03: Intro to Probability

Lisa Yan

September 27, 2019

Summary of Combinatorics

Review

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\, n_2! \cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct: r^n
  - Indistinct: $\frac{(n+r-1)!}{n!\,(r-1)!}$

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable)
  - Some distinct
- Choose k objects (combinations)
  - Distinct
    - 1 group
    - r groups
- Put objects in r buckets
  - Distinct
  - Indistinct

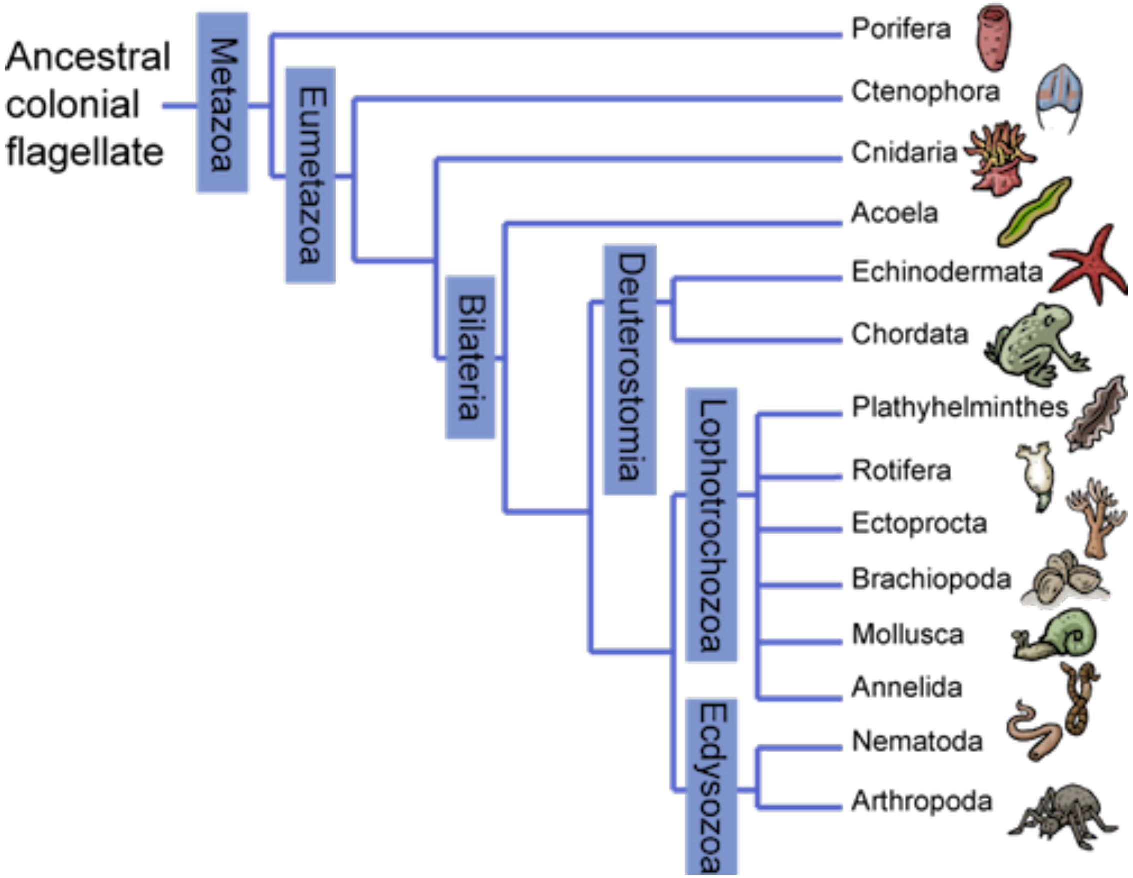
- Determine if objects are distinct
- Use Product Rule if several steps
- Use Inclusion-Exclusion if different cases

DNA distance

Review

For a DNA tree, we need to calculate the DNA distance between each pair of animals.

How many calculations are needed, i.e, how many distinct pairs of n animals are there?



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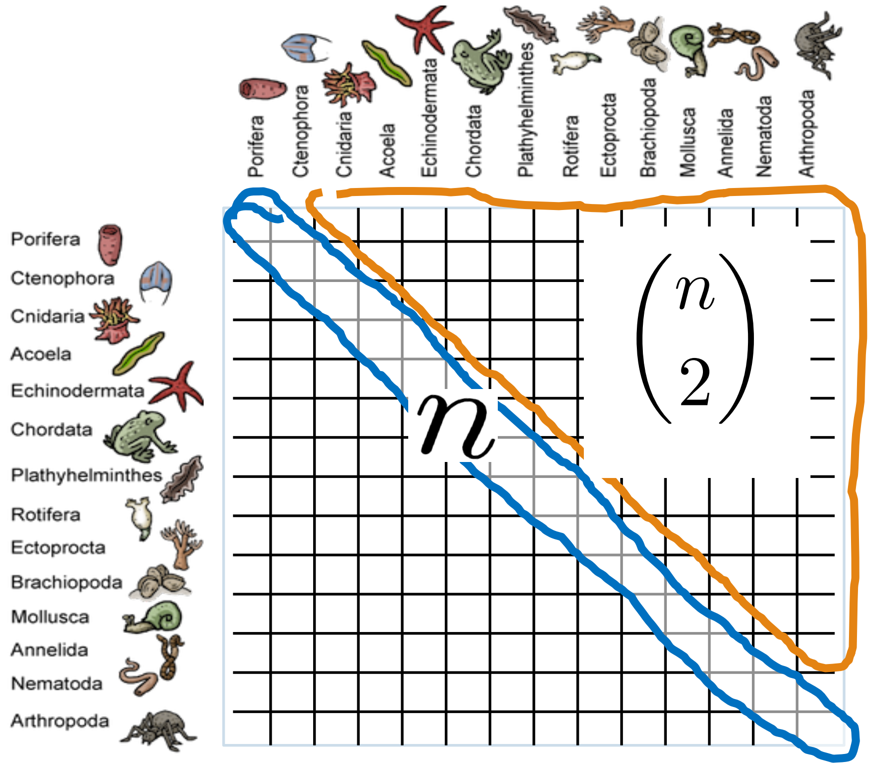


