02: Combinatorics

Lisa Yan
September 25, 2019

Takeaways from last time

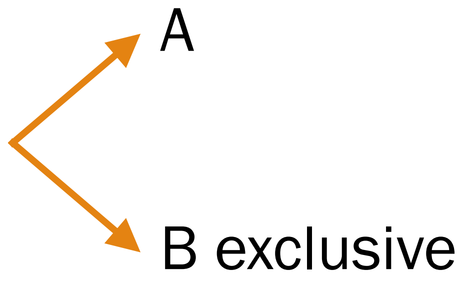
Review

Inclusion-Exclusion Principle (generalized Sum Rule)

If the outcome of an experiment can be either from Set A **or** set B, where A and B may overlap, then the total number of outcomes of the experiment is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

One-step experiment

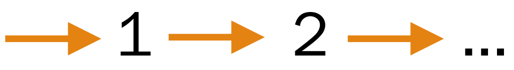


General Principle of Counting (generalized Product Rule)

If an experiment has r **steps**, such that step i has n_i outcomes for all $i = 1, \ldots, r$, then the total number of outcomes of the experiment is

$$n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i \, .$$

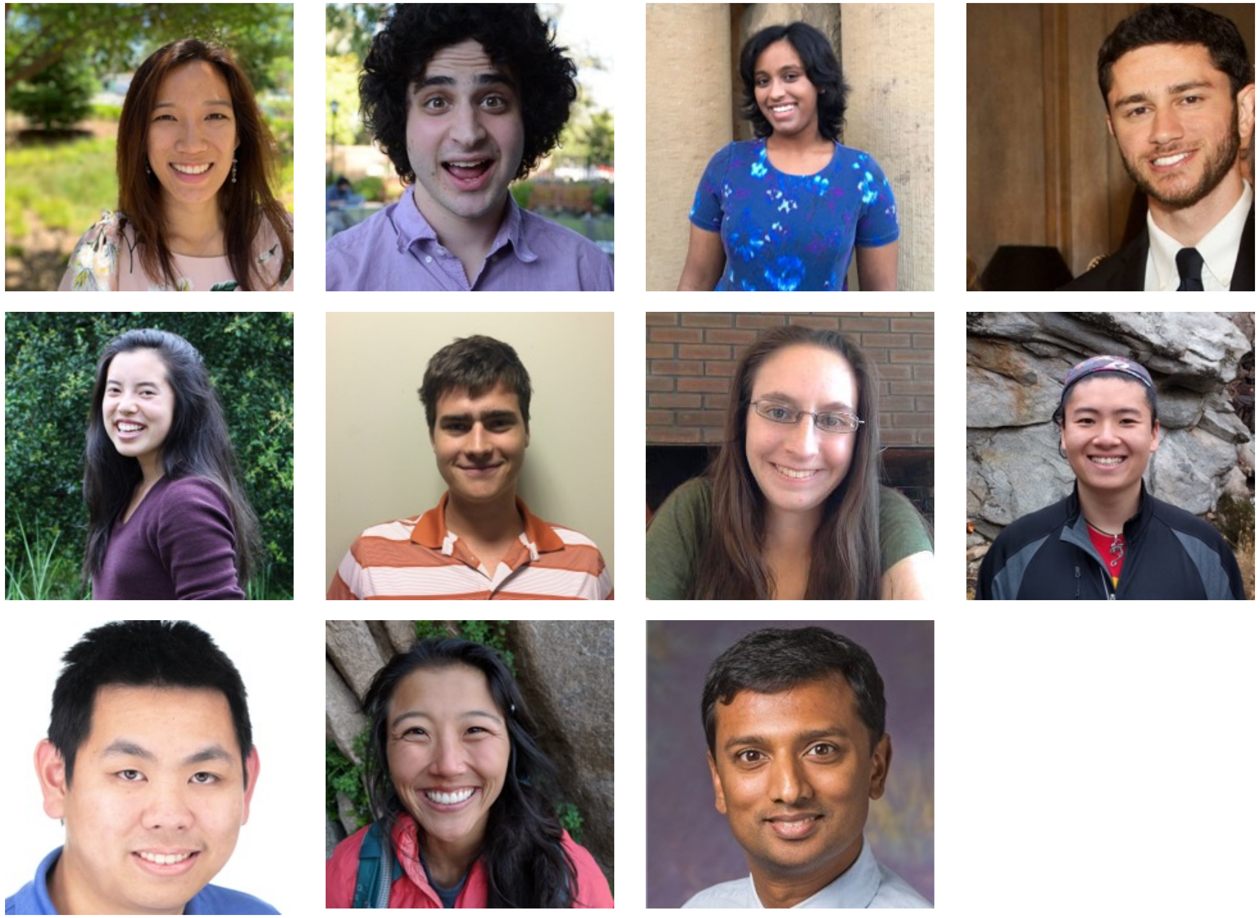
Multi-step experiment



Essential information

Website cs109.stanford.edu

Teaching Staff



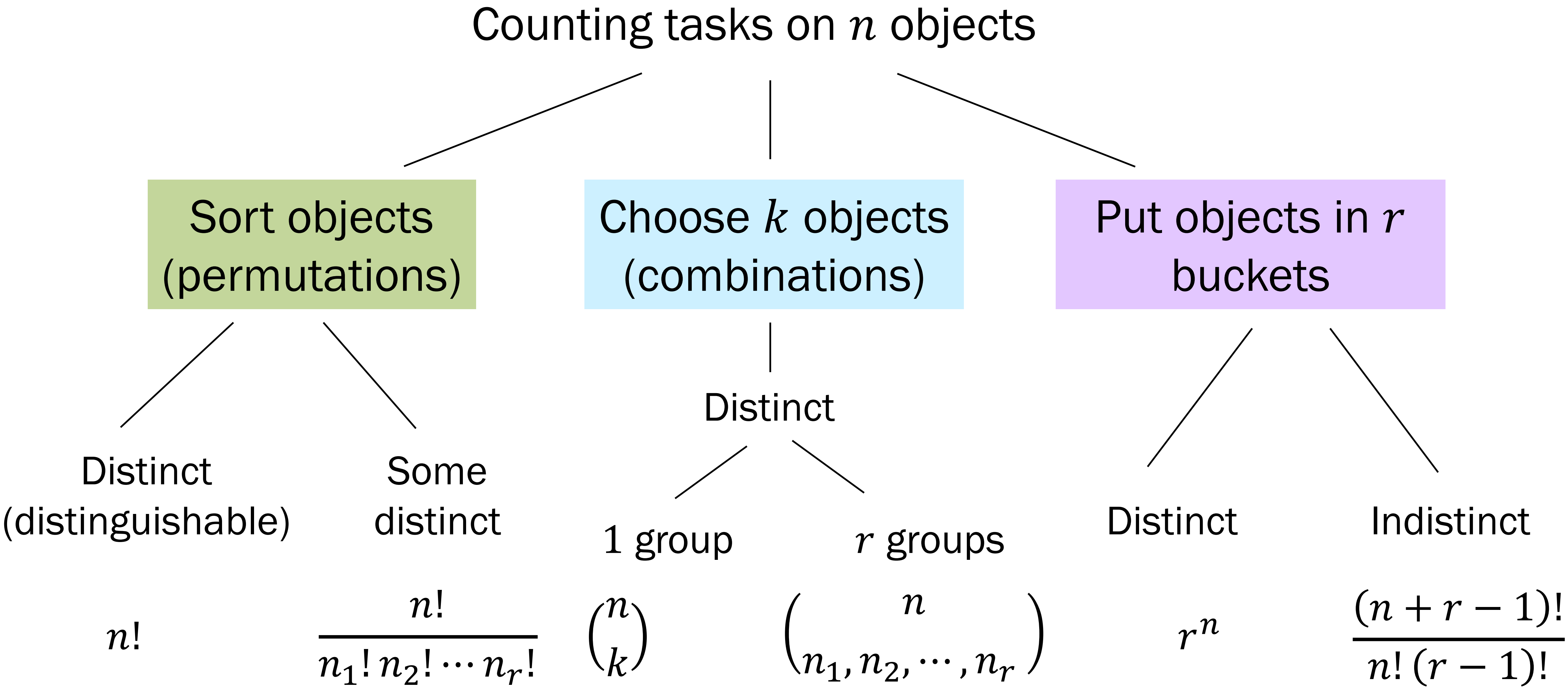
Today's plan

Permutations (sort objects)

Combinations (choose objects)

Put objects into buckets

Summary of Combinatorics



Counting tasks on n objects

Sort objects (permutations)

Distinct (distinguishable)

$$n!$$

Some distinct

$$\frac{n!}{n_1!\, n_2! \cdots n_r!}$$

Choose k objects (combinations)

Distinct

1 group

$$\binom{n}{k}$$

r groups

$$\binom{n}{n_1, n_2, \cdots, n_r}$$

Put objects in r buckets

Distinct

$$r^n$$

Indistinct

$$\frac{(n+r-1)!}{n!\,(r-1)!}$$

Today's plan

⇨ Permutations (sort objects)

