05: Independence

Lisa Yan October 2, 2019

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Two Dice

- Roll two 6-sided dice, yielding values D_1 and D_2 .
- Let event *E*: $D_1 = 5$ event *F*: $D_2 = 5$
- 1. Roll a 5 on one of the rolls
- 2. Roll a 5 on both rolls
- 3. Neither roll is 5
- 4. Roll a 5 on roll 2
- 5. Do not roll a 5 on one of the rolls

- A. P(F)
- $\mathsf{B.} \ P(E \cup F)$
- C. $P(E^C \cup F^C)$
- **D.** P(EF)
- E. $P(E^{C}F^{C})$



Two Dice

- Roll two 6-sided dice, yielding values D_1 and D_2 .
- Let event E: $D_1 = 5$ event F: $D_2 = 5$

1. Roll a 5 on one of the rollsA. P(F)2. Roll a 5 on both rollsB. $P(E \cup F)$ 3. Neither roll is 5C. $P(E^C \cup F^C)$ 4. Roll a 5 on roll 2D. P(EF)5. Do not roll a 5 on one of the rollsE. $P(E^C F^C)$



Monty Hall, 1000 envelope version



Start with 1000 envelopes (of which 1 is the prize).

1. You choose 1 envelope.

 $\frac{1}{1000}$ = P(envelope is prize)

 $\frac{999}{1000}$ = P(other 999 envelopes have prize)

2. I open 998 of remaining 999 (showing they are empty).

 $\frac{999}{1000} = P(998 \text{ empty envelopes had prize}) + P(\text{last other envelope has prize})$

= P(last other envelope has prize)

3. Should you switch?

P(you win without switching) = $\frac{1}{\text{original # envelopes}}$

P(you win with switching) = <u>original # envelopes - 1</u> original # envelopes

This class going forward

Last week Equally likely events For most of this course

Not equally likely events



 $P(E \cap F) \qquad P(E \cup F)$

(counting, combinatorics)

Probability of events



Probability of events



- P(student programs in Java) = 0.28
- P(student programs in Python) = 0.07
- P(student programs in Java and Python) = 0.05.

What is P(student does not program in (Java or Python))?

1. Define events & state goal

2. Identify <u>known</u> probabilities 3. Solve

- P(student programs in Java) = 0.28
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- P(student programs in Java and Python) = 0.05.

What is P(student does not program in (Java or Python))?

Define events
 & state goal

2. Identify <u>known</u> probabilities 3. Solve

Let: E: Student programs in Java F: Student programs in Python

- P(student programs in Java) = 0.28
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- P(student programs in Java and Python) = 0.05.

What is P(student does not program in (Java or Python))?

Define events
 & state goal

- 2. Identify <u>known</u> probabilities
- Let: E: Student programs in Java F: Student programs in Python

P(E) P(F) $P(E \cap F) = P(EF)$

3. Solve

- P(student programs in Java) = 0.28
- P(student programs in Python) = 0.07
- P(student programs in Java and Python) = 0.05.

What is P(student does not program in (Java or Python))?

P

Define events
 & state goal

- 2. Identify known
probabilities3. Solve
- Let: E: Student programs in Java F: Student programs in Python Want: $P((E \cup F)^C)$

$$\begin{pmatrix} (E \cup F)^C \end{pmatrix} = 1 - P(E \cup F) \\ = 1 - [P(E) + P(F) - P(E \cap F)] \\ = 1 - [0.28 + 0.07 - 0.05] \\ = 0.70$$

Review

P(E)

P(F)

 $P(E \cap F) = P(EF)$

Probability of events



Probability of events



Review

Definition of conditional probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

The Chain Rule:

P(EF) = P(E|F)P(F)

Generalized Chain Rule

$P(E_1 E_2 E_3 \dots E_n) = P(E_1) P(E_2 | E_1) P(E_3 | E_1 E_2) \dots P(E_n | E_1 E_2 \dots E_{n-1})$



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Quick check

You are going to a friend's Halloween party.