Approach 1: $\frac{n^2 - n}{2}$ $=$ Approach 2: $\binom{n}{2}$

123 SESAME STREET

The Count

Chance The Rapper

Today's plan

Key definitions: sample spaces and events

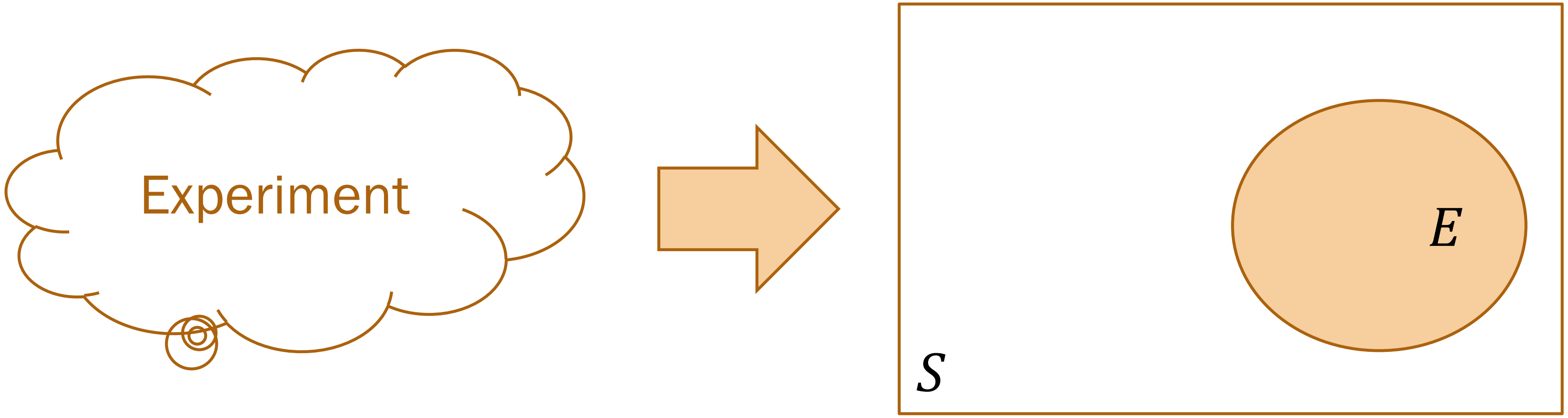
Axioms of Probability

Equally likely outcomes (counting)

Corollaries of Axioms of Probability

Key definitions

An experiment in probability:



Sample Space, S: The set of all possible outcomes of an experiment

Event, E: Some subset of S ($E \subseteq S$).

Key definitions

Sample Space, S

- Coin flip
  $S = \{\text{Heads}, \text{Tails}\}$
- Flipping two coins
  $S = \{(\text{H,H}), (\text{H,T}), (\text{T,H}), (\text{T,T})\}$
- Roll of 6-sided die
  $S = \{1, 2, 3, 4, 5, 6\}$
- # emails in a day
  $S = \{x \mid x \in \mathbb{Z},\ x \geq 0\}$
- YouTube hours in a day
  $S = \{x \mid x \in \mathbb{R}, 0 \leq x \leq 24\}$

Event, E

- Flip lands heads
  $E = \{\text{Heads}\}$
- $\geq$ 1 head on 2 coin flips
  $E = \{(\text{H,H}), (\text{H,T}), (\text{T,H})\}$
- Roll is 3 or less:
  $E = \{1, 2, 3\}$
- Low email day ($\leq$ 20 emails)
  $E = \{x \mid x \in \mathbb{Z},\ 0 \leq x \leq 20\}$
- Wasted day ($\geq$ 5 YT hours):
  $E = \{x \mid x \in \mathbb{R}, 5 \leq x \leq 24\}$

What is a probability?

A number between 0 and 1
to which we ascribe meaning.*

*our belief that an event E occurs.

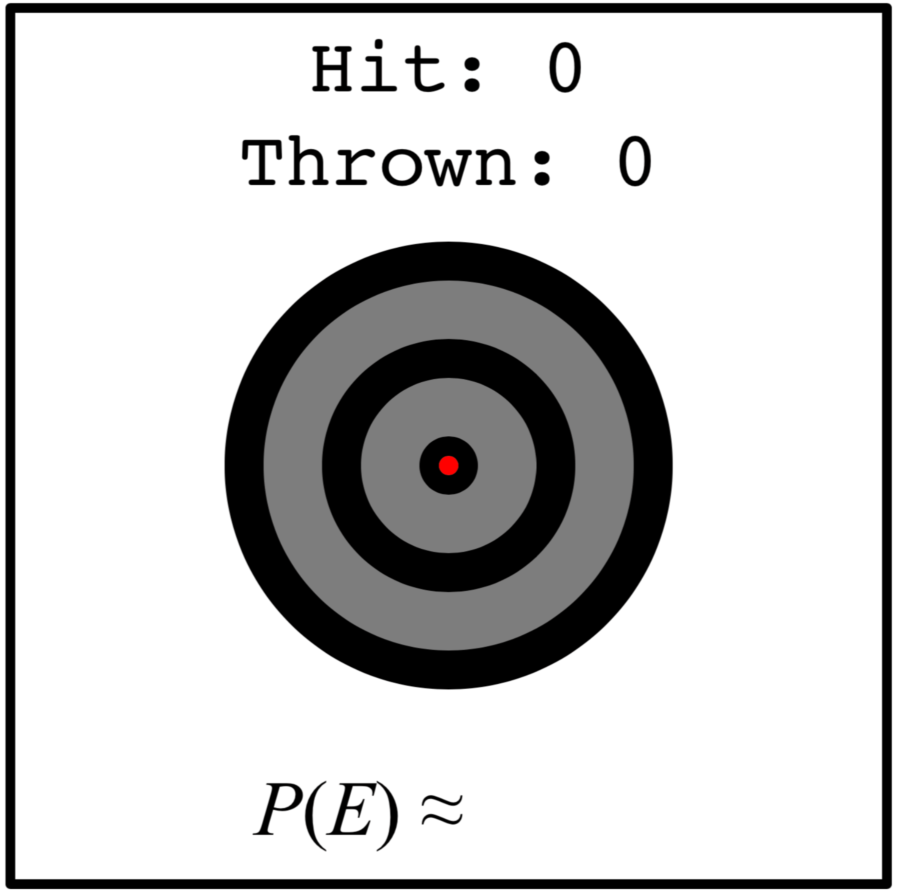
What is a probability?

Let E = the set of outcomes where you hit the target.

