05: Independence

Lisa Yan

October 2, 2019

Two Dice

Review

- Roll two 6-sided dice, yielding values D_1 and D_2.
- Let event E: $D_1 = 5$
  event F: $D_2 = 5$

1. Roll a 5 on one of the rolls
2. Roll a 5 on both rolls
3. Neither roll is 5
4. Roll a 5 on roll 2
5. Do not roll a 5 on one of the rolls

A. $P(F)$

B. $P(E \cup F)$

C. $P(E^C \cup F^C)$

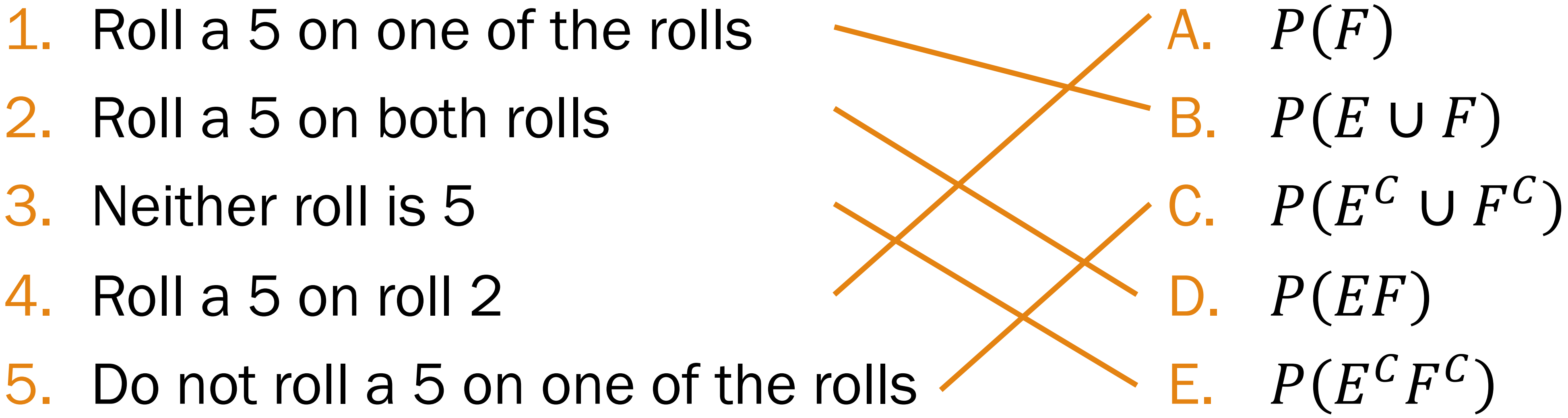
D. $P(EF)$

E. $P(E^C F^C)$

Two Dice

Review

- Roll two 6-sided dice, yielding values D_1 and D_2.
- Let event E: $D_1 = 5$
  event F: $D_2 = 5$



1. Roll a 5 on one of the rolls
2. Roll a 5 on both rolls
3. Neither roll is 5
4. Roll a 5 on roll 2
5. Do not roll a 5 on one of the rolls

A. $P(F)$

B. $P(E \cup F)$

C. $P(E^C \cup F^C)$

D. $P(EF)$

E. $P(E^C F^C)$

Monty Hall, 1000 envelope version

Review

Start with 1000 envelopes
(of which 1 is the prize).

1. You choose 1 envelope.

$$\frac{1}{1000} = P(\text{envelope is prize})$$

$$\frac{999}{1000} = P(\text{other 999 envelopes have prize})$$

2. I open 998 of remaining 999 (showing they are empty).

$$\frac{999}{1000} = P(\text{998 empty envelopes had prize}) + P(\text{last other envelope has prize}) = P(\text{last other envelope has prize})$$

3. Should you switch?

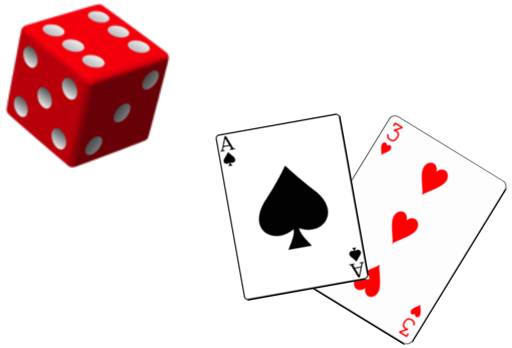
$$P(\text{you win without switching}) = \frac{1}{\text{original \# envelopes}}$$

$$P(\text{you win with switching}) = \frac{\text{original \# envelopes - 1}}{\text{original \# envelopes}}$$

