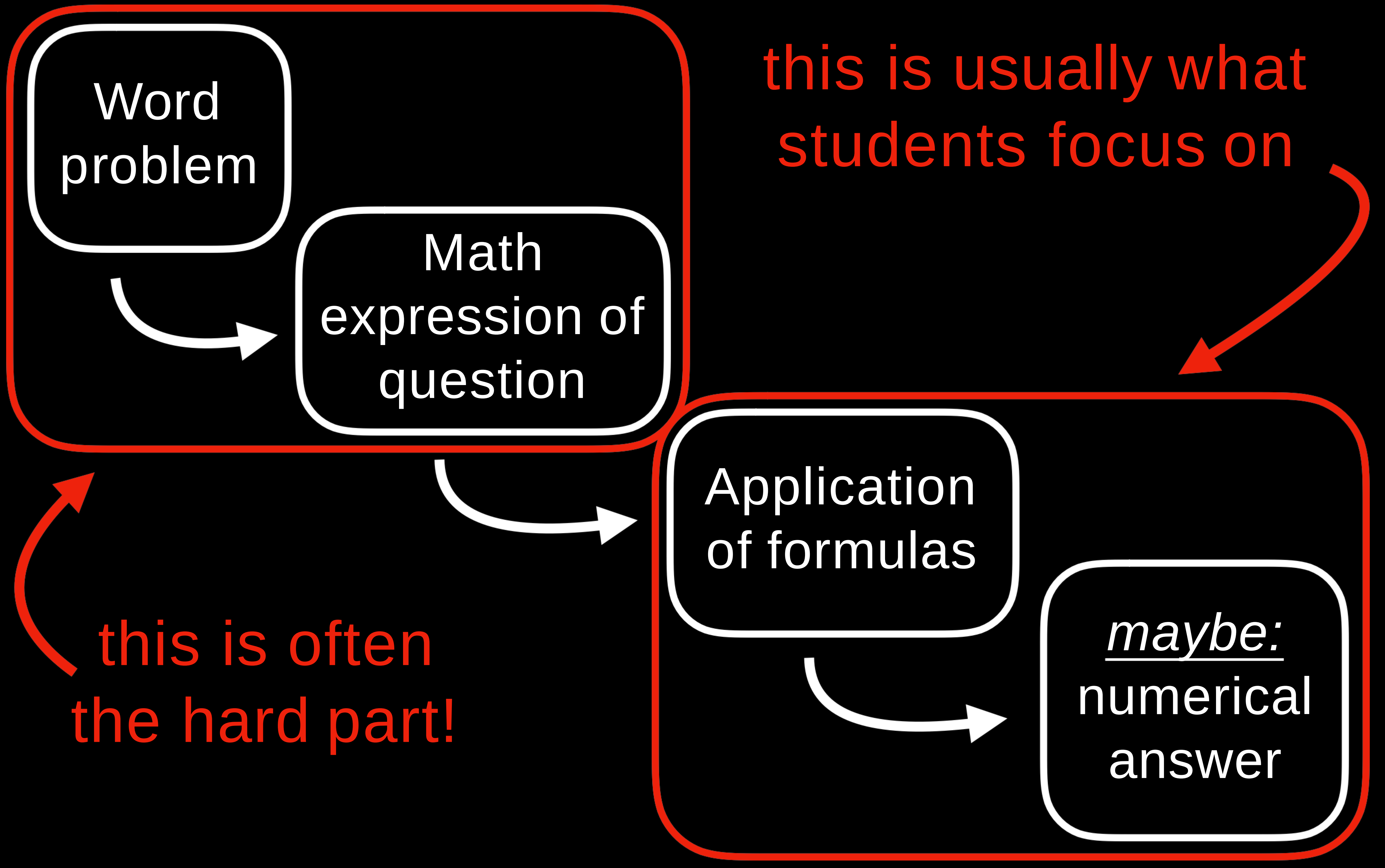
Solving a CS109 problem



Solving a CS109 problem



Solving a CS109 problem



Step 1: Defining Your Terms

- Counting: What is distinct? Which orders do I care about? Can I come up with a generative process?
- Probability: Are events independent? Def conditional probability? Bayes? Law of total probability? What’s a ‘success’? What’s the event space?
  - WRITE DOWN what your variables mean
- Random variables: What values does it take on? How is it distributed?
  - Make sure time intervals and units match - particularly important for Poisson and Exponential

Translating English to Probability

What the problem asks:	What you should immediately think:
“What’s the probability of__________”	P( )
“___given______”, “____if____”	___ | ___
“at least_____”	could we use what we know about everything less than ________________________________ ?
“approximate_______.”	use an approximation!
“How many ways…”	combinatorics

these are just a few, and these are why practice is the best way to prepare for the exam!