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

n = # of total trials

$n(E)$ = # trials where E occurs



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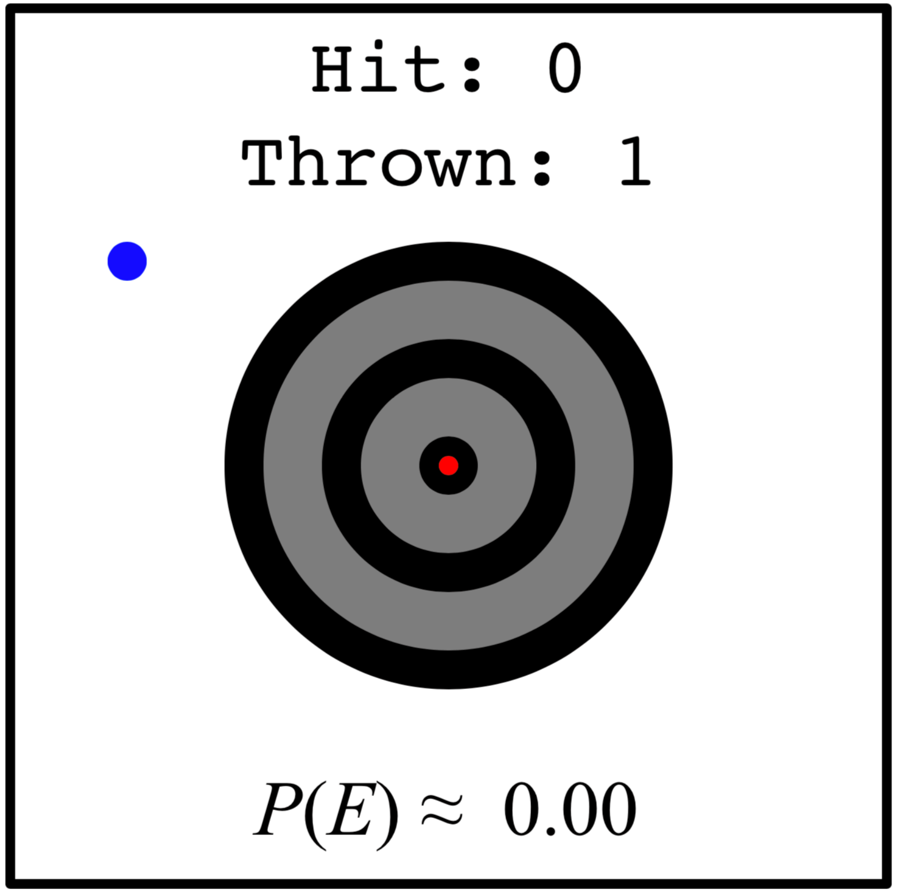

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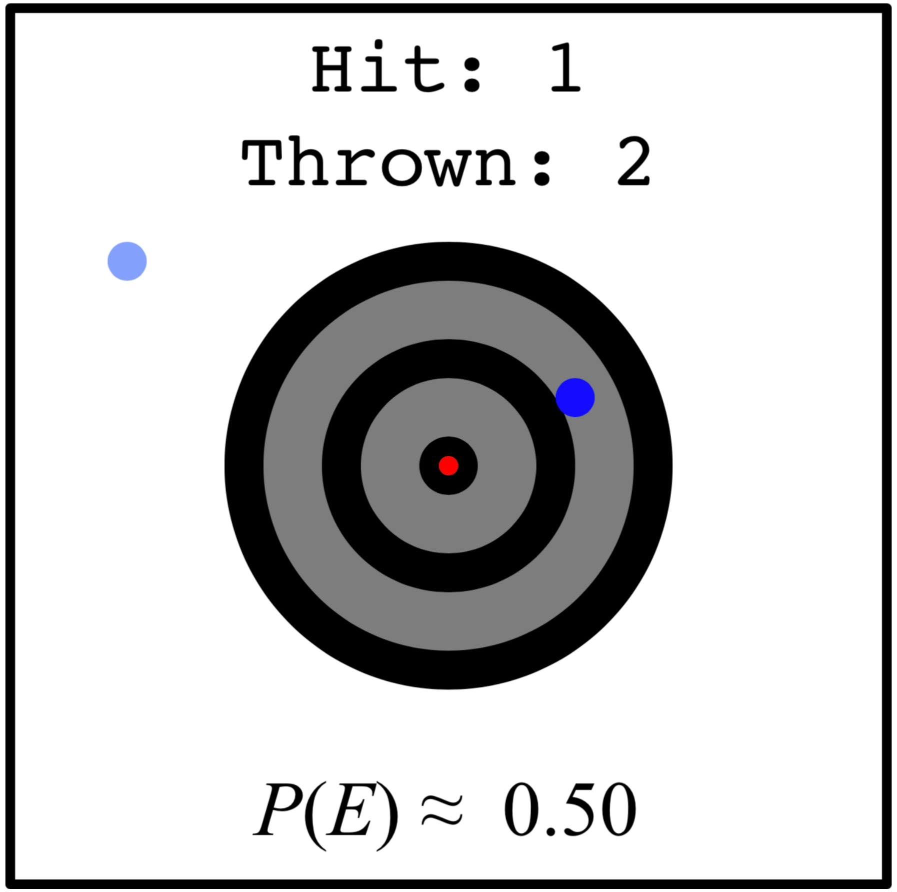

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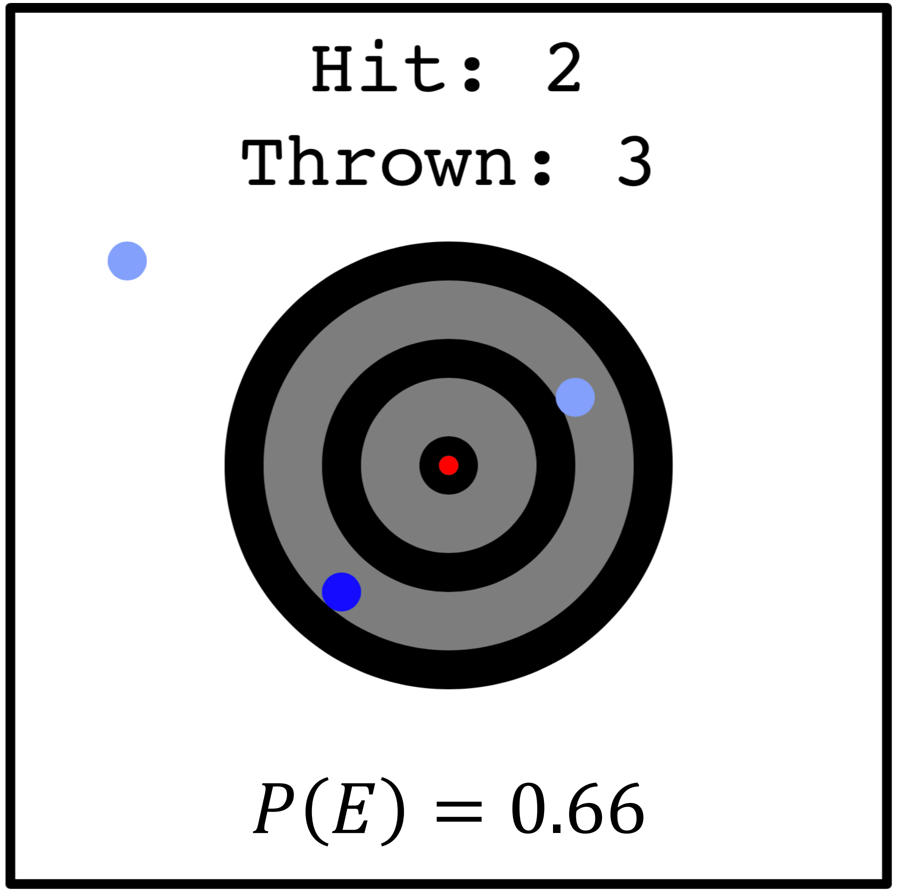