This class going forward

Last week

Equally likely events



$P(E \cap F)$ $P(E \cup F)$

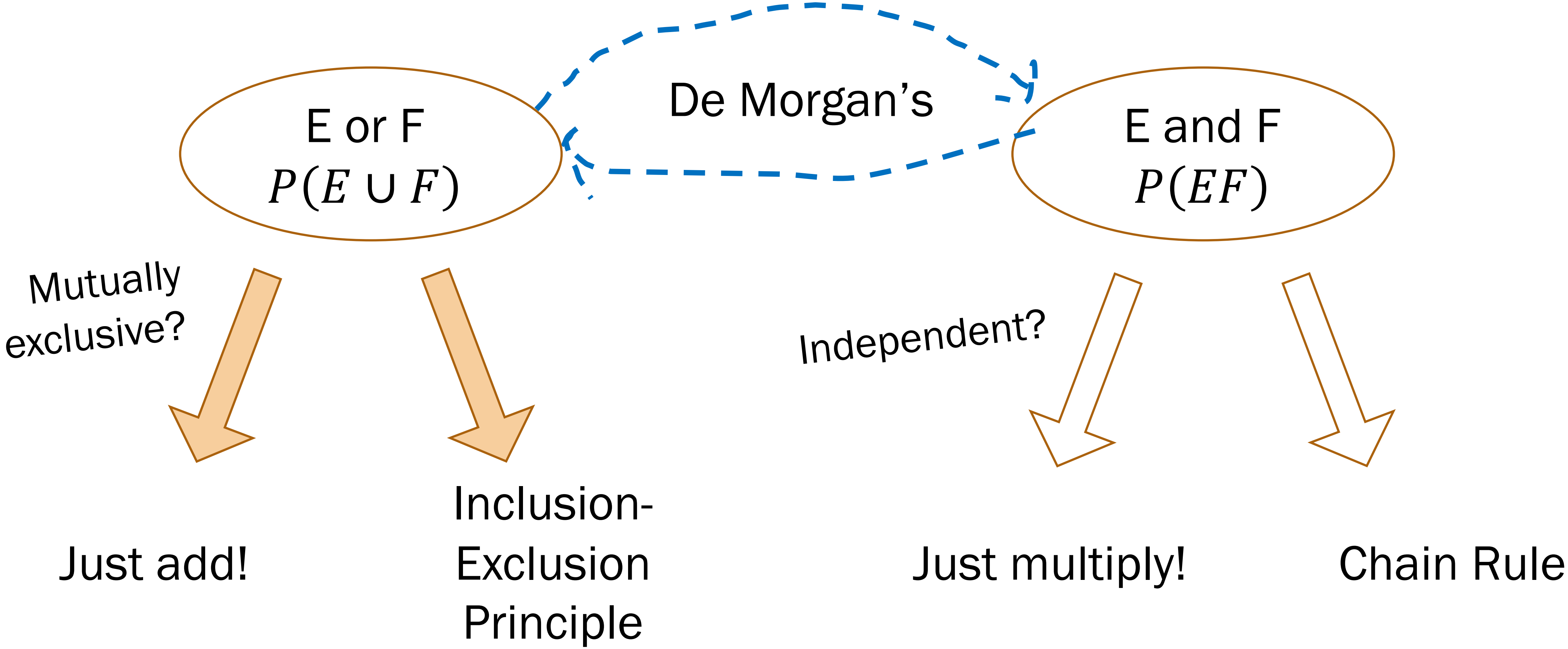
(counting, combinatorics)

For most of this course

Not equally likely events

$$P\left(\begin{array}{l}E \text{ given some evidence} \\ \quad\text{has been observed}\end{array}\right)$$

Probability of events



E or F

$P(E \cup F)$

De Morgan's

E and F

$P(EF)$

Mutually exclusive?

Just add!