LetC = there is candyW = you wear a costumeM = there is musicE = no one wears your costume

An awesome party means that all of these events must occur.

What is P(awesome party) = P(CMWE)?

- A. P(C)P(M|C)P(W|CM)P(E|CMW)
- B. P(M)P(C|M)P(W|MC)P(E|MCW)
- C. P(W)P(E|W)P(CM|EW)
- D. A, B, and C
- E. None/other



Quick check

You are going to a friend's Halloween party.

LetC = there is candyE = noM = there is musicW = yo

E = no one wears your costume W = you wear a costume

An awesome party means that all of these events must occur.

What is P(awesome party) = P(CMEW)?

- A. P(C)P(M|C)P(E|CM)P(W|CME)
- B. P(M)P(C|M)P(E|MC)P(W|MCE)
- C. P(W)P(E|W)P(CM|EW)
- D. A, B, and C
- E. None/other

Chain Rule is a way of introducing "order" and "procedure" into probability.

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Probability of events



Probability of events



Independent trials

De Morgan's Laws

Conditional independence (if time)

CONGRESS, July 4, 1776. DECLARATI the REPRESENTATIVES UNITED STATES OF AMERICA: IN GENERAL CONGRESS ASSEMBLED.

If IE N in the Courie of human Events, it becomes necellary fut now People to defider the Policical Bands which have conof the Earth, the feperate and equal Station to which the Laws of Nature and of Nature's God untills than, a decent Relpolt up the Opinions of Markind requires that they fhould declare the Caufes which impel them the site Reperiors.

We hald their Truths to be fell-avident, that all Men are excited num .-- That to ferore these Rights, Governments are indicated among abdulate Rale into these Colonies Men, desiring their juft Paners from the Confert of the Governal, that hanceser any Form of Government becomes deflor flore of their Ends, shaling fundamentally the Forms of our Opvernments it is the Right of the Propie to also us to thebds it, and to infante new comments, laying its feandation on fach Principles, and organizing its Pewers in 1ach Form, in to them duall from most likely to cheft their Schere and Happingh. Producer, indeed, will distant that, Governments' and waging War agoled on larg effectivities droudd nut be changed tay light and standard Caulon, and apposingly all Experience hach threes, that Mankind are most dif- defroged the Lives of our People.

For quartering large Bodies of armed Troops among us : For prosofting them, by a muck Total, from Publishment for any Murpartied shops with another, and to allower among the Powers' devises they flexible contails on the labolitates of their featers For cutting off our Trafe with all Parts of the World : For impoling Taxes on as without our Confess For depuising us, in many Calix, of the Bandia of Trial by Jury : For transposting as beyond from in he wind for presended Offences

For abalifying the free fyllow of English Laws in a neighbouring Prevince erest, this they are endowed by their Greater with orthon uncleanable schulding therein an arbitrary Goutermont, and enlarging indicatedness, for lights, that among their are Life, Liberry, and the Partials of Happi- as secondar is at once an Europic and its infromment for increducing the laws

For taking away our Churten, sholiding our molt valuable Lows, and

For folgerding our own Legilarores, and declaring themfolges levelled with Power to legitlate for an in all Calls whattiener.

He has abdiened Government here, by declaring us out of his Prototion

He has ploudened nor Size, earaged our Casfle, haven our. Towns, and paid to failer, while furth we followhile, thus to sight themselves by their or the Time, washes ing large to size of faring Mesenaries to con-

Eur when a long, plan the Works of Darb, Deletainy and Tytaney, alwady began wh

On this day in 1958, Guinea declared independence from France.

Two events *E* and *F* are defined as <u>independent</u> if: P(EF) = P(E)P(F)

Otherwise E and F are called <u>dependent</u> events.

An equivalent definition:

$$P(E|F) = P(E)$$

Intuition through proof

Statement:

If E and F are independent, then P(E|F) = P(E).