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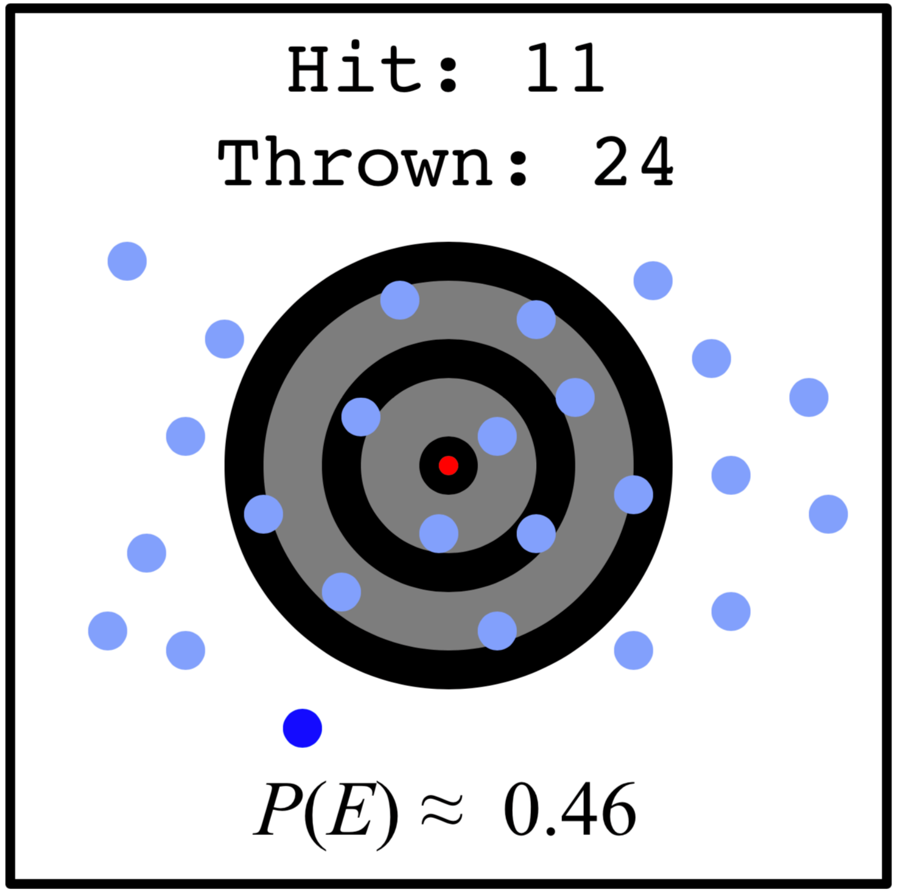


Not just yet...

Today's plan

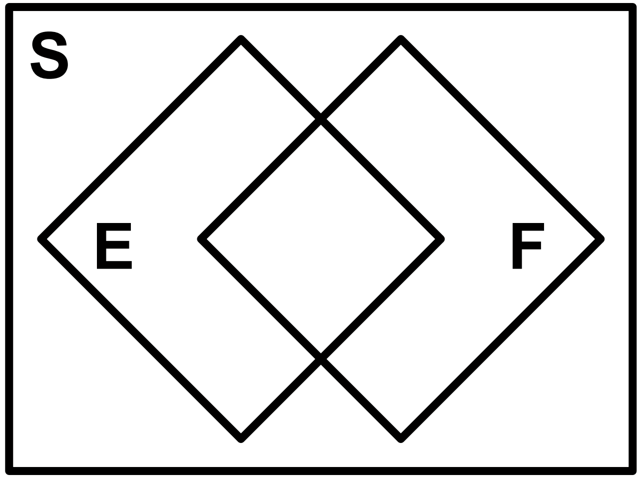
Key definitions: sample spaces and events

Axioms of Probability

Equally likely outcomes (counting)

Corollaries of Axioms of Probability

Quick review of sets



E and F are events in S.

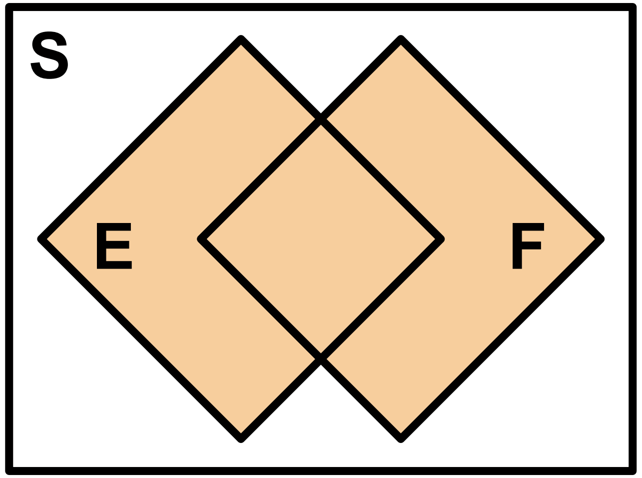
Experiment:

Dice roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

Quick review of sets



E and F are events in S.

Experiment:

Dice roll

$S = \{1, 2, 3, 4, 5, 6\}$

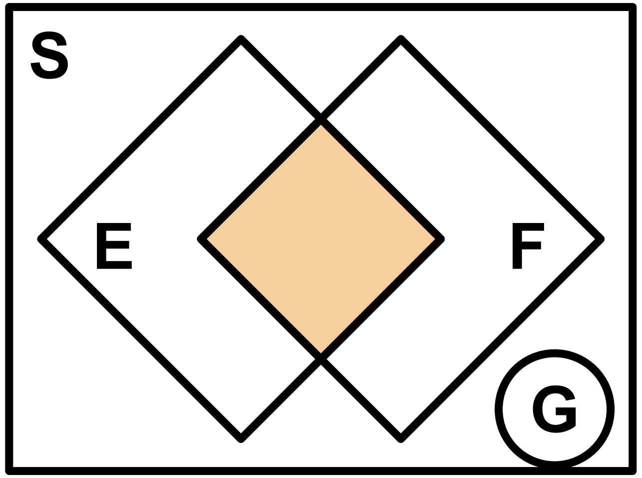
Let $E = \{1, 2\}$, and $F = \{2,3\}$

def Union of events, $E \cup F$

The event containing all outcomes in E or F.

$$E \cup F = \{1,2,3\}$$

Quick review of sets



E and F are events in S.