Inclusion-Exclusion Principle

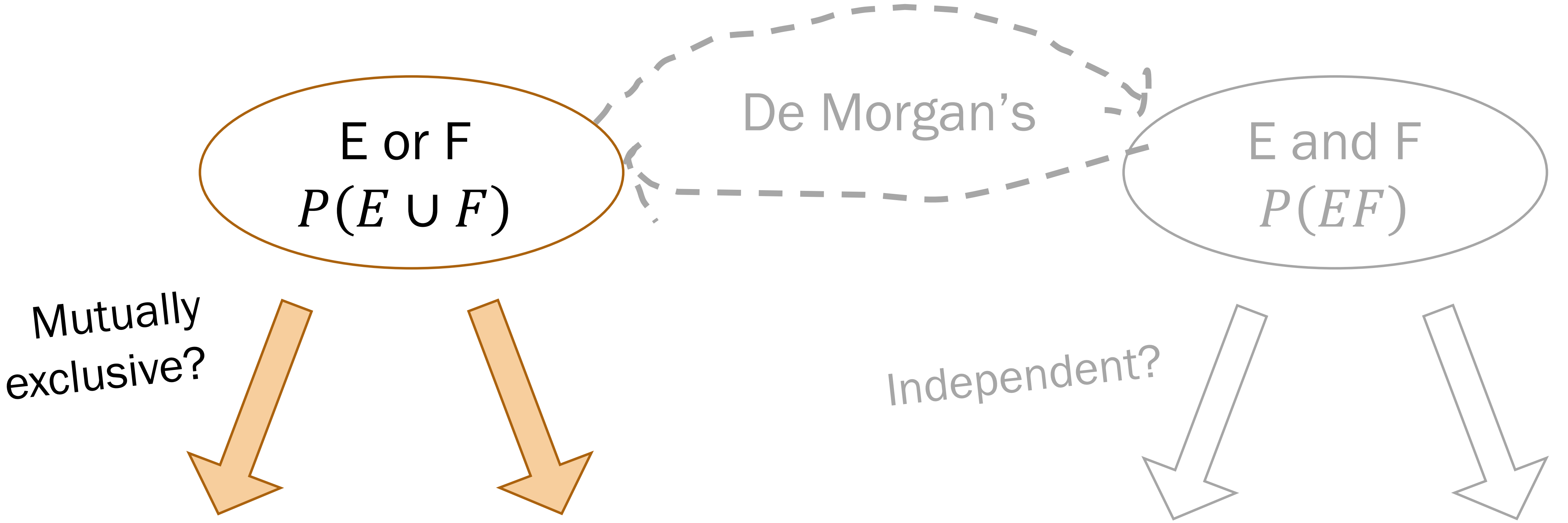
Independent?

Just multiply!

Chain Rule

Probability of events

Review



E or F
$P(E \cup F)$

De Morgan's

E and F
$P(EF)$

Mutually exclusive?

Independent?

Just add!

$P(E) + P(F)$

Inclusion-Exclusion Principle

$P(E) + P(F) - P(E \cap F)$

Just multiply!

Chain Rule

Selecting Programmers

Review

- P(student programs in Java) = 0.28
- P(student programs in Python) = 0.07
- P(student programs in Java and Python) = 0.05.

What is P(student does not program in (Java or Python))?

1. Define events & state goal
2. Identify known probabilities
3. Solve

Selecting Programmers

- P(student programs in Java) = 0.28
- P(student programs in Python) = 0.07
- P(student programs in Java and Python) = 0.05.

What is P(student does not program in (Java or Python))?

1. Define events & state goal
2. Identify known probabilities
3. Solve

Let: E: Student programs in Java

F: Student programs in Python

Want: $P\left((E \cup F)^C\right)$

Selecting Programmers

Review

- P(student programs in Java) = 0.28 $P(E)$
- P(student programs in Python) = 0.07 $P(F)$
- P(student programs in Java and Python) = 0.05. $P(E \cap F) = P(EF)$

What is P(student does not program in (Java or Python))?

1. Define events & state goal
2. Identify known probabilities
3. Solve

Let: E: Student programs in Java

F: Student programs in Python

Want: $P\left((E \cup F)^C\right)$

Selecting Programmers

Review

- P(student programs in Java) = 0.28
- P(student programs in Python) = 0.07
- P(student programs in Java and Python) = 0.05.

$P(E)$

$P(F)$

$P(E \cap F) = P(EF)$

What is P(student does not program in (Java or Python))?