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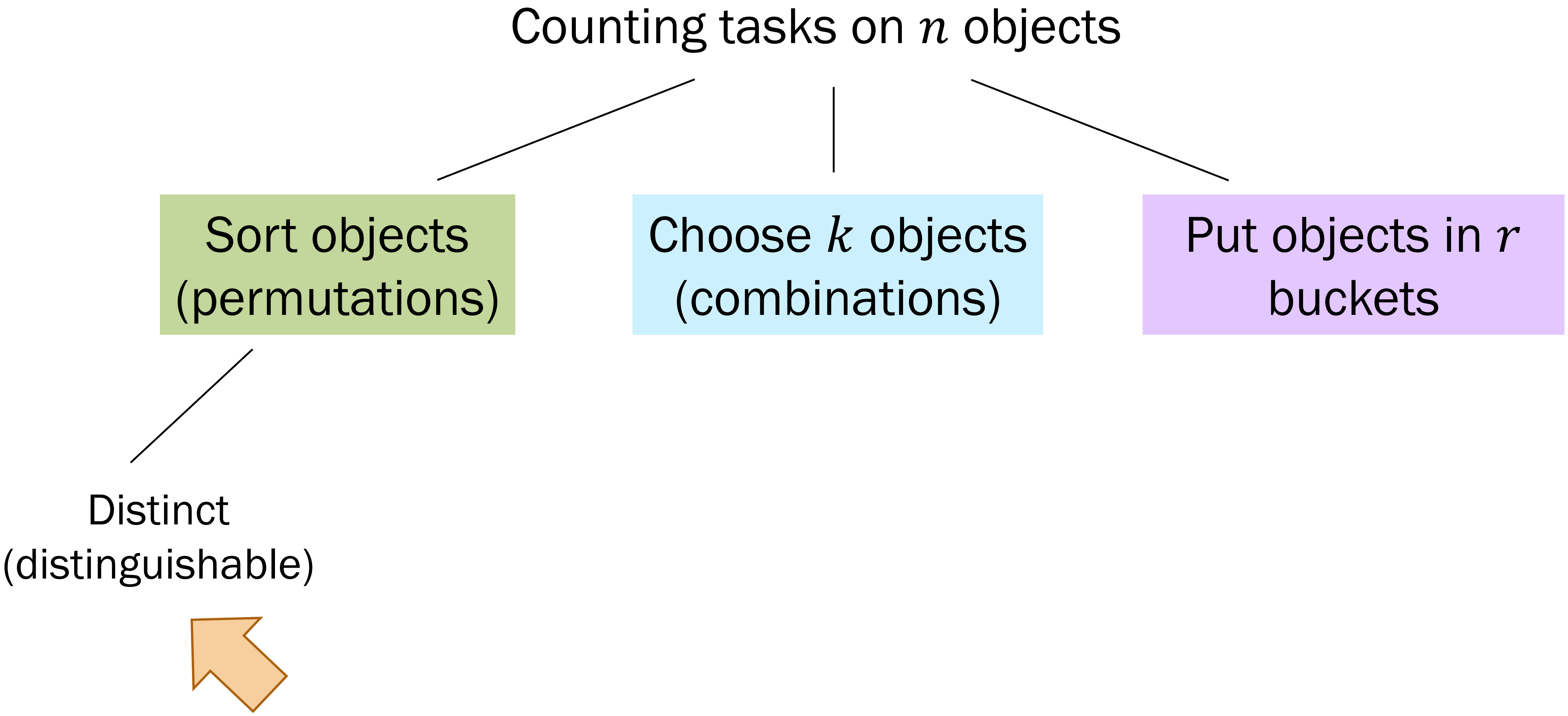


Counting tasks on n objects

Sort objects (permutations)

Choose k objects (combinations)

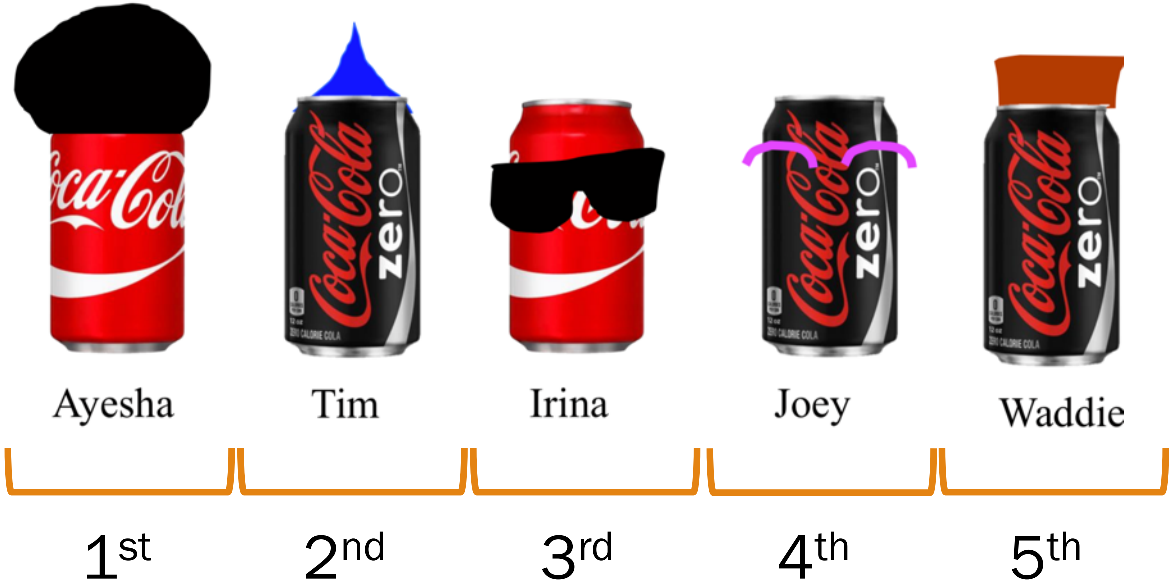
Put objects in r buckets

Distinct (distinguishable)

Sort n distinct objects

Ayesha Tim Irina Joey Waddie

Sort n distinct objects



Steps:

1. Choose 1st can 5 options
2. Choose 2nd can 4 options

…

5. Choose 5th can 1 option

$$\text{Total} = 5 \times 4 \times 3 \times 2 \times 1$$
$$= 120$$

Permutations

A permutation is an <u>ordered</u> arrangement of <u>distinct</u> objects.

The number of unique orderings (permutations) of n distinct objects is

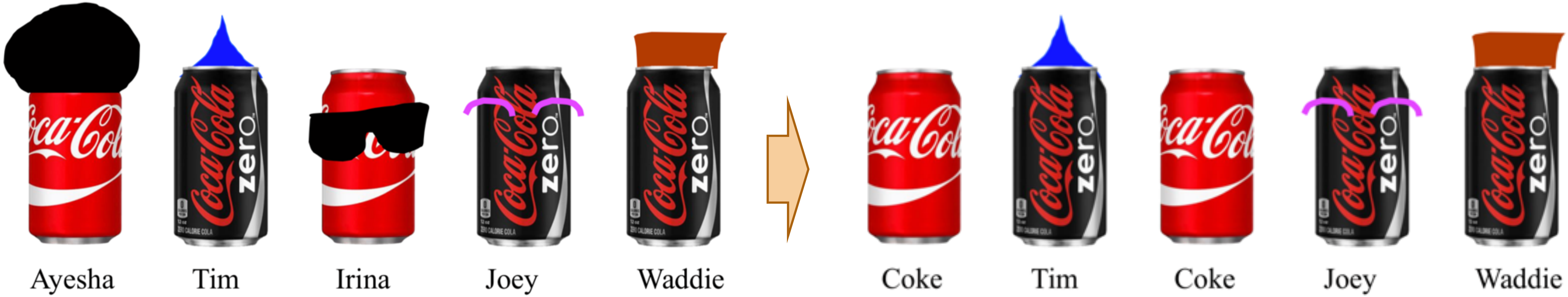
$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

Sort semi-distinct objects

Order n distinct objects $n!$

All distinct

Some indistinct



$$= 5! = 120$$

$$= 120/2$$

Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

permutations of distinct objects = permutations considering some objects are indistinct × Permutations of just the indistinct objects

Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

$$\frac{\text{permutations of distinct objects}}{\text{Permutations of just the indistinct objects}} = \text{permutations considering some objects are indistinct}$$

General approach to counting permutations

When there are n objects such that

n_1 are the same (indistinguishable or indistinct), and
n_2 are the same, and
…
n_r are the same,

The number of unique orderings (permutations) is

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}.$$

👉 For each group of indistinct objects,
Divide by the overcounted permutations

Sort semi-distinct objects

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many permutations?

$$= \frac{5!}{3!2!} = 10$$

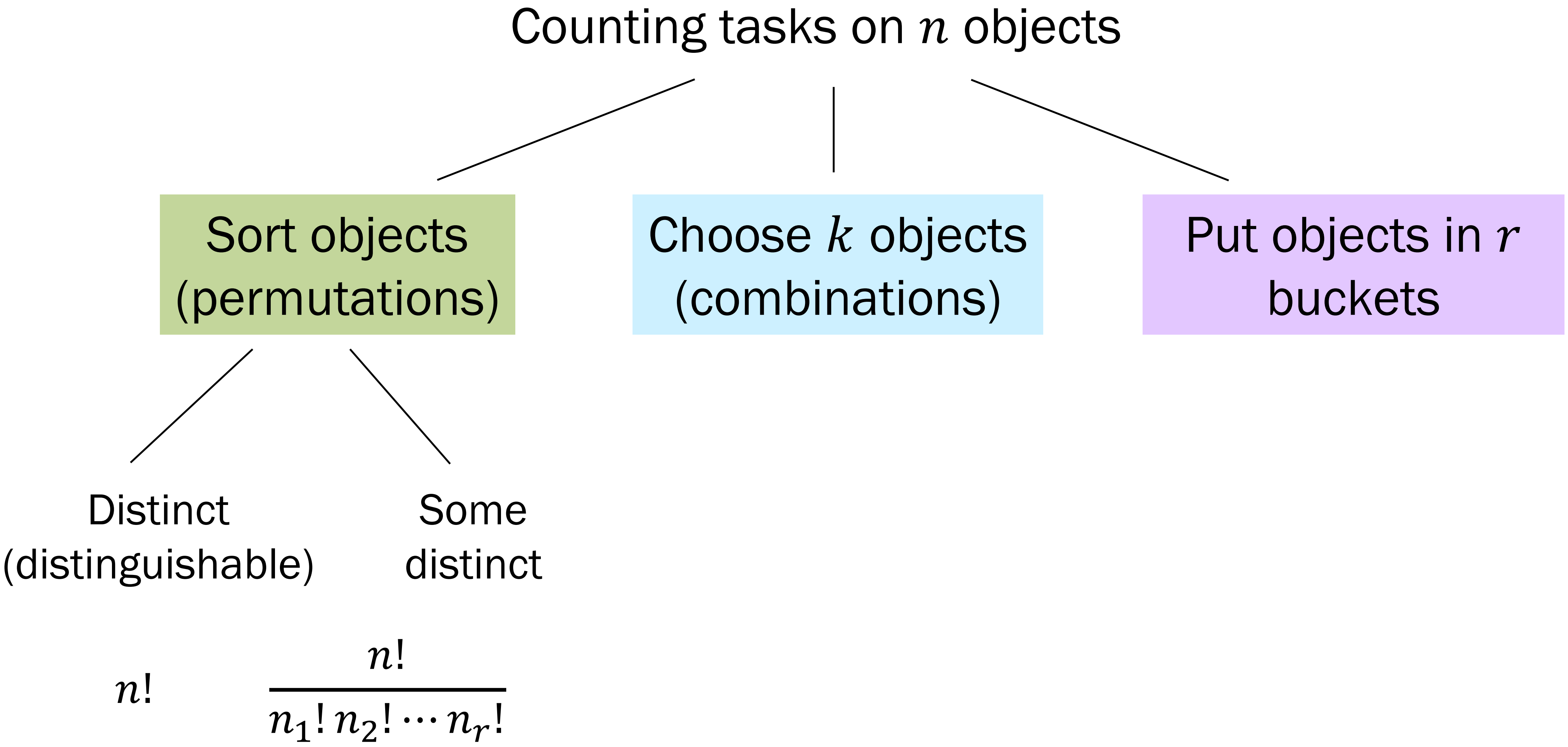
Strings

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many orderings of letters are possible for the following strings?