Translating English to Probability

People can have blue or brown eyes. What’s the probability John has blue eyes if his mother has brown eyes?

Translating English to Probability

People can have blue or brown eyes. What's the probability John has blue eyes if his mother has brown eyes?

1. **What events are we given?**
2. **What are we asked to solve?**

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Counting

Sum Rule
$outcomes = \|A\| + \|B\|$ $if \|A \cap B\| = 0$
I can choose to dress up as one of 5 superheroes or one of 4 farm animals. How many costume choices?

Counting

Sum Rule	Inclusion-Exclusion Principle
$outcomes = \|A\| + \|B\|$ $if\ \|A \cap B\| = 0$	$\|A\| + \|B\| - \|A \cap B\|$ $for\ any\ \|A \cap B\|$
I can choose to dress up as one of 5 superheroes or one of 4 farm animals. How many costume choices?	**I can choose to dress up as one of 5 superheroes or one of 6 strong female movie leads. 2 of the superheroes are female movie leads. How many costume choices?**

Counting

Product Rule
$outcomes = \|A\| \times \|B\|$ **if all outcomes of B are possible regardless of the outcome of A**
I can choose to go to one of 3 parties and then trick-or-treat in one of 5 neighborhoods. How many different ways to celebrate?

Counting

Product Rule	Pigeonhole Principle
$outcomes = \|A\| \times \|B\|$ if all outcomes of B are possible regardless of the outcome of A	If m objects are placed into n buckets, then at least one bucket has at least *ceiling(m / n)* objects.
I can choose to go to one of 3 parties and then trick-or-treat in one of 5 neighborhoods. How many different ways to celebrate?	If you have an infinite number of red, white, blue, and green socks in a drawer, how many must you pull out before being guaranteed a pair?

Combinatorics: Arranging Items

	Permutations (ordered)	**Combinations (unordered)**
Distinct	$n!$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
Indistinct	$\frac{n!}{k_1!k_2!\ldots k_n!}$	$\binom{n+r-1}{r-1}$ the divider method!

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Probability basics

$$\text{Probability} = \frac{\text{Event space}}{\text{Sample space}}$$

if all outcomes are equally likely!

(use counting with distinct objects)

Probability basics

$$\text{Probability} = \frac{\text{Event space}}{\text{Sample space}}$$

if all outcomes are equally likely!

(use counting with distinct objects)