Experiment:

Dice roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

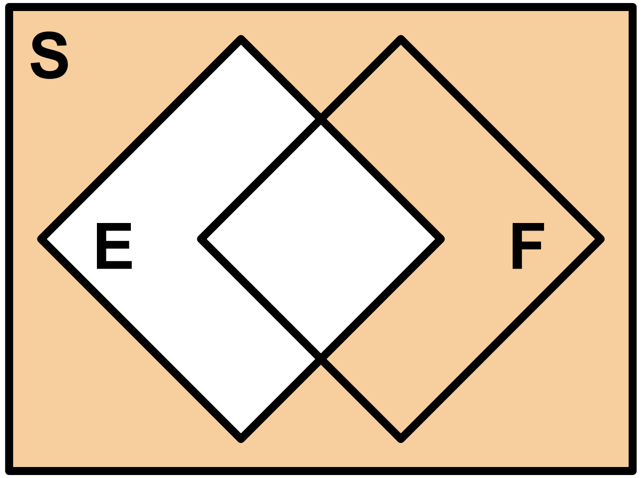
def Intersection of events, $E \cap F$

The event containing all outcomes in E *and* F.

$$E \cap F = EF = \{2\}$$

def Mutually exclusive events F and G means that $F \cap G = \emptyset$

Quick review of sets



E and F are events in S.

Experiment:

Dice roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

<u>def</u> Complement of event E, E^C

The event containing all outcomes in that are ***not*** in E.

$$E^C = \{3, 4, 5, 6\}$$

3 Axioms of Probability

Definition of probability: $P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$

Axiom 1: $0 \leq P(E) \leq 1$

Axiom 2: $P(S) = 1$

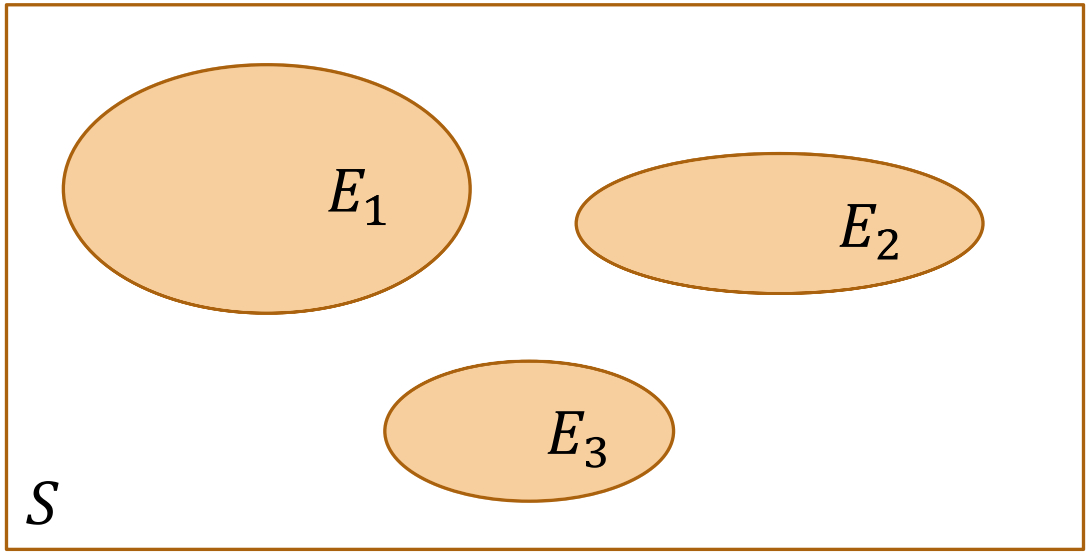
Axiom 3: If E and F are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$

Axiom 3 is the (analytically) useful Axiom

Axiom 3: If E and F are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$

More generally, for any sequence of mutually exclusive events $E_1, E_2, \ldots$:

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$



(like the Sum Rule of Counting, but for probabilities)

Today's plan

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Axioms of Probability

Equally likely outcomes (counting)

Corollaries of Axioms of Probability

Equally Likely Outcomes

Some sample spaces have equally likely outcomes.

- Coin flip: S = {Head, Tails}
- Flipping two coins: S = {(H, H), (H, T), (T, H), (T, T)}
- Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}

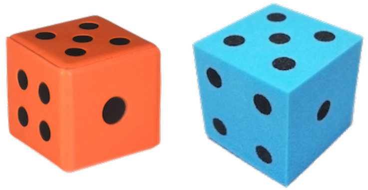
P(Each outcome) $= \dfrac{1}{|S|}$

In that case, $P(E) = \dfrac{\text{\# outcomes in E}}{\text{\# outcomes in S}} = \dfrac{|E|}{|S|}$ (by Axiom 3)

Roll two dice

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Roll two 6-sided dice. What is P(sum = 7)?



$S =$ { (1, 1) , (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1) , (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1) , (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1) , (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1) , (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
(6, 1) , (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