1. BOBA $$= \frac{4!}{2!} = 12$$
2. MISSISSIPPI $$= \frac{11!}{1!4!4!2!} = 34{,}650$$

Summary of Combinatorics



Counting tasks on n objects

Sort objects (permutations)

Choose k objects (combinations)

Put objects in r buckets

Distinct (distinguishable)

Some distinct

$n!$

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

Unique 6-digit passcodes

How many unique 6-digit passcodes are possible?

$$\begin{aligned}\text{Total} &= n_1 \times n_2 \times \cdots \times n_r \\ &= 10 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &= 10^6 \text{ passcodes}\end{aligned}$$

Unique 6-digit passcodes with six smudges

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many unique 6-digit passcodes are possible if a phone password uses each of six distinct numbers?

$$\text{Total} = 6!$$
$$= 720 \text{ passcodes}$$

Unique 6-digit passcodes with five smudges

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many unique 6-digit passcodes are possible if a phone password uses each of five distinct numbers?

Steps:

1. Choose digit to repeat — 5 outcomes
2. Create passcode — (permute 4 distinct, 2 indistinct)

$$\text{Total} = 5 \times \frac{6!}{2!}$$

$$= 1{,}800 \text{ passcodes}$$

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Permutations (sort objects)

➡ Combinations (choose objects)

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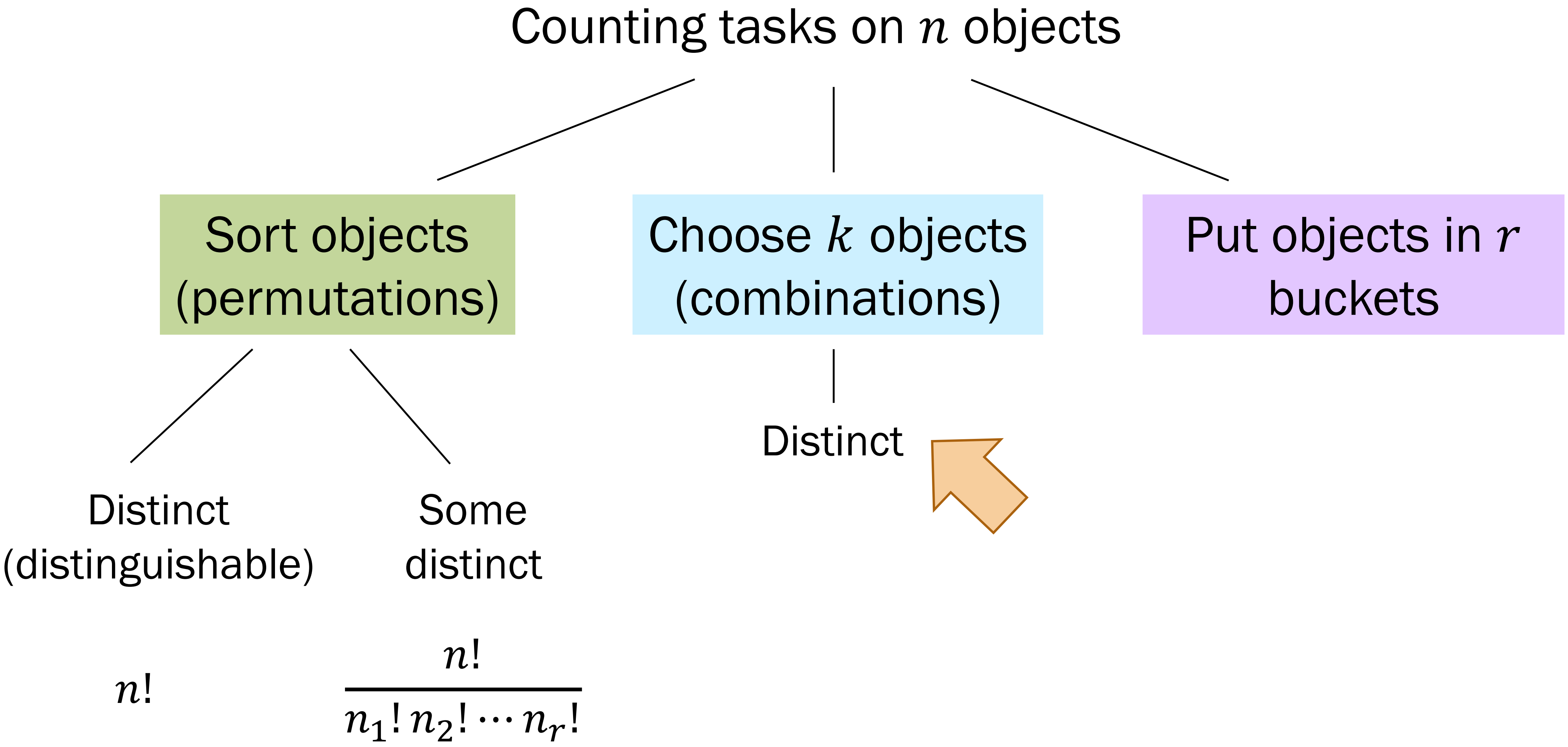
Summary of Combinatorics



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Distinct (distinguishable)

Some distinct

Distinct

$$n!$$

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

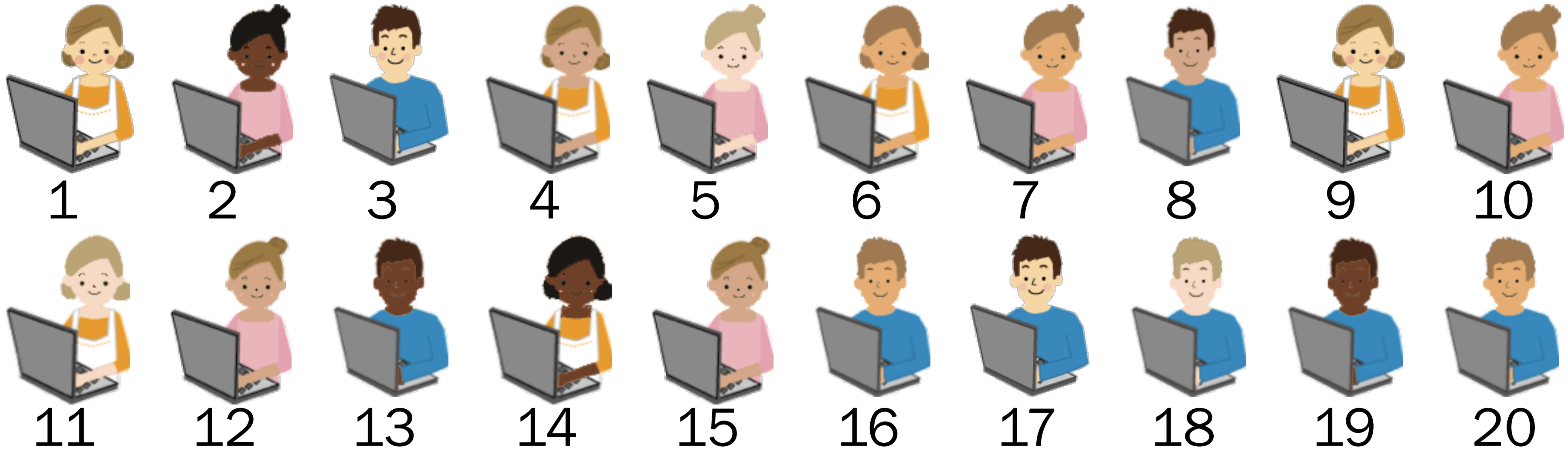
Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

Consider the following generative process...

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

$n!$ ways

Combinations with cake

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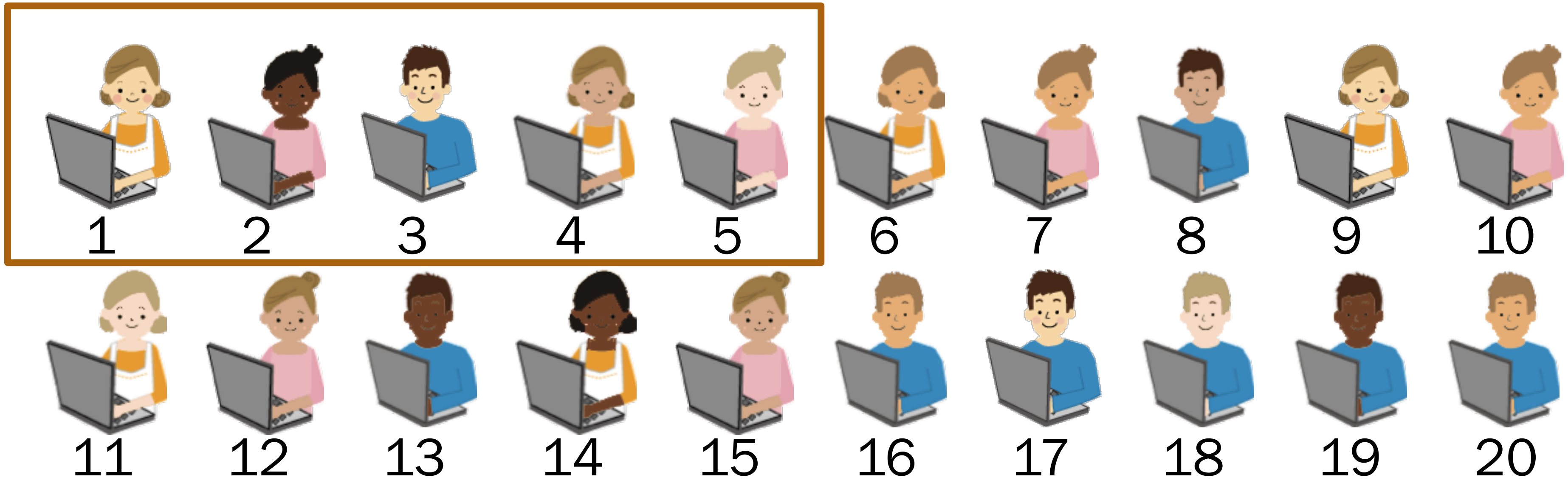