Proof:

$$P(E|F) = \frac{P(EF)}{P(F)}$$
$$= \frac{P(E)P(F)}{P(F)}$$
$$= P(E)$$

Definition of conditional probability

Independence of E and F

Taking the bus to cancellation city



Knowing that *F* happened **does not change** our belief that *E* happened.

Independent P(EF) = P(E)P(F)

events E and F P(E|F) = P(E)

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Dice, our misunderstood friends

- Roll two 6-sided dice, yielding values D_1 and D_2 .
- Let event E: $D_1 = 1$ event F: $D_2 = 6$ event G: $D_1 + D_2 = 5$ $G = \{(1,4), (2,3), (3,2), (4,1)\}$
- **1.** Are *E* and *F* independent?

P(E) = 1/6P(F) = 1/6P(EF) = 1/36

independent

2. Are *E* and *G* independent?

P(E) = 1/6 P(G) = 4/36 = 1/9 $P(EG) = 1/36 \neq P(E)P(G)$

<u>dependent</u>



Independent events *E* and *F* P(EF) = P(E)P(F)P(E|F) = P(E)













Independent events E and F P(EF) = P(E)P(F)P(E|F) = P(E)

Statement:

If E and F are independent, then E and F^{C} are independent.

Proof:

 $P(EF^{C}) = P(E) - P(EF)$ = P(E) - P(E)P(F)= P(E)[1 - P(F)]= $P(E)P(F^{C})$

E and F^{C} are independent

Intersection

Independence of E and F

Factoring

Complement

Definition of independence



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Independent trials

De Morgan's Laws

Conditional independence (if time)

Generalizing independence

Three events *E*, *F*, and *G* are independent if:

n events E_1, E_2, \ldots, E_n are

$$P(EFG) = P(E)P(F)P(G), \text{ and}$$

$$P(EF) = P(E)P(F), \text{ and}$$

$$P(EG) = P(E)P(G), \text{ and}$$

$$P(FG) = P(F)P(G)$$
for $r = 1, ..., n$:
for every subset $E_1, E_2, ..., E_r$:
$$P(E_1, E_2, ..., E_r) = P(E_1)P(E_2) \cdots P(E_r)$$

Independent trials:

independent if:

Outcomes of n separate flips of a coin are all independent of one another. Each flip in this case is a <u>trial</u> of the experiment.

Dice, increasingly misunderstood (still our friends)

- Each roll of a 6-sided die is an independent trial.
- Two rolls: D_1 and D_2 .
- Let event E: $D_1 = 1$ event F: $D_2 = 6$ event *G*: $D_1 + D_2 = 7$ $G = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

v independent?

independent?

1. Are E and F **2.** Are E and G **3.** Are F and G **4.** Are E, F, G independent? independent?

P(E) = 1/6P(F) = 1/6P(EF) = 1/36



Dice, increasingly misunderstood (still our friends)

- Each roll of a 6-sided die is an independent trial.
- Two rolls: D_1 and D_2 .
- Let event *E*: $D_1 = 1$ event F: $D_2 = 6$ event *G*: $D_1 + D_2 = 7$ $G = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$
- **1.** Are *E* and *F* **2.** Are *E* and *G* independent?

independent?

3. Are F and G **4.** Are E, F, Gindependent? independent?

P(E) = 1/6P(F) = 1/6P(EF) = 1/36

- P(E) = 1/6P(F) = 1/6P(G) = 1/6P(G) = 1/6P(EG) = 1/36P(FG) = 1/36
- P(EFG) = 1/36 $\neq \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$



Pairwise independence is not sufficient to prove independence of >2 events!

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Network reliability

Consider the following parallel network:

- *n* independent routers, each with probability p_i of functioning (where $1 \le i \le n$)
- E = functional path from A to B exists.