Axioms: $0 \leq P(E) \leq 1$ $\quad P(S) = 1$ $\quad P(E^C) = 1 - P(E)$

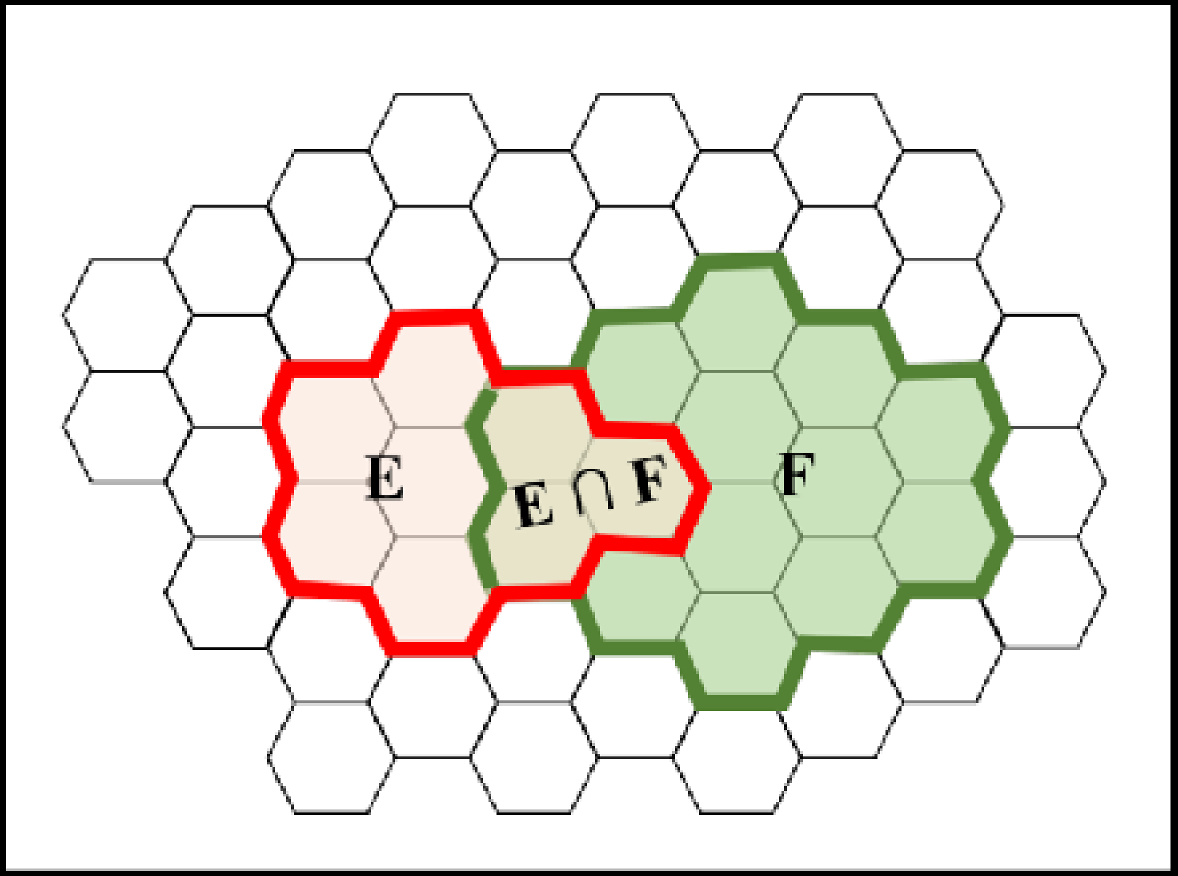
Conditional Probability

definition:

$$P(A|B) = \frac{P(AB)}{P(B)}$$

Chain Rule:

$$P(AB) = P(A|B)P(B)$$



* $P(EF) = P(E \cap F)$

Law of Total Probability

$$P(A) = P(A|B)P(B) + P(A|B^C)P(B^C)$$

Bayes' Rule

$$P(E|F) = \frac{P(F|E)P(E)}{P(F)}$$

Bayes' Rule

posterior

likelihood

prior

$$P(E|F) = \frac{P(F|E)P(E)}{P(F)}$$

normalization constant

Bayes' Rule

$$P(E|F) = \frac{P(F|E)P(E)}{P(F)}$$

$$P(F|E)P(E) + P(F|E^C)P(E^C)$$

divide the event F into all the possible ways it can happen; use LoTP

Bayes' Rule

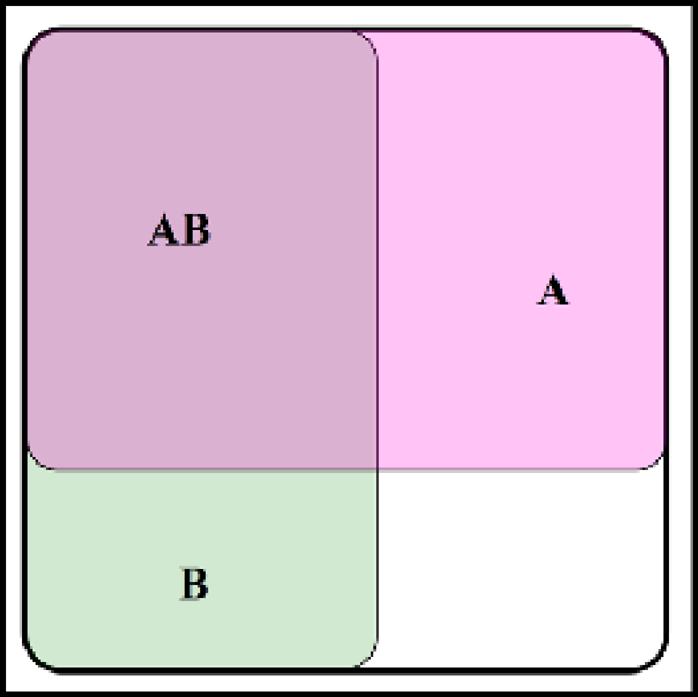
$$P(E|F) = \frac{P(F|E)P(E)}{P(F)}$$

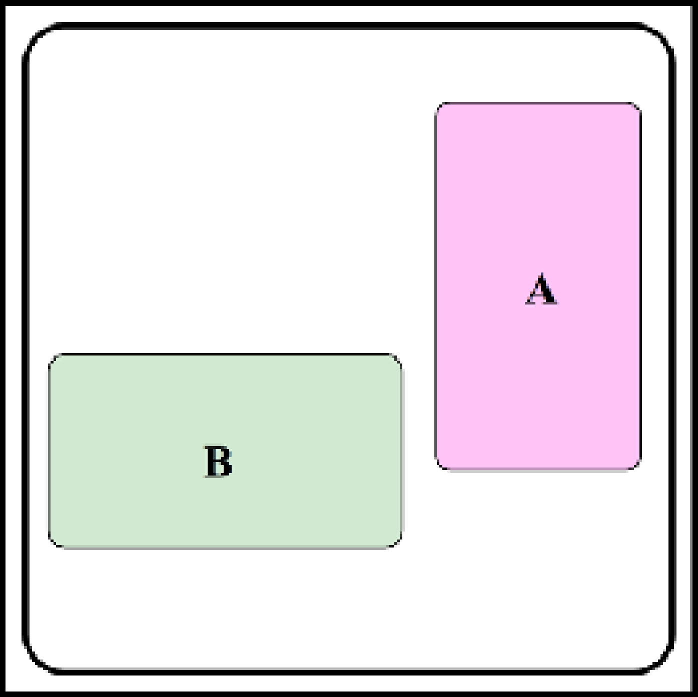
$$P(F|E)P(E) + P(F|E^C)P(E^C)$$

divide the event F into all the possible ways it can happen; use LoTP

Independence + Mutual Exclusion

Independence	Mutual Exclusion
$P(EF) = P(E)P(F)$	$\lvert E \cap F \rvert = 0$
“AND”	**“OR”**





Notes for last slide

- Two events can be independent, but not mutually exclusive. E.g. What is the probability that it's raining and I am eating a pomegranate? Both events are independent, but they can both occur simultaneously. Sometimes it's raining, sometimes I'm eating a pomegranate, and sometimes It's raining and I'm eating a pomegranate.
- Two sets of events can be mutually exclusive but not necessarily independent. I am either studying or sleeping, these are mutually exclusive events. However, how much I study affects how much I sleep, so therefore they are not necessarily independent.
- Call back to your definitions whenever trying to prove independence.

Independence of events

Independence	Conditional Independence
$P(EF) = P(E)P(F)$	$P(EF \mid G) = P(E \mid G) P(F \mid G)$
$P(E \mid F) = P(E)$	$P(E \mid FG) = P(E \mid G)$

If E and F are independent…..

…..that does not mean they'll be independent if another event happens!

& vice versa

Independence of Events

Example:

Normally, the event that I wear a costume (E) and that I am in front of a large group of people (F) are independent events.

Given:

P(wear costume) = .001

P(in front of people)= .2

What is the probability that I wear a costume and I'm in front of a large group of people?

P(costume and in front of people) = p(costume) * p(in front of people) = .0002

However, given that it is October 25 (G), I have the following conditional probabilities:

P(costume | October 25) = .2

P(in front of people|October 25) = 1

P(costume and in front of people| October 25) = 1

Are these events, wearing a costume and being in front of people, independent events?

1 != (.2 * 1), so no!