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

Combinations with cake

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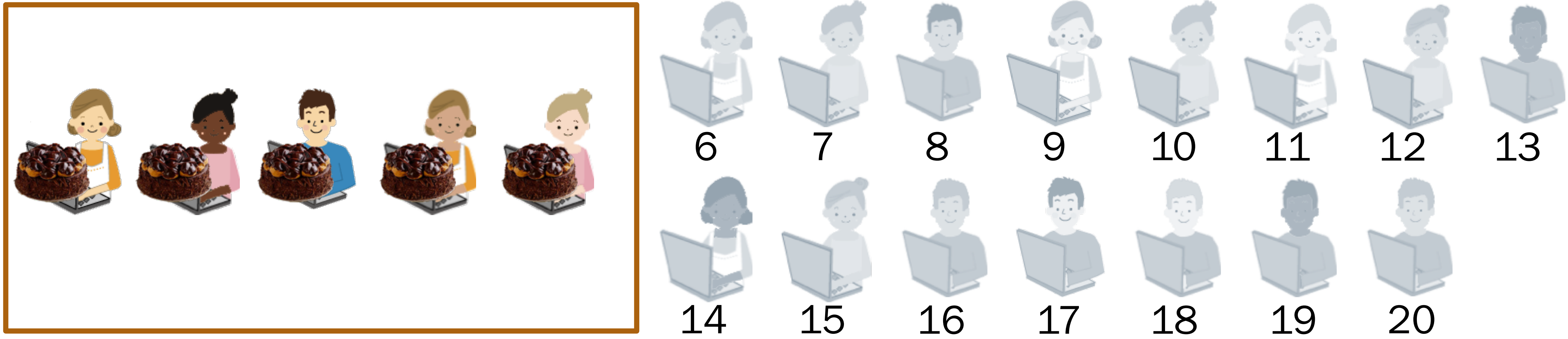

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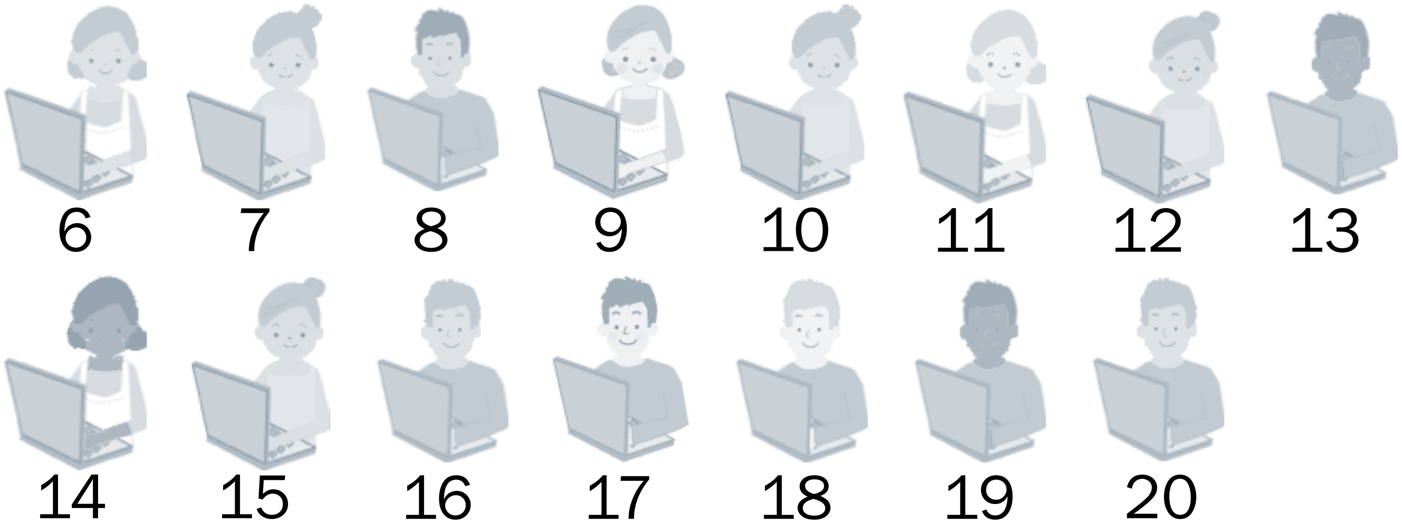

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

Combinations with cake

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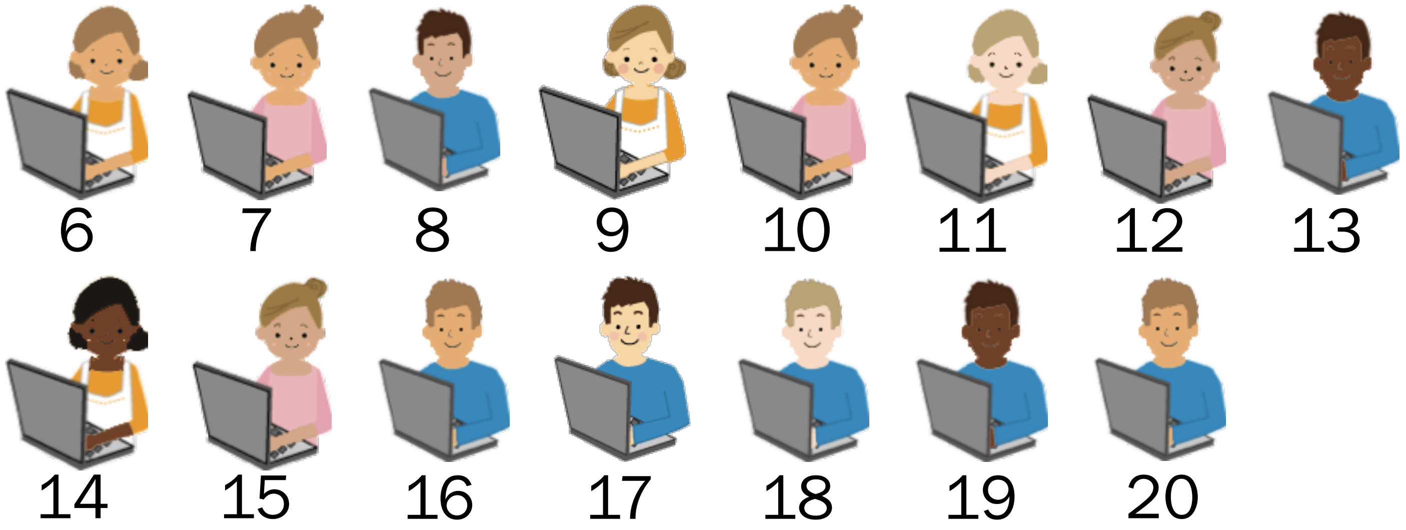

1. n people get in line
2. Put first k in cake room
3. Allow cake group to mingle
4. Allow non-cake group to mingle

$n!$ ways

1 way

$k!$ different permutations lead to the same mingle

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

   $(n-k)!$ different permutations lead to the same mingle

Combinations

A combination is an <u>unordered</u> selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!}$$

1. Order n distinct objects
2. Take first k as chosen
3. Overcounted: any ordering of chosen group is same choice
4. Overcounted: any ordering of unchosen group is same choice

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!} = \binom{n}{k}$$

Binomial coefficient

The Binomial Theorem
(if you're interested)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{r} x^k y^{n-k}$$

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

🤔

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

2. What if we do not want to read both the 9th and 10th edition of Ross?

A. $\binom{6}{3} - \binom{6}{2} = 5$ ways

B. $\frac{6!}{3!3!2!} = 10$

C. $2 \cdot \binom{4}{2} + \binom{4}{3} = 16$

D. $\binom{6}{3} - \binom{4}{1} = 16$

E. Both C and D

🤔

Probability textbooks

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[✓ E.] Both C and D

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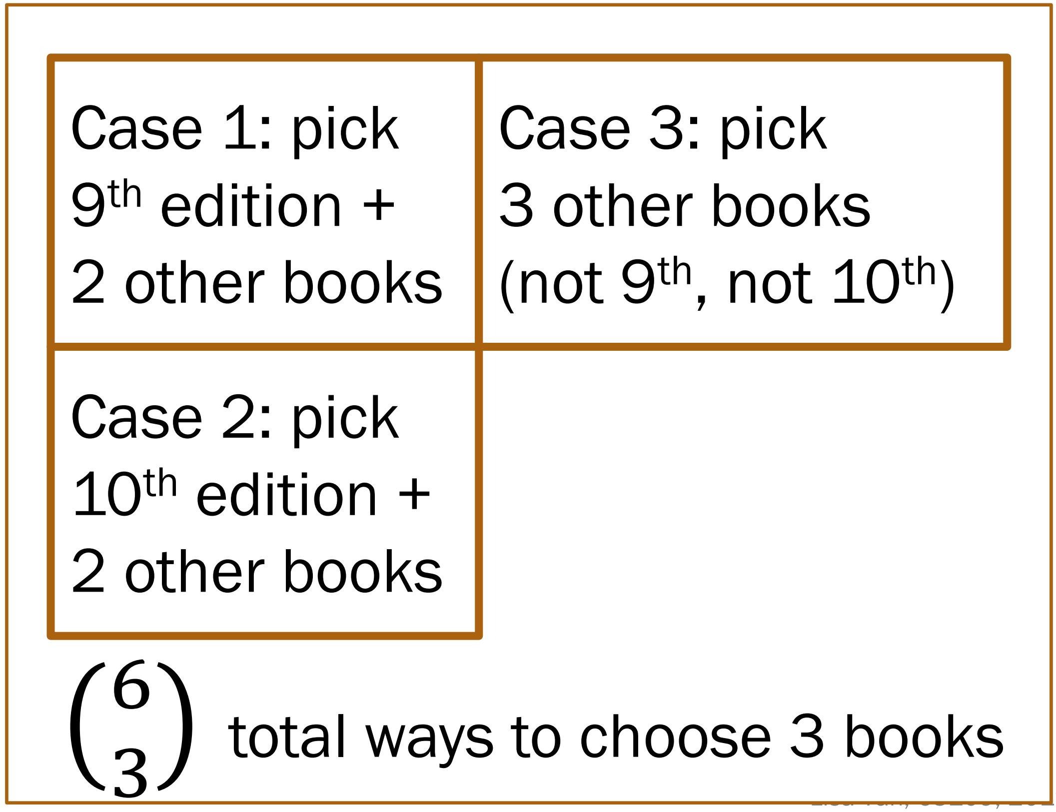


$1 \cdot \binom{4}{2}$ Case 1: pick 9th edition + 2 other books

$1 \cdot \binom{4}{2}$ Case 2: pick 10th edition + 2 other books

$\binom{4}{3}$ Case 3: pick 3 other books (not 9th, not 10th)

$\binom{6}{3}$ total ways to choose 3 books

C. $2 \cdot \binom{4}{2} + \binom{4}{3} = 16$

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

2. What if we do not want to read both the 9th and 10th edition of Ross?

$1 \cdot \binom{4}{2}$	Case 1: pick 9th edition + 2 other books	Case 3: pick 3 other books (not 9th, not 10th)	$\binom{4}{3}$
$1 \cdot \binom{4}{2}$	Case 2: pick 10th edition + 2 other books	~~Forbidden: pick 9th & 10th editions + 1 other book~~	$\binom{4}{1}$

$\binom{6}{3}$ total ways to choose 3 books

D. $\binom{6}{3} - \binom{4}{1} = 16$

👉 Forbidden method: It is sometimes easier to ***exclude*** invalid cases than to ***include*** cases.