1. Define events & state goal
2. Identify known probabilities
3. Solve

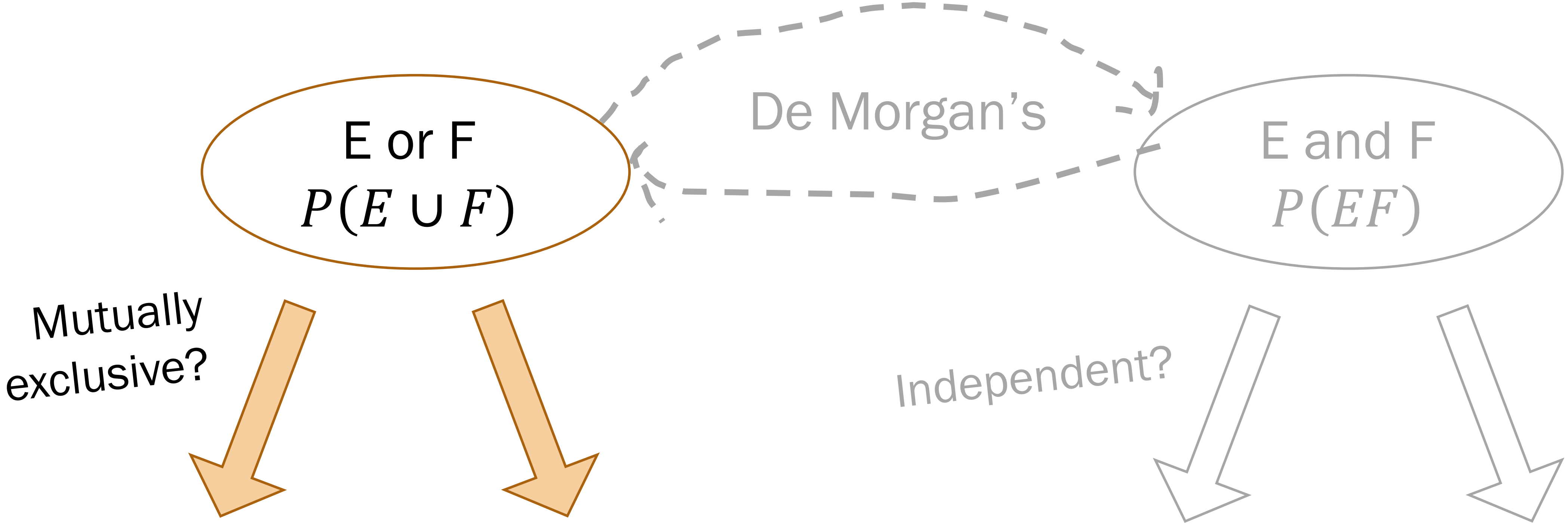
Let: E: Student programs in Java
F: Student programs in Python

Want: $P\left((E \cup F)^C\right)$

$$
\begin{aligned}
P\left((E \cup F)^C\right) &= 1 - P(E \cup F) \\
&= 1 - [P(E) + P(F) - P(E \cap F)] \\
&= 1 - [0.28 + 0.07 - 0.05] \\
&= 0.70
\end{aligned}
$$

Probability of events

Review



E or F

$P(E \cup F)$

De Morgan's

E and F

$P(EF)$

Mutually exclusive?

Independent?

Just add!