Extending our Rules to Multiple Events

For any events A, B, and E, you can condition consistently on E, and all formulas still hold:

Axiom 1	$0 \le P(A\|E) \le 1$
Corollary 1 (complement)	$P(A\|E) = 1 - P(A^C\|E)$
Transitivity	$P(AB\|E) = P(BA\|E)$
Chain Rule	$P(AB\|E) = P(B\|E)P(A\|BE)$
Bayes' Theorem	$P(A\|BE) = \frac{P(B\|AE)P(A\|E)}{P(B\|E)}$

BAE's theorem?

Independence relationships can change with conditioning.

A and B independent does NOT necessarily mean A and B independent given E.

Notice that A, B, and E can be any events

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But first, a minute break.

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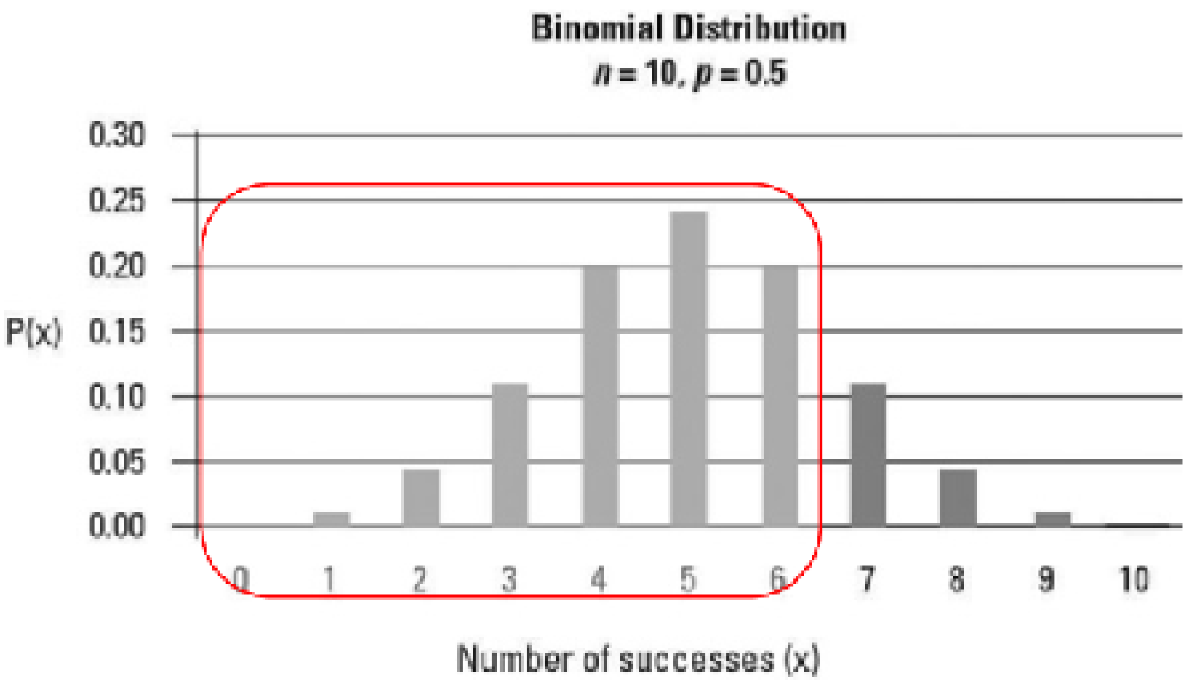
Probability Distributions