$E =$ { (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

$$P(E) = \frac{|E|}{|S|}$$

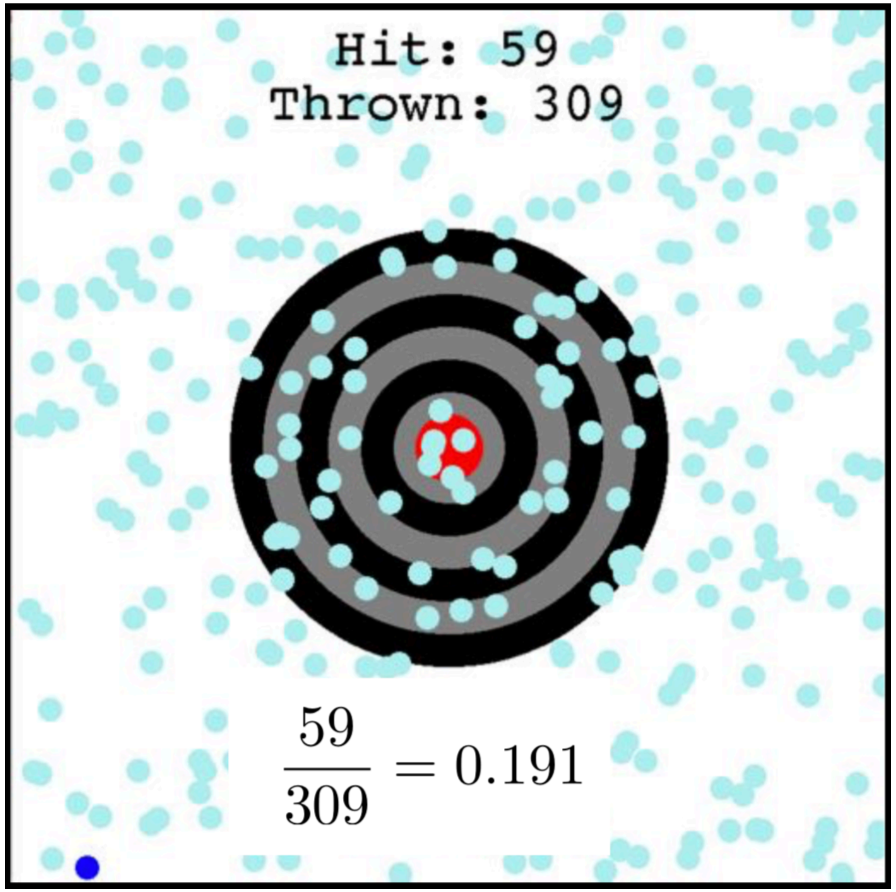
$$= \frac{6}{36} = \frac{1}{6}$$

Target revisited

Target revisited

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Let E = the set of outcomes where you hit the target.



The dart is equally likely to land anywhere on the screen.

What is $P(E)$, the probability of hitting the target?

Screen size = 800 ×800 $\quad |S| = 800^2$

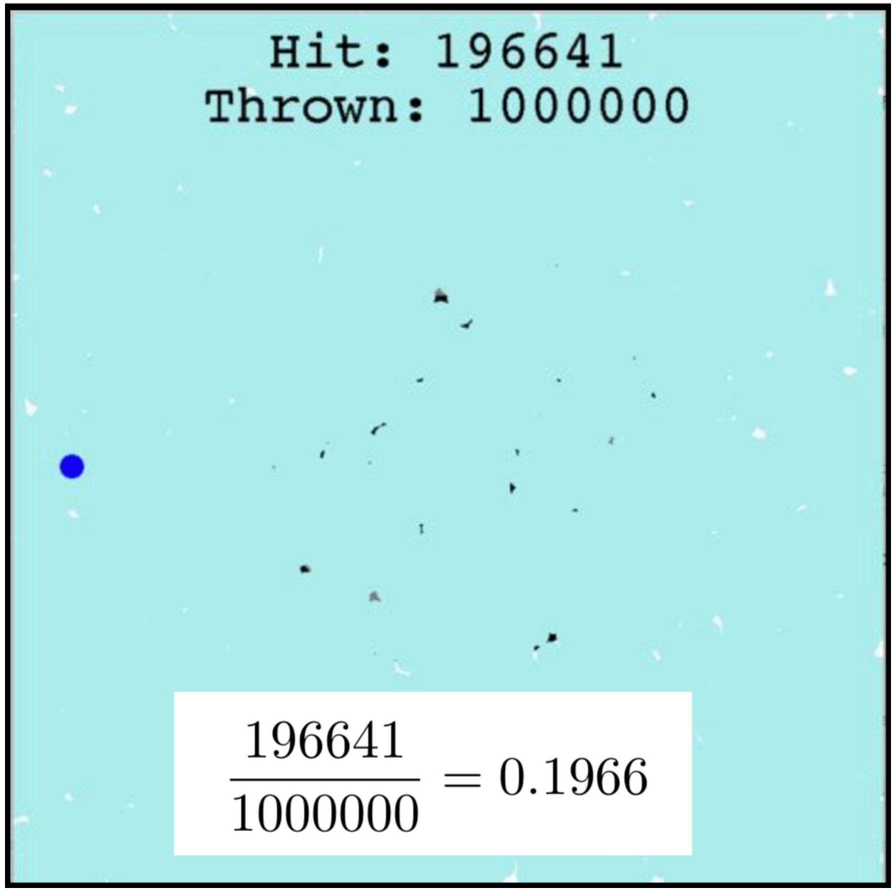
Radius of target: 200 $\quad |E| = \pi \cdot 200^2$

$$P(E) = \frac{|E|}{|S|} = \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$

Target revisited

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Let E = the set of outcomes where you hit the target.



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Not equally likely outcomes

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Play the lottery.

What is $P(\text{win})$?

$S = \{\text{Lose}, \text{Win}\}$

$E = \{\text{Win}\}$

$$P(E) = \frac{|E|}{|S|} = \frac{1}{2} = 50\%?$$

The hard part: defining equally likely outcomes ***consistently*** across sample space and events

Cats and carrots

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 carrots in a bag. 3 drawn.

What is P(1 cat and 2 carrots drawn)?

Note: Do indistinct objects give you an equally likely sample space?

A. $\frac{3}{7}$

B. $\frac{1}{4} \cdot \frac{2}{3}$

C. $\frac{4}{7} + 2 \cdot \frac{3}{6}$

D. $\frac{12}{35}$

E. Zero/other

Cats and carrots

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Note: Do indistinct objects give you an equally likely sample space?

Make indistinct items distinct to get equally likely outcomes.

Cats and carrots

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 carrots in a bag. 3 drawn.

What is P(1 cat and 2 carrots drawn)?

D. $\frac{12}{35}$

<u>Ordered</u> | <u>Unordered</u>

Define

- S = Pick 3 distinct items — $|S| = 7 \cdot 6 \cdot 5 = 210$
- E = 1 distinct cat, 2 distinct carrots — Pick Cat 1st, 2nd, or 3rd

$$|E| = 4 \cdot 3 \cdot 2 + 3 \cdot 4 \cdot 2 + 3 \cdot 2 \cdot 4$$

$$= 72$$

Compute

$$P(E) = 72/210 = 12/35$$

Cats and carrots

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 carrots in a bag. 3 drawn.

What is P(1 cat and 2 carrots drawn)?

D. $\frac{12}{35}$

Define

- S = Pick 3 distinct items
- E = 1 distinct cat, 2 distinct carrots

Ordered

$|S| = 7 \cdot 6 \cdot 5 = 210$

Pick Cat 1st, 2nd, or 3rd

$|E| = 4 \cdot 3 \cdot 2 + 3 \cdot 4 \cdot 2 + 3 \cdot 2 \cdot 4$

$= 72$

Unordered

$|S| = \binom{7}{3}$

$|E| = \binom{4}{1}\binom{3}{2}$

Compute

Ordered: $P(E) = 72/210 = 12/35$

Unordered: $P(E) = 12/35$

🤔

Break for Friday/ announcements

Announcements

Section sign-ups

Preference form:	out
Due:	Saturday 9/28
Results:	latest Monday

Concept check

Due:	Tuesday 1:00pm

Python tutorial

When:	Friday 3:30-4:20pm
Location:	Hewlett 102
Recorded?	Yes!
Notes:	to be posted online
Installation:	On Piazza

Any Poker Straight

Consider 5-card poker hands.