Break for jokes/announcements

Announcements

PS#1

Out: today
Due: Friday 10/4, 1:00pm
Covers: through Friday

Staff help

Piazza policy: student discussion
Office hours: start today
cs109.stanford.edu/handouts/staff.html

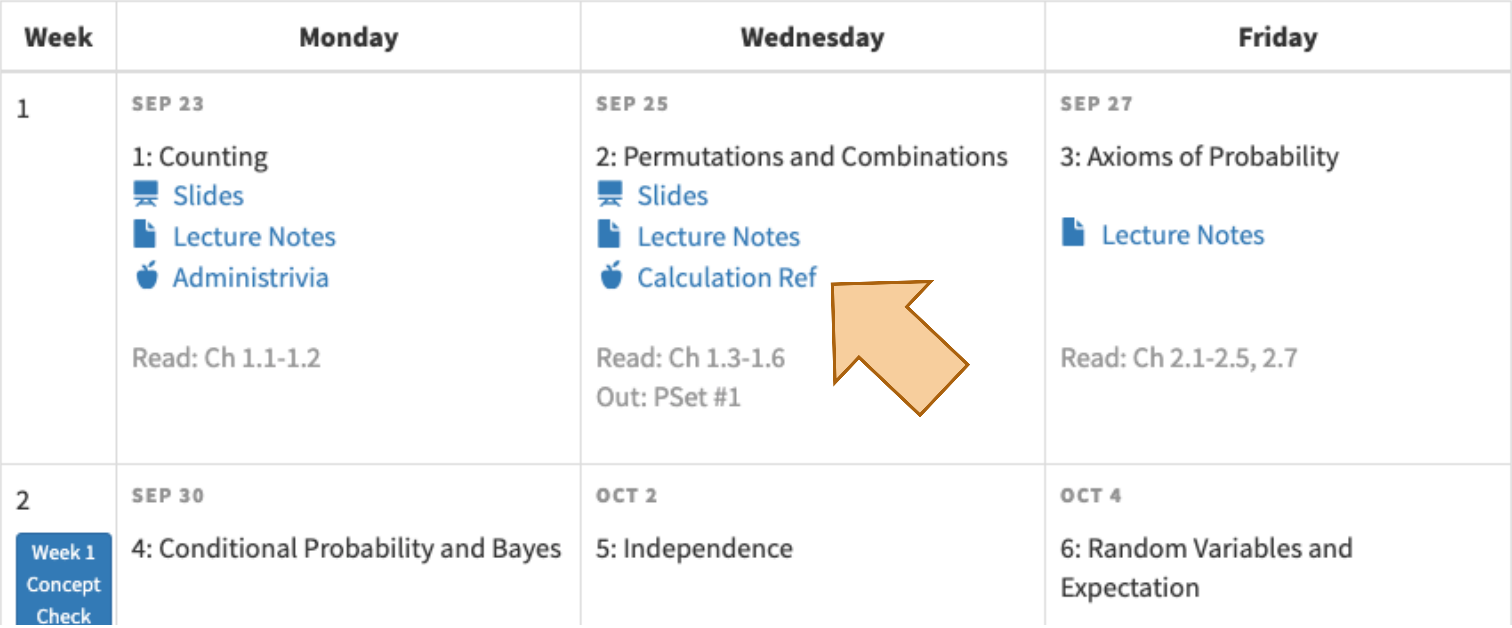
Python tutorial

When: Friday 3:30-4:20pm
Location: Hewlett 102
Recorded? maybe
Notes: to be posted online

Section sign-ups

Preference form: later today
Due: Saturday 9/28
Results: latest Monday

Handout: Calculation Reference



Week	Monday	Wednesday	Friday
1	SEP 23 1: Counting Slides Lecture Notes Administrivia Read: Ch 1.1-1.2	SEP 25 2: Permutations and Combinations Slides Lecture Notes Calculation Ref Read: Ch 1.3-1.6 Out: PSet #1	SEP 27 3: Axioms of Probability Lecture Notes Read: Ch 2.1-2.5, 2.7
2 Week 1 Concept Check	SEP 30 4: Conditional Probability and Bayes	OCT 2 5: Independence	OCT 4 6: Random Variables and Expectation

Geometric series:

$$\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$$

$$\sum_{i=m}^{n} x^i = \frac{x^{n+1}-x^m}{x-1}$$

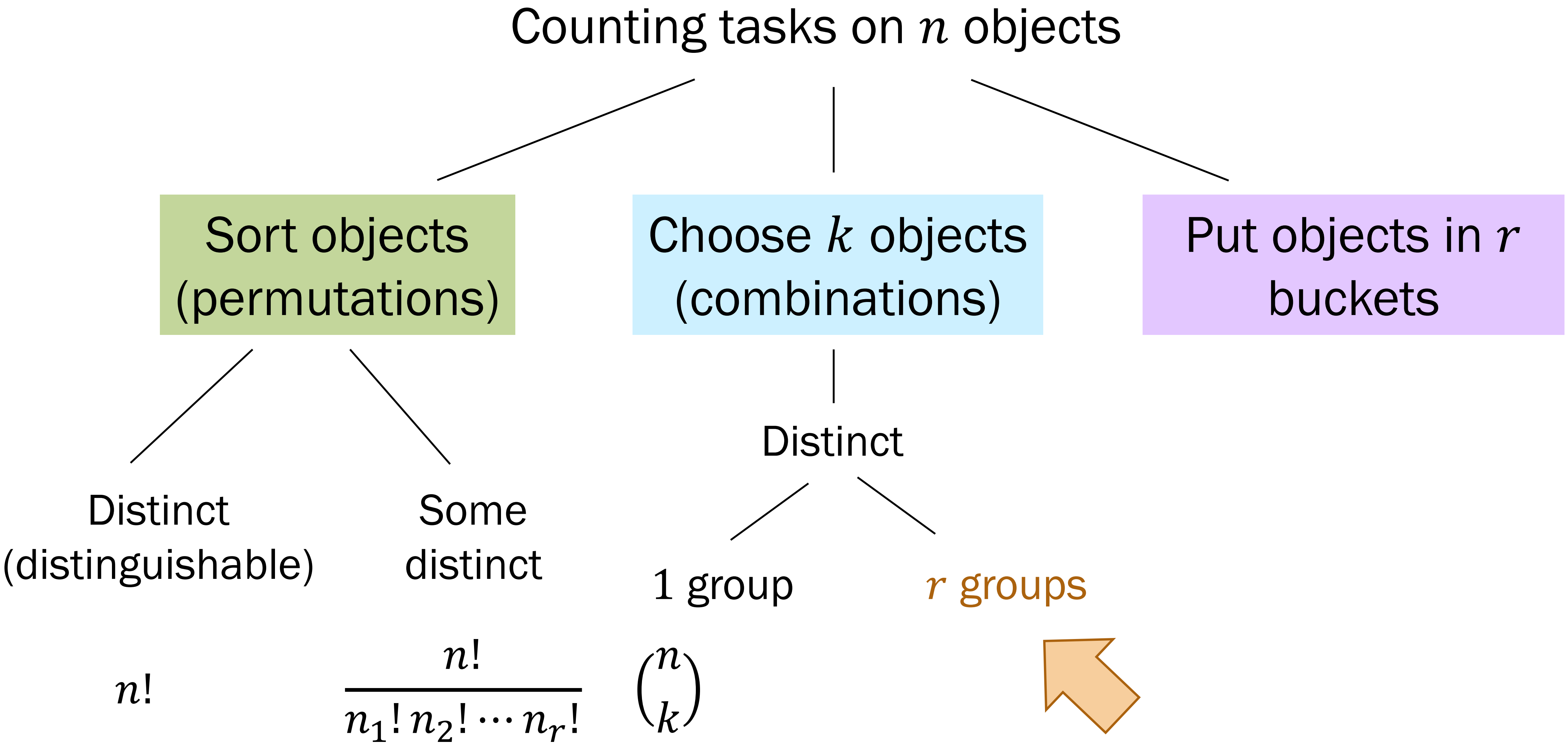
$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \text{ if } |x| < 1$$

Integration by parts (everyone's favorite!):

Choose a suitable u and dv to decompose the integral of interest:

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

Summary of Combinatorics



Counting tasks on n objects

Sort objects (permutations)

Choose k objects (combinations)

Put objects in r buckets

Distinct (distinguishable)

$n!$

Some distinct

$$\frac{n!}{n_1!\, n_2! \cdots n_r!}$$

Distinct

1 group

$\binom{n}{k}$

r groups

General approach to combinations

The number of ways to choose r groups of n distinct objects such that

For all $i = 1, \ldots, r$, group i has size n_i, and

$\sum_{i=1}^{r} n_i = n$ (all objects are assigned), is

$$\frac{n!}{n_1!\, n_2! \cdots n_r!} = \binom{n}{n_1, n_2, \cdots, n_r}$$

Multinomial coefficient

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \dots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	6
B	4
C	3

A. $\binom{13}{6,4,3} = 60{,}060$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

C. $6 \cdot 1001 \cdot 10 = 60{,}060$

D. A and B

E. All of the above

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[✓ D.] A and B

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🤔

Datacenters, Solution 1

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	6
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C	3