What is P(E)?

$$P(E) = P(\ge 1 \text{ one router works})$$

= 1 - P(all routers fail)
= 1 - (1 - p₁)(1 - p₂) \dots (1 - p_n)
= 1 - $\prod_{i=1}^{n} (1 - p_i)$





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(biased) Coin Flips

Suppose we flip a coin n times.

- A coin comes up heads with probability p.
- Each coin flip is an independent trial.
- **1.** P(n heads on n coin flips)
- **2.** P(n tails on n coin flips)



(biased) Coin Flips

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- **1.** P(n heads on n coin flips)
- 2. P(n tails on n coin flips)
- **3.** P(first k heads, then n k tails)
- **4.** *P*(exactly *k* heads on *n* coin flips)





(biased) Coin Flips

Suppose we flip a coin n times.

- A coin comes up heads with probability p.
- Each coin flip is an independent trial.
- 1. P(n heads on n coin flips)
- 2. P(n tails on n coin flips)
- **3.** P(first k heads, then n k tails)
- **4.** *P*(exactly *k* heads on *n* coin flips)

$$p^{n}$$

$$(1-p)^{n}$$

$$p^{k}(1-p)^{n-k}$$

$$\binom{n}{k} p^k (1-p)^{n-k}$$

of mutually P(a particular outcome's
 exclusive k heads on n coin flips)
 outcomes



Break for jokes/ announcements

Announcements

Section

Starts: Late signups/changes: b Solutions:

today by end of day end of week

Problem Set 1

Gradescope: entry code M7B45K Assignment portal: available

Concept checks

Due date:every Tuesday 1:00pmYou can edit your response, so don'tbe afraid of submitting multiple times.

This quarterBeginning:fast-pacedLater:deep into conceptsCounting:the hardest part!

Independent trials

De Morgan's Laws

Conditional independence

Augustus De Morgan

Augustus De Morgan (1806–1871):

British mathematician who wrote the book Formal Logic (1847).





He looked remarkably similar to Jason Alexander (George from Seinfeld) (but that's not important right now)

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De Morgan's Laws

$$\underbrace{\mathsf{S}}_{\mathsf{E}} \underbrace{\mathsf{F}}_{\mathsf{E}} \underbrace{(E \cap F)^{C}}_{i=1} E_{i}^{C} \cup F^{C} = \bigcup_{i=1}^{n} E_{i}^{C} \underbrace{\mathsf{S}}_{i=1} \underbrace{\mathsf{F}}_{i}^{C} \underbrace{(E \cup F)^{C}}_{i=1} E_{i}^{C} \cap F^{C} = \bigcup_{i=1}^{n} E_{i}^{C} \underbrace{(\bigcup_{i=1}^{n} E_{i})^{C}}_{i=1} = \bigcap_{i=1}^{n} E_{i}^{C} \underbrace{\mathsf{F}}_{i}^{C} \underbrace{(\bigcup_{i=1}^{n} E_{i})^{C}}_{i=1} = \bigcap_{i=1}^{n} E_{i}^{C} \underbrace{\mathsf{F}}_{i}^{C} \underbrace{\mathsf{F}}_{i=1}^{C} \underbrace$$

In probability:

 $P(E_1 E_2 \cdots E_n) = 1 - P(E_1^C \cup E_2^C \cup \cdots \cup E_n^C)$ Gr $P(E_1 \cup E_2 \cup \cdots \cup E_n) = 1 - P(E_1^C E_2^C \cdots E_n^C)$ Gr

Great if E_i^C mutually exclusive!

Great if *E_i* independent!

$$\longrightarrow De Morgan's: AND \leftrightarrow OR$$

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Hash table fun

- *m* strings are hashed (unequally) into a hash table with *n* buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket *i*.
- **1.** E = bucket 1 has \geq 1 string hashed into it.



More hash table fun

- *m* strings are hashed (unequally) into a hash table with *n* buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket *i*.