- "straight" is 5 consecutive rank cards of any suit

What is P(Poker straight)?

- What is an example of an outcome?
- Is each outcome equally likely?
- Should objects be ordered or unordered?

Any Poker Straight

Consider 5-card poker hands.

- "straight" is 5 consecutive rank cards of any suit

What is P(Poker straight)?

- What is an example of an outcome?
- Is each outcome equally likely?
- Should objects be ordered or unordered?

Define

- S (unordered) $\quad |S| = \binom{52}{5}$
- E (unordered, consistent with S) $\quad |E| = 10 \cdot \binom{4}{1}^5$

Compute $\quad P(\text{Poker straight}) = \frac{|E|}{|S|} = \frac{10 \cdot \binom{4}{1}^5}{\binom{52}{5}} \approx 0.00394$

"Official" Poker Straight

Consider 5-card poker hands.

- "straight" is 5 consecutive rank cards of any suit
- "straight flush" is 5 consecutive rank cards of same suit

What is P(Poker straight, but not straight flush)?

Define

- S (unordered) $\quad |S| = \binom{52}{5}$
- E (unordered, consistent with S) $\quad |E| = 10 \cdot \binom{4}{1}^5 - 10 \cdot \binom{4}{1}$

Compute $\quad P(\text{Official Poker straight}) = \frac{|E|}{|S|} = \frac{10 \cdot \binom{4}{1}^5 - 10 \cdot \binom{4}{1}}{\binom{52}{5}} \approx 0.00392$

Chip defect detection

n chips are manufactured, 1 of which is defective.
k chips are randomly selected from n for testing.

What is P(defective chip is in k selected chips?)

Define

- S (unordered) $\quad |S| = \binom{n}{k}$
- E (unordered, consistent with S) $\quad |E| = \binom{1}{1}\binom{n-1}{k-1}$
  1. Choose defective chip
  2. Choose $k-1$ other chips

Compute

$$P(E) = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{\frac{(n-1)!}{(k-1)!(n-k)!}}{\frac{n!}{k!(n-k)!}} = \frac{(n-1)!\,k!}{n!\,(k-1)!} = \boxed{\frac{k}{n}}$$

Chip defect detection, solution #2

n chips are manufactured, 1 of which is defective.
k chips are randomly selected from n for testing.

What is P(defective chip is in k selected chips?)

Redefine experiment

1. Choose k indistinct chips (1 way)
2. Throw a dart and make one defective

Define

- S (unordered) $\quad |S| = 1 \cdot n$
- E (unordered, consistent with S) $\quad |E| = 1 \cdot k$

$$P(E) = \boxed{\frac{k}{n}}$$

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Equally likely outcomes (counting)

Corollaries of Axioms of Probability

Axioms of Probability

Review

Definition of probability: $P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$

Axiom 1: $0 \le P(E) \le 1$

Axiom 2: $P(S) = 1$

Axiom 3: If E and F are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$

3 Corollaries of Axioms of Probability

Corollary 1: $P(E^C) = 1 - P(E)$

Proof of Corollary 1

Corollary 1: $P(E^C) = 1 - P(E)$

Proof:

E, E^C are mutually exclusive	Definition of E^C
$P(E \cup E^C) = P(E) + P(E^C)$	Axiom 3
$S = E \cup E^C$	Everything must either be in E or E^C, by definition
$1 = P(S) = P(E) + P(E^C)$	Axiom 2
$P(E^C) = 1 - P(E)$	Rearrange

3 Corollaries of Axioms of Probability

Corollary 1: $P(E^C) = 1 - P(E)$

Corollary 2: If $E \subseteq F$, then $P(E) \leq P(F)$

Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$

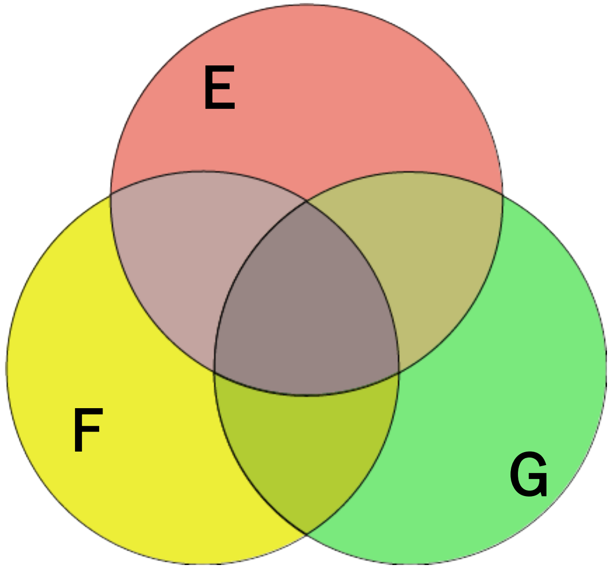
(Inclusion-Exclusion Principle for Probability)

Inclusion-Exclusion Principle (Corollary 3)

Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$

(Inclusion-Exclusion Principle for Probability)

General form: $$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{(r+1)} \sum_{i_1 < \cdots < i_r} P\left(\bigcap_{j=1}^{r} E_{i_j}\right)$$