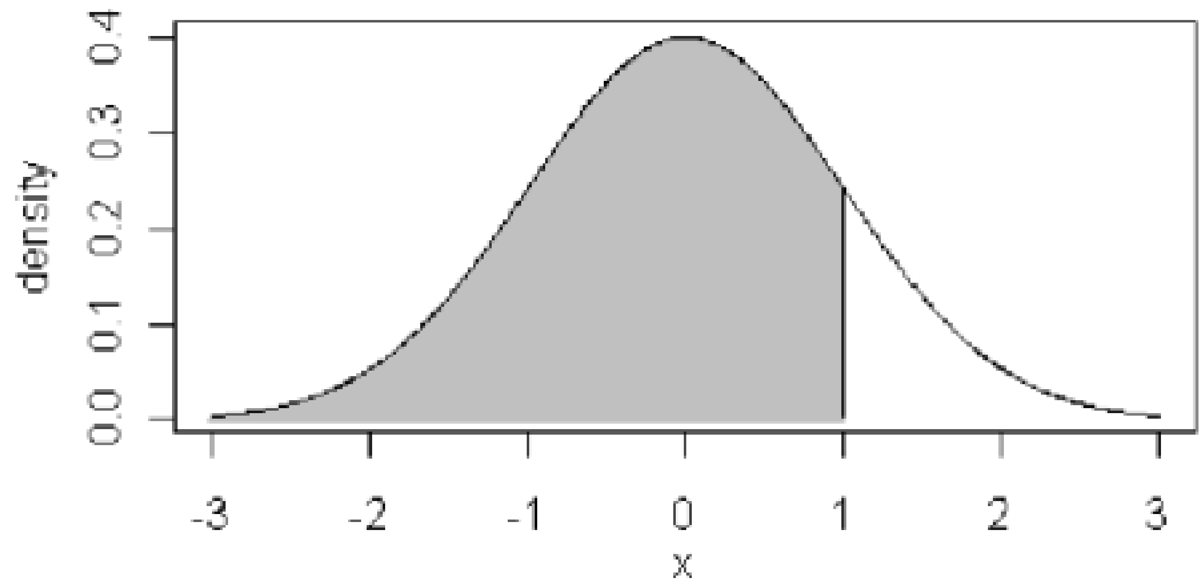
Discrete

CDF:



Continuous

CDF:



Expectation & Variance

Discrete definition

$$E[X] = \sum_{x:P(x)>0} x * P(x)$$

Continuous definition

$$E[X] = \int_x x * p(x) dx$$

Expectation & Variance

Discrete definition

$$E[X] = \sum_{x:P(x)>0} x * P(x)$$

Continuous definition

$$E[X] = \int_x x * p(x)dx$$

Properties of Expectation

$$E[X + Y] = E[X] + E[Y]$$

$$E[aX + b] = aE[X] + b$$

$$E[g(X)] = \sum_x g(x) * p_X(x)$$

Properties of Variance

$$Var(X) = E[(X - \mu)^2]$$

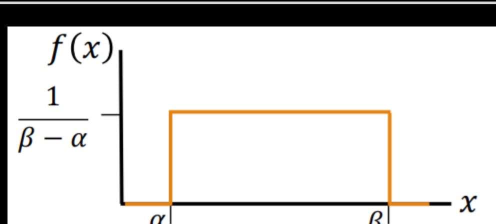
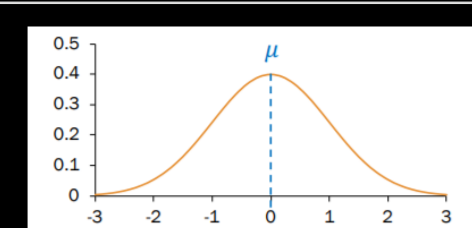
$$Var(X) = E[X^2] - E[X]^2$$

$$Var(aX + b) = a^2 Var(X)$$

All our (discrete) friends

Ber(p)	Bin(n, p)	Poi(λ)	Geo(p)	NegBin (r, p)
P(X) = p	$\binom{n}{k} p^k (1-p)^{n-k}$	$\frac{\lambda^k e^{-\lambda}}{k!}$	$(1-p)^{k-1} p$	$\binom{k-1}{r-1} p^r (1-p)^{k-r}$
E[X] = p	E[X] = np	E[X] = λ	E[X] = 1 / p	E[X] = r / p
Var(X) = p(1-p)	Var(X) = np(1-p)	Var(X) = λ	$\frac{1-p}{p^2}$	$\frac{r(1-p)}{p^2}$
1 experiment with prob p of success	n independent trials with prob p of success	Number of success over experiment duration, λ rate of success	Number of independent trials until first success	Number of independent trials until r successes

All our (continuous) friends

Uni(α, β)	Exp(λ)	N(μ, σ)
$f(x) = \frac{1}{\beta - \alpha}$	$f(x) = \lambda e^{-\lambda x}$	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$
$P(a \leq X \leq b) = \frac{b-a}{\beta - \alpha}$	$F(x) = 1 - e^{-\lambda x}$	$F(x) = \Phi(\frac{x-\mu}{\sigma})$
$E(x) = \frac{\alpha + \beta}{2}$	E[x] = 1 / λ	E[x] = μ
$Var(x) = \frac{(\beta - \alpha)^2}{12}$	$Var(x) = \frac{1}{\lambda^2}$	$Var(x) = \sigma^2$
	Duration of time until success occurs. λ is rate of success	

Approximations

When can we approximate a binomial?