1. E = bucket 1 has ≥ 1 string hashed into it.

2. E = at least 1 of buckets 1 to k has ≥ 1 string hashed into it.

What is P(E)?

WTF (not-real acronym for Want To Find)

Define F_i = bucket *i* has at least one string in it

$$P(E) = A. P(F_1F_2...F_k)$$

B. $1 - P(F_1^C)P(F_2^C)\cdots P(F_k^C)$
C. $P(F_1 \cup F_2 \cup \cdots \cup F_k)$
D. $P(F_1) + P(F_2) + \cdots + P(F_k)$
E. None/other



More hash table fun

- *m* strings are hashed (unequally) into a hash table with *n* buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket *i*.

1. E = bucket 1 has ≥ 1 string hashed into it.

2. E = at least 1 of buckets 1 to k has ≥ 1 string hashed into it.

 $\underline{\mathsf{B}} \quad 1 - P(F_1^{\mathcal{C}})P(F_2^{\mathcal{C}}) \cdots P(F_k^{\mathcal{C}})$

D. $P(F_1) + P(F_2) + \dots + P(F_k)$

C.) $P(F_1 \cup F_2 \cup \cdots \cup F_k)$

E. None/other

What is P(E)?

WTF (not-real acronym for Want To Find)

 $P(E) = A. P(F_1F_2...F_k)$

Define F_i = bucket *i* has at least one string in it

define well before

complementing!

Bucket events F_i are not independent



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More hash table fun

- *m* strings are hashed (unequally) into a hash table with *n* buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket *i*.
- 1. E = bucket 1 has ≥ 1 string hashed into it.
- 2. E = at least 1 of buckets 1 to k has ≥ 1 string hashed into it.

What is
$$P(E)$$
?
WTF: $P(E) = P(F_1 \cup F_2 \cup \dots \cup F_k)$
 $= 1 - P((F_1 \cup F_2 \cup \dots \cup F_k)^C)$
 $= 1 - P(F_1^C F_2^C \cdots F_k^C)$
 $P(E) = 1 - (1 - p_1 - p_2 \dots - p_k)^m$
Define F_i = bucket *i* has at least one string in it
 $P(E) = 1 - (1 - p_1 - p_2 \dots - p_k)^m$
 $P(E) = 1 - (1 - p_1 - p_2 \dots - p_k)^m$

The fun never stops with hash tables

- *m* strings are hashed (unequally) into a hash table with *n* buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket *i*.

1. E = bucket 1 has \geq 1 string hashed into it.2. E = at least 1 of buckets 1 to k has \geq 1 string hashed into it.3. E = each of of buckets 1 to k has \geq 1 string hashed into it.What is P(E)?



The fun never stops with hash tables

- *m* strings are hashed (unequally) into a hash table with *n* buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

1. E = bucket 1 has \ge 1 string hashed into it. 2. E = at least 1 of buckets 1 to k has \ge 1 string hashed into it. 3. E = each of of buckets 1 to k has \ge 1 string hashed into it. What is P(E)? Define F_i = bucket i has at

WTF:
$$P(E) = P(F_1F_2 \cdots F_k)$$

$$= 1 - P((F_1F_2 \cdots F_k)^C)$$
 Complement

$$= 1 - P(F_1^C \cup F_2^C \cup \cdots \cup F_k^C)$$
 De Morgan's Law

$$= 1 - P\begin{pmatrix}k \\ \cup \\ i=1 \end{pmatrix} = 1 - \sum_{r=1}^k (-1)^{(r+1)} \sum_{i_1 < \cdots < i_r} P(F_{i_1}^c F_{i_2}^c \dots F_{i_r}^c)$$

where $P(F_{i_1}^c F_{i_2}^c \dots F_{i_r}^c) = (1 - p_{i_1} - p_{i_2} \dots - p_{i_r})^m$

It is expected that this last example will need some review!

Probability of events



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Independent trials

De Morgan's Laws