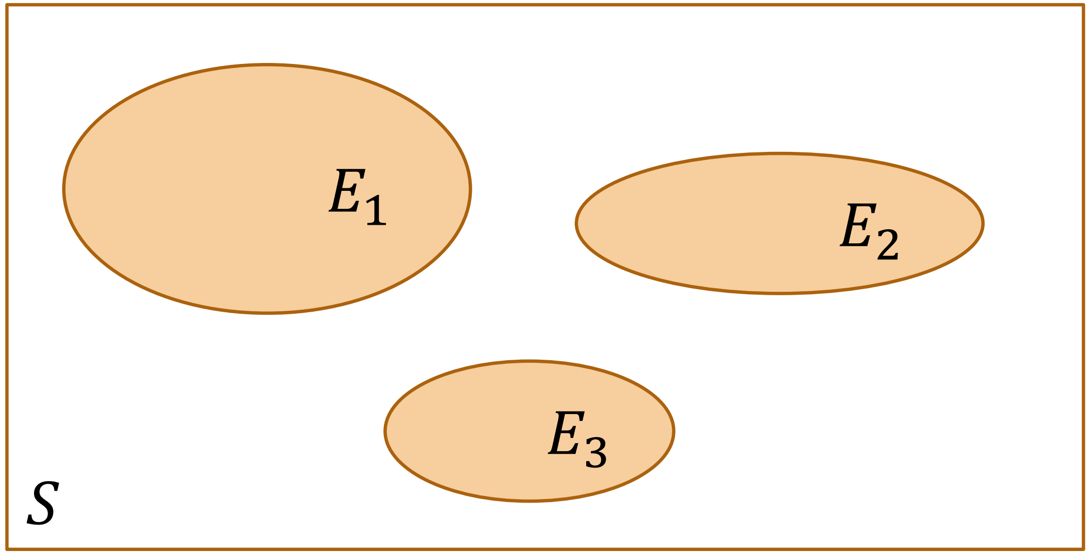
$P(E \cup F \cup G) =$

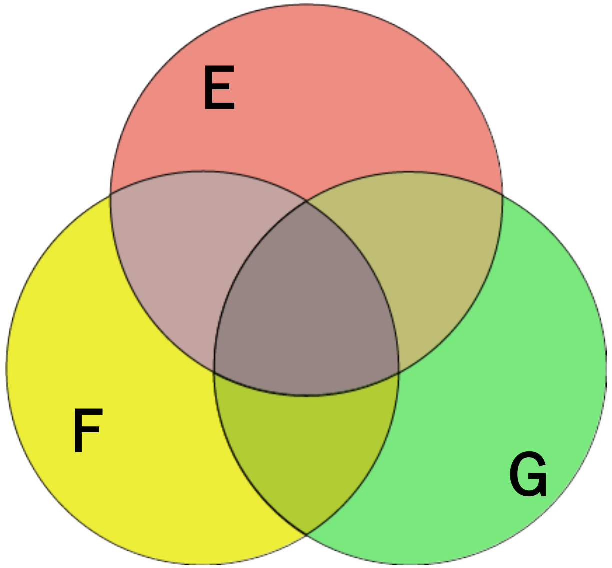
r = 1: $P(E) + P(F) + P(G)$

r = 2: $- P(E \cap F) - P(E \cap G) - P(F \cap G)$

r = 3: $+ P(E \cap F \cap G)$

Takeaway: Mutually exclusive events





Axiom 3,
Mutually exclusive events

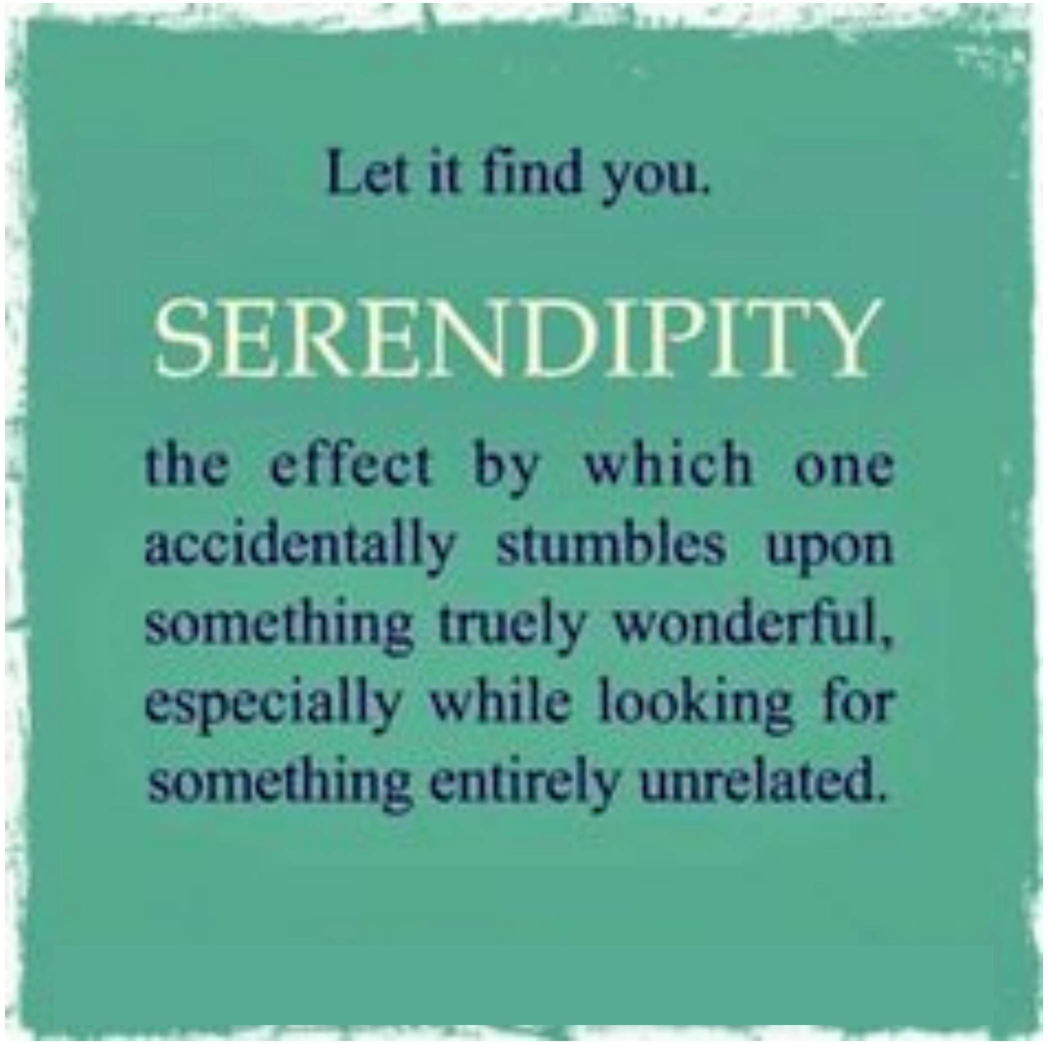
$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Inclusion-Exclusion Principle

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{(r+1)} \sum_{i_1 < \cdots < i_r} P\left(\bigcap_{j=1}^{r} E_{i_j}\right)$$

Design your experiment to compute easier probabilities.

Serendipity



Serendipity

- The population of Stanford is $n = 17{,}000$ people.
- You are friends with $r = ?$ people.
- Walk into a room, see $k = 268$ random people.
- Assume you are equally likely to see each person at Stanford.

What is the probability that you see someone you know?

Define

- S (unordered)
- E: see ≥ 1 friend in the room

$$|S| = \binom{n}{k} = \binom{17000}{268}$$

How should we compute $P(E)$?

A. $P(\text{exactly } 1) + P(\text{exactly } 2)$
$P(\text{exactly } 3) + \cdots$

B. $1 - P(\text{see no friends})$

It is often much easier to compute $P(E^c)$.

The Birthday ~~Paradox~~ Problem

What is the probability that in a set of *n* people, <u>at least one</u> pair of them will share the same birthday?

For you to think about (and discuss in section!)