Group 1 (datacenter A): $n_1 = 6$

Group 2 (datacenter B): $n_2 = 4$

Group 3 (datacenter C): $n_3 = 3$

$$\binom{n}{n_1, n_2, n_3} = \binom{13}{6,4,3}$$

A. $\binom{13}{6,4,3} = 60{,}060$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

Datacenters, Solution 2

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	6
B	4
C	3

Steps:

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 2 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

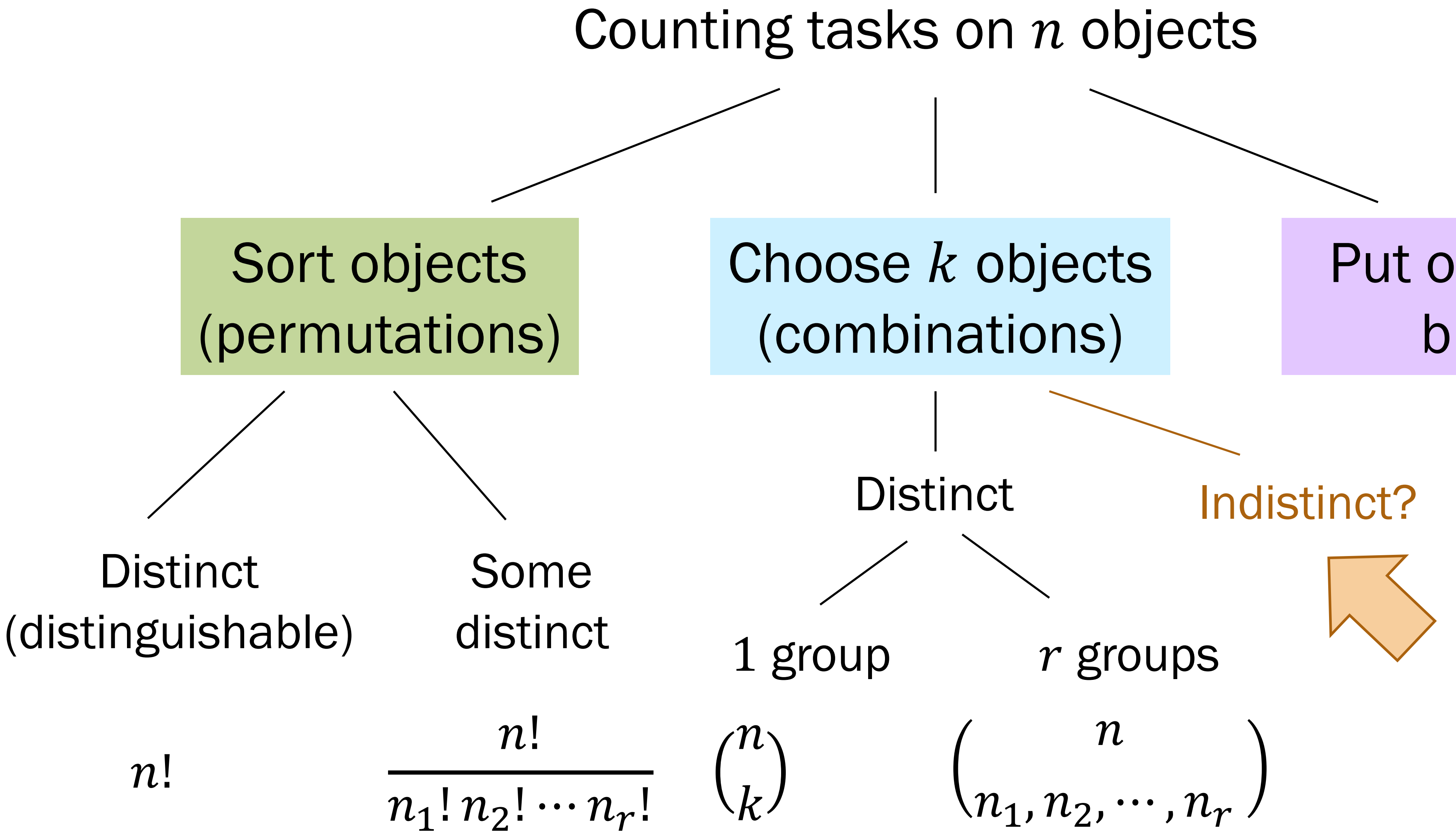
Product Rule to combine

A. $\binom{13}{6,4,3} = 60{,}060$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

👉 Your approach will determine if you use binomial/multinomial coefficients or factorials (often not both).

Summary of Combinatorics



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Distinct

1 group

$$\binom{n}{k}$$

r groups

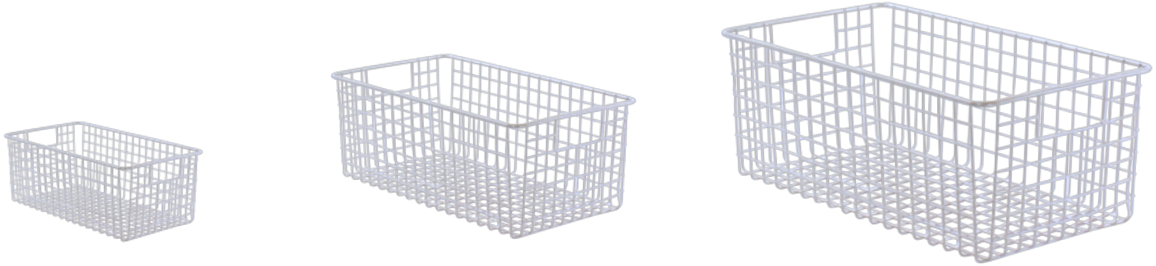
$$\binom{n}{n_1, n_2, \cdots, n_r}$$

Indistinct?

A trick question

Choose k of n distinct objects into r groups of size $n_1, \dots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

How many ways are there to group 6 indistinct (indistinguishable) objects into 3 groups, where group A, B, and C have size 1, 2, and 3, respectively?



A (fits 1) B (fits 2) C (fits 3)

A. $\binom{6}{1,2,3}$

B. $\frac{6!}{1!2!3!}$

C. 0

D. 1

E. Both A and B

A trick question

Choose k of n distinct objects into r groups of size $n_1, \dots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

How many ways are there to group 6 indistinct (indistinguishable) objects into 3 groups, where group A, B, and C have size 1, 2, and 3, respectively?

A (fits 1)

B (fits 2)

C (fits 3)

A. $\binom{6}{1,2,3}$

B. $\frac{6!}{1!2!3!}$

C. 0

[✓ D.] 1

E. Both A and B

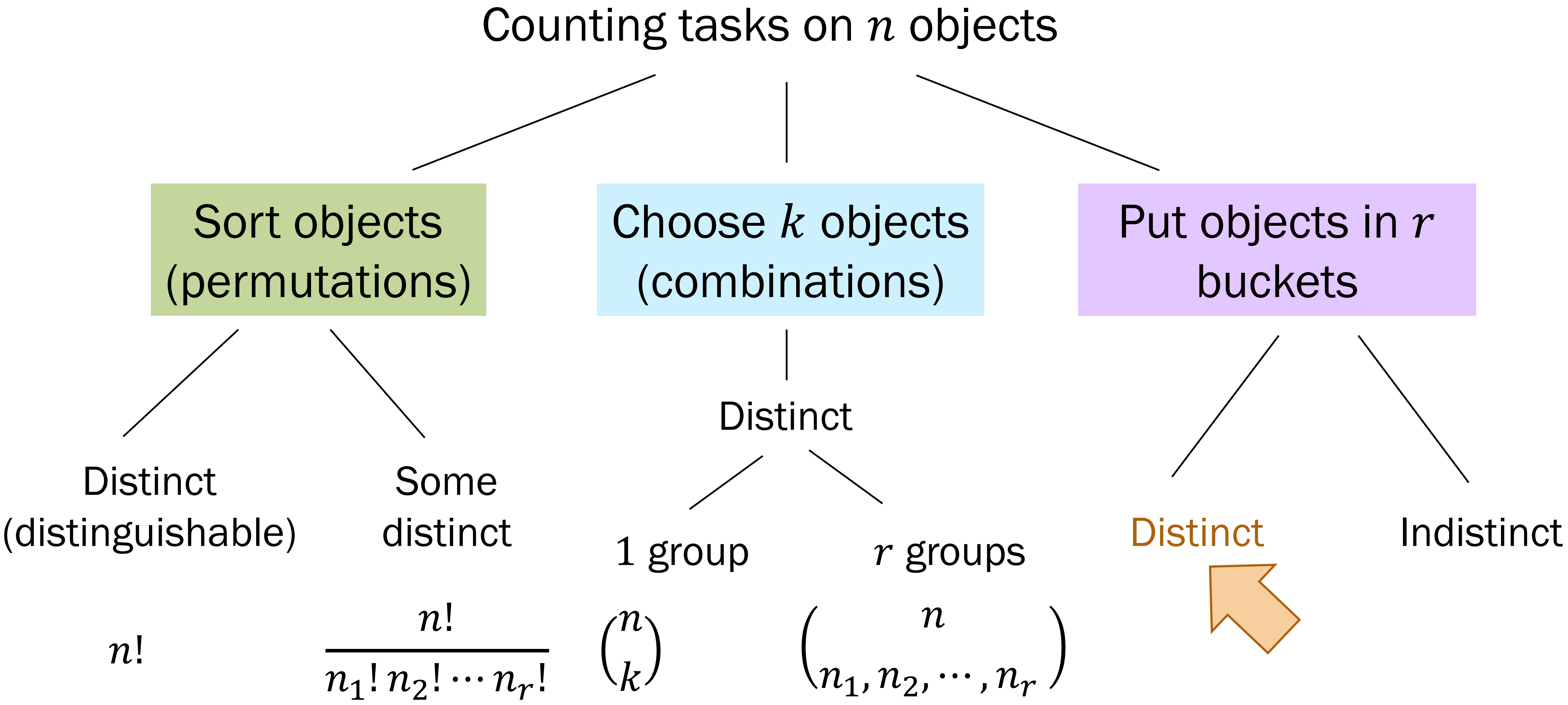
Today’s plan

Permutations (sort objects)

Combinations (choose objects)

➡ Put objects into buckets

Summary of Combinatorics



Counting tasks on n objects

Sort objects (permutations)

- Distinct (distinguishable): $n!$
- Some distinct: $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

Choose k objects (combinations)