Poisson

- **$n > 20$**
- **p is small**
- **$\lambda = np$ is moderate**
  - **$n > 20$ and $p < 0.05$**
  - **$n > 100$ and $p < 0.1$**
- **Slight dependence ok**

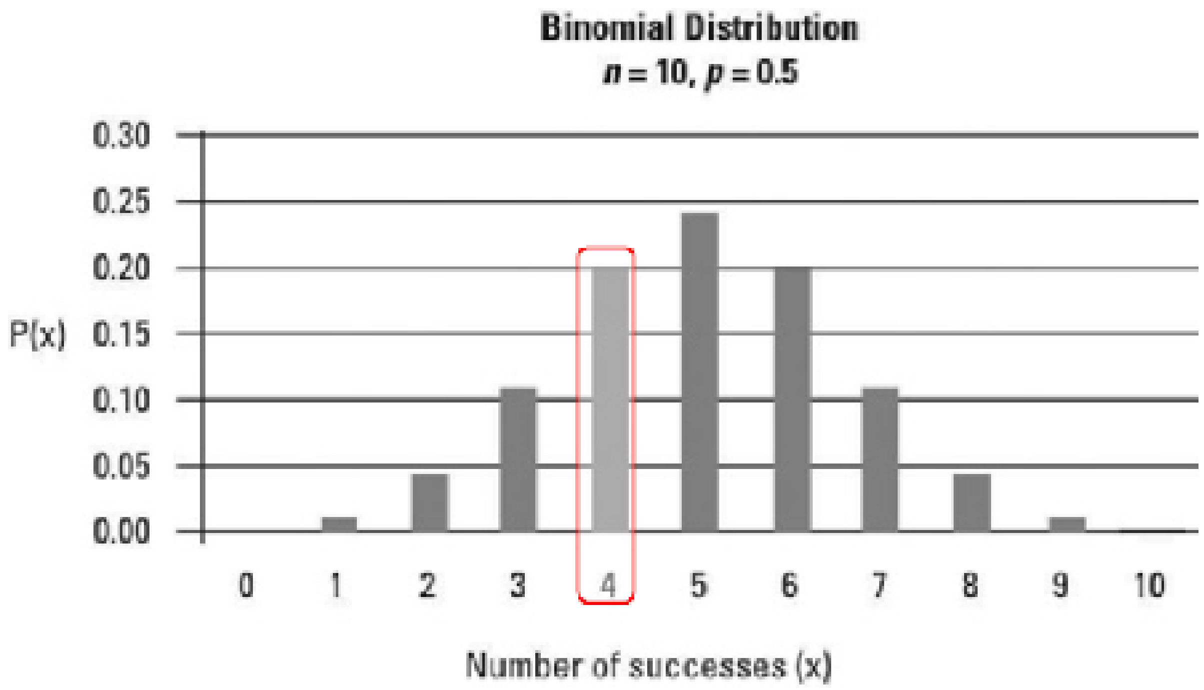
Normal

- **$n > 20$**
- **p is moderate**
  - **$np(1-p) > 10$**
- **Independent trials**

Continuity correction

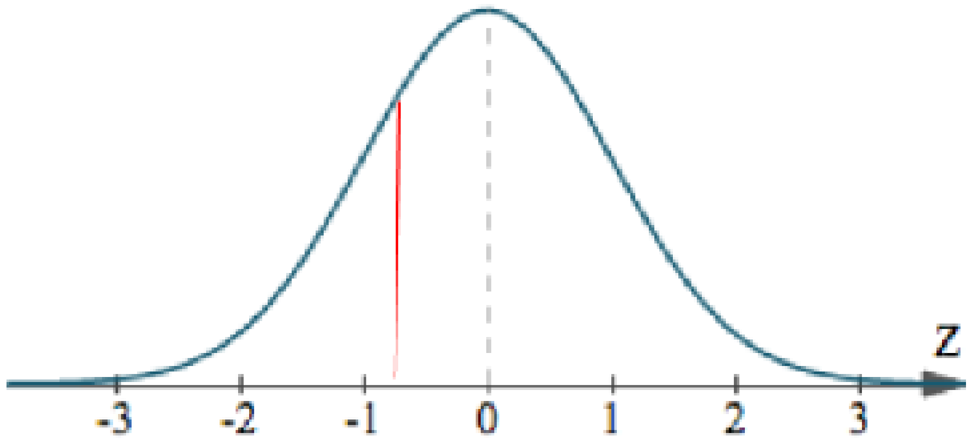
Discrete

PMF:



Continuous

PDF:



Joint PDF's

- Discrete case: $p_{x,y}(a, b) = P(X = a, Y = b)$. $P_x(a) = \sum_y P_{x,y}(a, y)$
- Continuous case:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)dy\,dx = 1$$

$$P(a_1 < X \leq a_2,\ b_1 < Y \leq b_2) = \int_{a_1}^{a_2}\int_{b_1}^{b_2} f_{X,Y}(x, y)dy\,dx$$

- Marginal Distribution:

$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y)dy \qquad f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x, b)dx$$

Joint CDF's

For discrete X and Y:

$$F_{X,Y}(a,b) = \sum_{x \le a} \sum_{y \le b} p_{X,Y}(x,y)$$

For continuous X and Y:

$$F_{X,Y}(a,b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x,y) dy\, dx$$

$$f_{X,Y}(a,b) = \frac{\partial^2}{\partial a\, \partial b} F_{X,Y}(a,b)$$

Joint CDF: $P(X \le x, Y \le y) = F_{X,Y}(x,y)$

$$P(a_1 < X \le a_2, b_1 < Y \le b_2) =$$
$$F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$

Independent RV’s

Two continuous random variables X and Y are independent if:

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

Equivalently:

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$
$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

More generally, X and Y are independent if joint density factors separately:

👉 $f_{X,Y}(x,y) = g(x)h(y)$, where $-\infty < x, y < \infty$

Examples!

Sums of random variables

For any discrete random variables X and Y:

$$P(X+Y=n)=\sum_k P(X=k, Y=n-k)$$

In particular, for independent discrete random variables X and Y:

$$P(X+Y=n)=\underbrace{\sum_k P(X=k)P(Y=n-k)}_{\text{the convolution of } p_X \text{ and } p_Y}$$

the convolution of p_X and p_Y

For independent continuous random variables X and Y:

$$f_{X+Y}(\alpha)=\int_{-\infty}^{\infty} f_X(x)f_Y(\alpha-x)dx$$

the convolution of f_X and f_Y

An Aside Convolving Uniform random variables

[Link]

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Practice problems

I fly a lot, and the last three times I've flown, I walked through the metal detector and been randomly selected to be searched by the machine. I asked the TSA officer, and he said the probability that the machine dings is .06. It has no knowledge of who you are, and goes off randomly.
Given that I fly 10 times in a year, what is the probability that I get dinged 3 times, and what is the probability that I get dinged 3 times in a row?
(This is a true story of Julie's life)

Practice problems

Probability that I get dinged 3 times in the 10 times I fly?
Model as a binomial with n = 10, k = 3.
(10 choose 3) * .06 ^ 3 * .94 * 7 = .0168

Probability that I get dinged 3 times in a row in the 10 times I fly?
Let's group the 3 times I get dinged into a single unit, then I need to arrange the 7 other times I don't get dinged around it. You can view this as 8 slots, and I need to pick the slot where the times I get dinged happen. That is (8 choose 1). Then, calculate the probability of 3 dings and 7 non dings, and get:
(8 choose 1) * .06^3 * .93^7 = .00112

This makes sense, I'm less likely to get dinged three times in a row as opposed to 3 times in any arrangement.