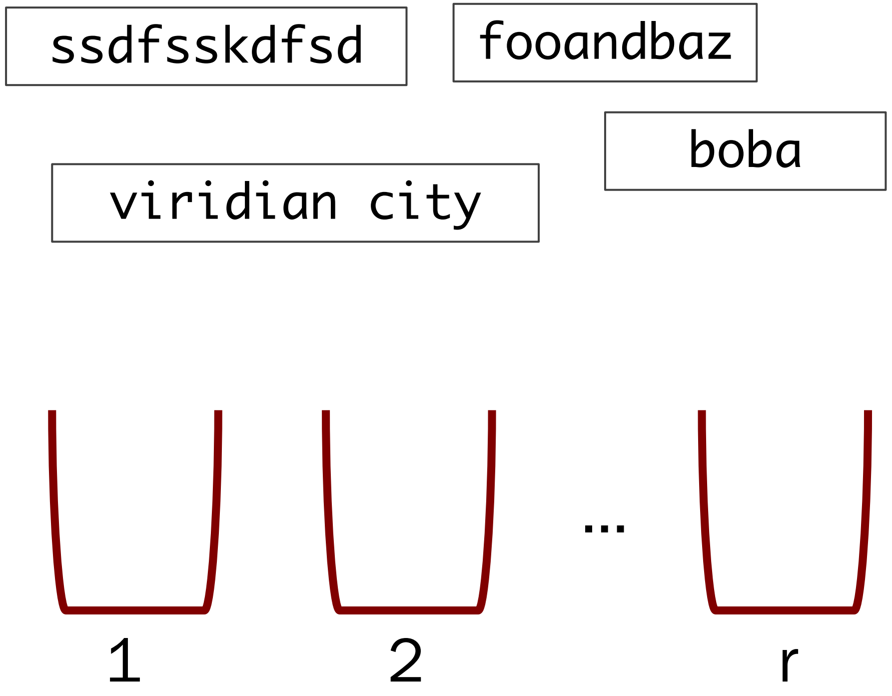
- Distinct
  - 1 group: $\binom{n}{k}$
  - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$

Put objects in r buckets

- Distinct
- Indistinct

~~Balls and urns~~ Hash tables and distinct strings

How many ways are there to hash n distinct strings to r buckets?



Steps:

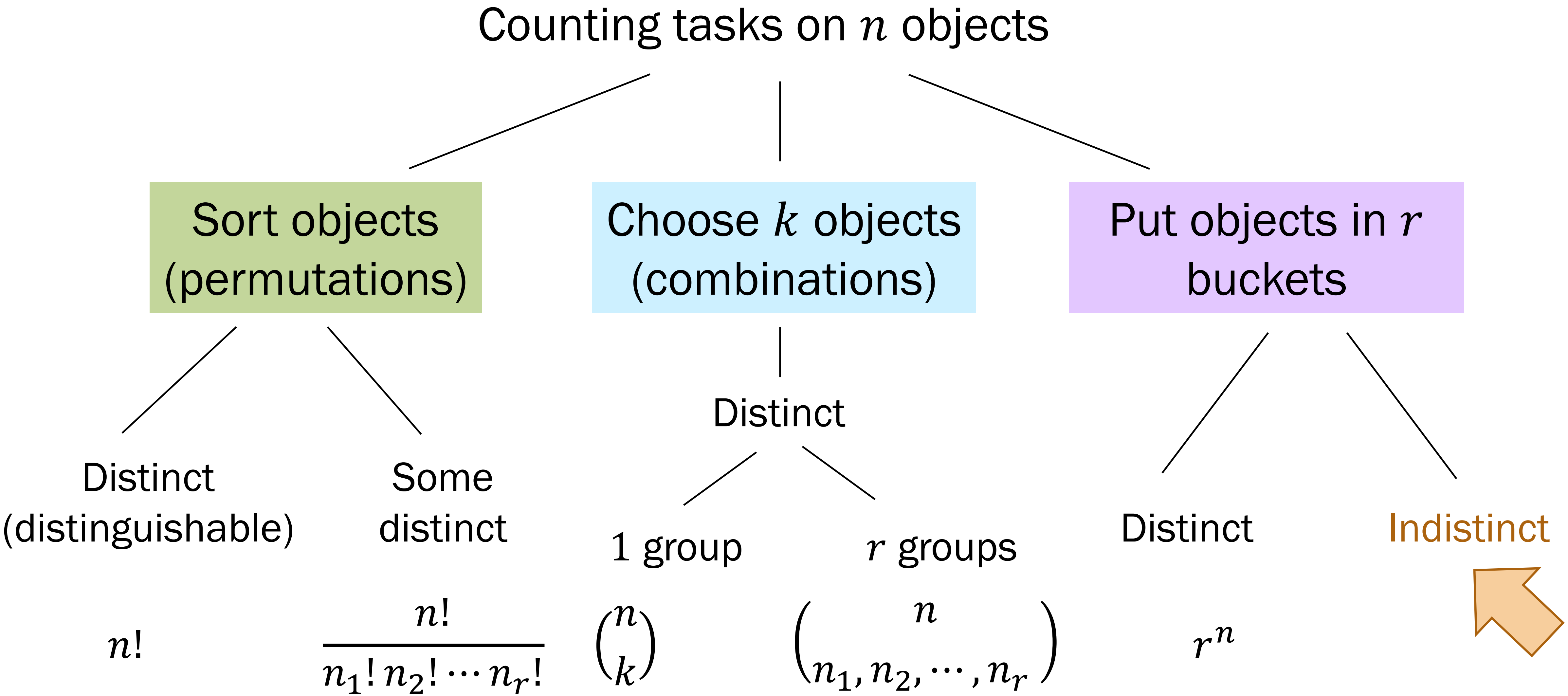
1. Bucket 1st string r options
2. Bucket 2nd string r options

…

n. Bucket nth string r options

Total $= \boxed{r^n \text{ ways}}$

Summary of Combinatorics



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- Distinct (distinguishable): $n!$
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Choose k objects (combinations)

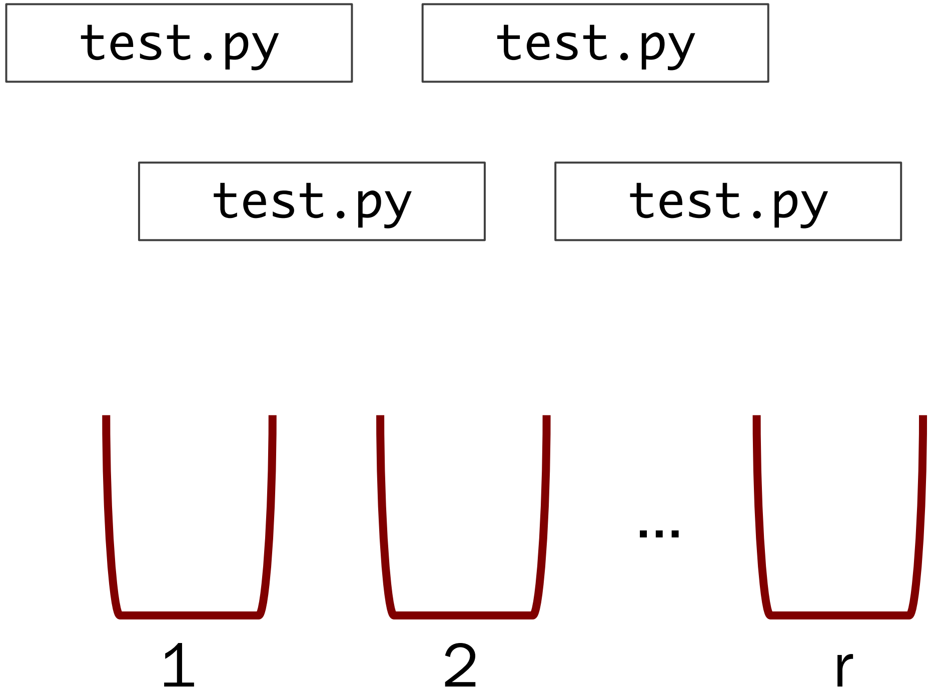
- Distinct
  - 1 group: $\binom{n}{k}$
  - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$

Put objects in r buckets

- Distinct: r^n
- Indistinct

Hash tables and indistinct strings

How many ways are there to distribute n indistinct strings to r buckets?



Goal

Bucket 1 has x_1 strings,

Bucket 2 has x_2 strings,

…

Bucket r has x_r strings (the rest)

Bicycle helmet sales

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

Consider the following generative process…

Bicycle helmet sales: 1 possible assignment outcome

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

$n = 5$ indistinct objects

$r = 4$ distinct buckets

How many ways can we assign $n = 5$ indistinguishable children to $r =$ 4
distinct bicycle helmet styles?
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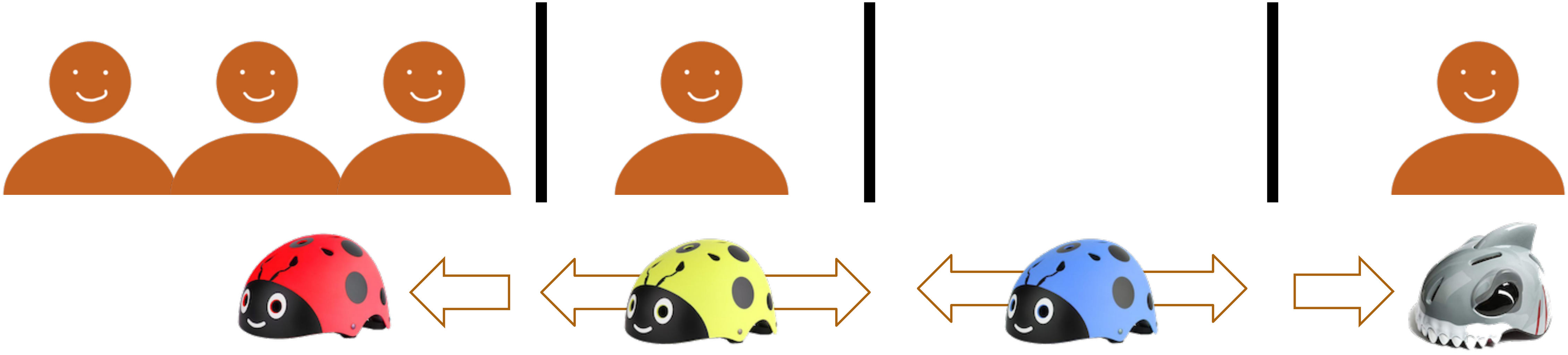


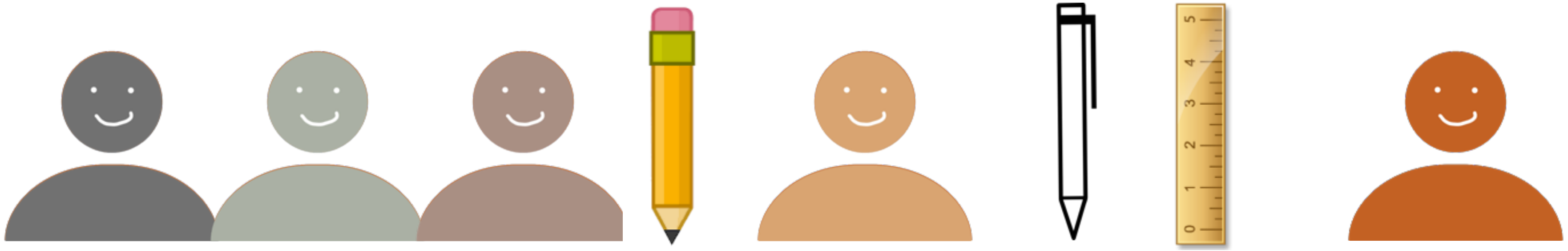
Goal Order n indistinct objects and $r - 1$ indistinct dividers.

Bicycle helmet sales: A generative proof

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

Goal Order n indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct



Bicycle helmet sales: A generative proof

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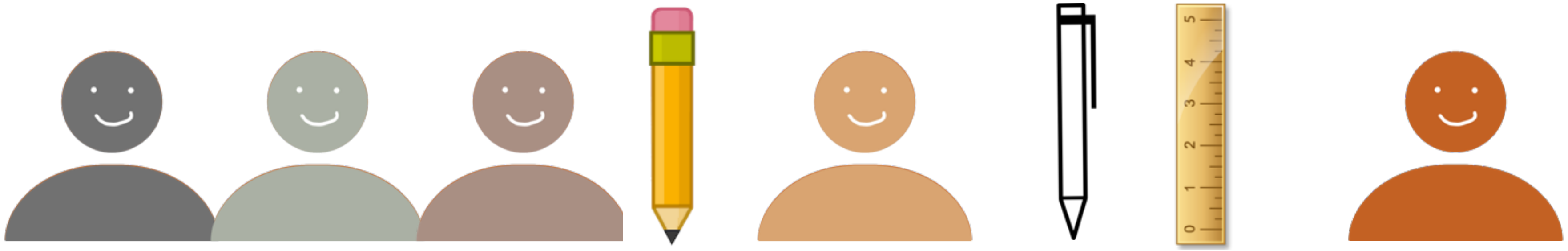

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

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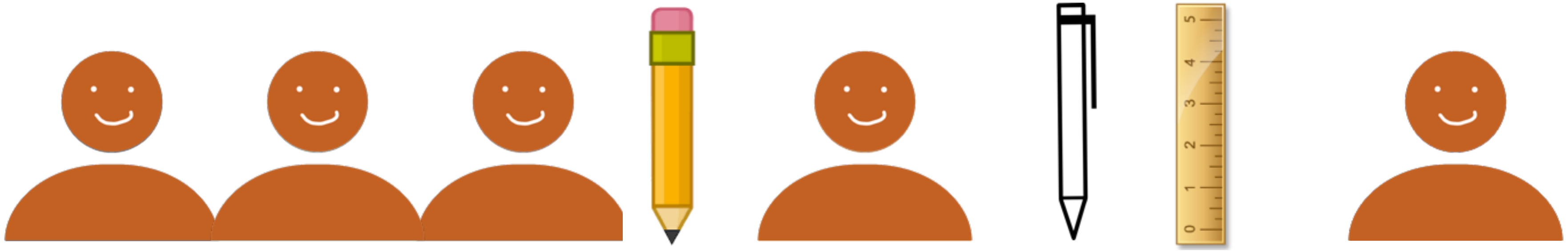


1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

Bicycle helmet sales: A generative proof

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

Goal Order n indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

3. Make $r - 1$ dividers indistinct

$$\frac{1}{(r-1)!}$$

Divider method

The number of ways to distribute n indistinct objects into r buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that

n are indistinct objects, and
$r - 1$ are indistinct dividers:

$$\text{Total} = (n + r - 1)! \times \frac{1}{n!} \times \frac{1}{(r-1)!}$$

$$= \binom{n + r - 1}{r - 1}$$

Integer solutions to equations

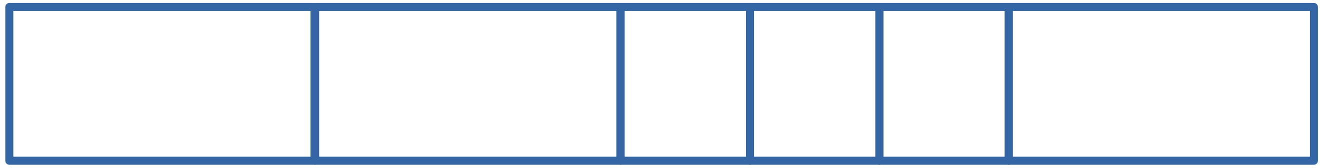
Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

How many integer solutions are there to the following equation:

$$x_1 + x_2 + \cdots + x_r = n,$$

where for all i, x_i is an integer such that $0 \leq x_i \leq n$?

Treat any solution as an integer array:



x[1] x[2] … x[r]

n increments (objects)
r array elements (buckets)

👉 Positive integer equations can be solved with the divider method.

Venture capitalists

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

$$x_i \geq 0$$

Solve

$n = 10$ increments
$r = 4$ companies

$$\binom{10+4-1}{4-1} = \binom{13}{3} = 286$$

Venture capitalists

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

⚠ $3 \le x_1$

$x_i \ge 0$ for $i = 2, 3, 4$

Fix x_1's bound

$$x_1 + x_2 + x_3 + x_4 = 7$$

x_i: amount invested in company i

$x_i \ge 0$

Solve

$n = 7$ increments
$r = 4$ companies

$$\binom{7+4-1}{4-1} = \binom{10}{3} = 120$$

Venture capitalists

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?
3. What if you don't invest all your money?

Set up ⚠

$$x_1 + x_2 + x_3 + x_4 \leq 10$$

x_i: amount invested in company i

$$x_i \geq 0$$

Add another bucket

$$x_1 + x_2 + x_3 + x_4 + x_5 = 10$$

x_i: amount invested in company i

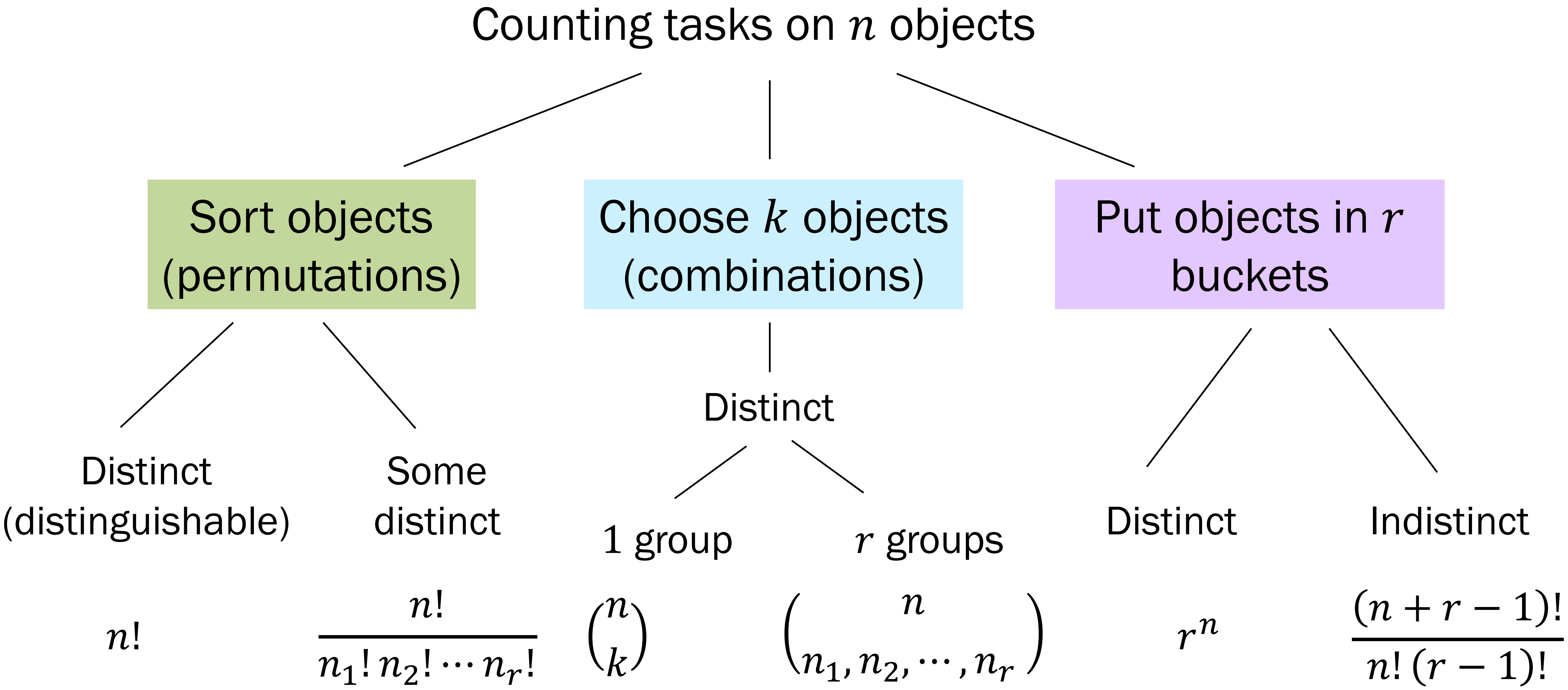
$$x_i \geq 0$$

Solve

$n = 10$ increments
$r = 5$ companies
(including yourself)

$$\binom{10+5-1}{5-1} = \binom{14}{4} = 1001$$

Summary of Combinatorics



Counting tasks on n objects

Sort objects (permutations)

Choose k objects (combinations)

Put objects in r buckets

Distinct (distinguishable)

$n!$

Some distinct

$$\frac{n!}{n_1!\, n_2! \cdots n_r!}$$

Distinct

1 group

$$\binom{n}{k}$$

r groups

$$\binom{n}{n_1, n_2, \cdots, n_r}$$

Distinct

r^n

Indistinct

$$\frac{(n+r-1)!}{n!\,(r-1)!}$$

See you next time…