Inclusion-Exclusion Principle

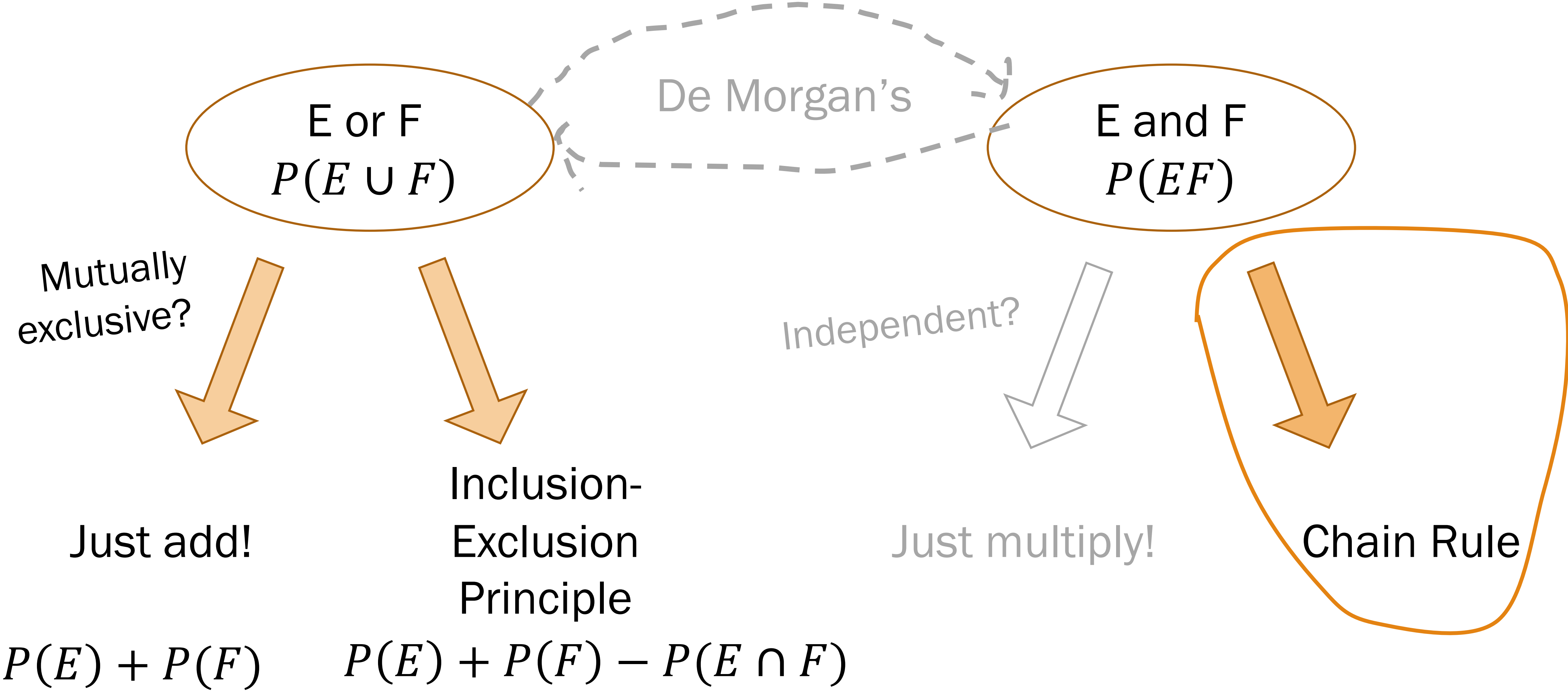
Just multiply!

Chain Rule

$P(E) + P(F)$

$P(E) + P(F) - P(E \cap F)$

Probability of events



E or F
$P(E \cup F)$

De Morgan's

E and F
$P(EF)$

Mutually exclusive?

Independent?

Just add!

$P(E) + P(F)$

Inclusion-Exclusion Principle

$P(E) + P(F) - P(E \cap F)$

Just multiply!

Chain Rule

Chain Rule

Definition of conditional probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

The Chain Rule:

$$P(EF) = P(E|F)P(F)$$

Generalized Chain Rule

$$P(E_1 E_2 E_3 \dots E_n) \\ = P(E_1)P(E_2|E_1)P(E_3|E_1 E_2) \dots P(E_n|E_1 E_2 \dots E_{n-1})$$

Quick check

$P(E_1 E_2 E_3 \dots E_n) =$
$P(E_1) P(E_2|E_1) \dots P(E_n|E_1 E_2 \dots E_{n-1})$ Chain Rule

You are going to a friend's Halloween party.

Let C = there is candy
M = there is music
W = you wear a costume
E = no one wears your costume

An awesome party means that all of these events must occur.

What is $P(\text{awesome party}) = P(CMWE)$?

A. $P(C)P(M|C)P(W|CM)P(E|CMW)$
B. $P(M)P(C|M)P(W|MC)P(E|MCW)$
C. $P(W)P(E|W)P(CM|EW)$
D. A, B, and C
E. None/other

Quick check

$$P(E_1 E_2 E_3 \ldots E_n) = P(E_1) P(E_2|E_1) \ldots P(E_n|E_1 E_2 \ldots E_{n-1})$$ Chain Rule

You are going to a friend's Halloween party.

Let C = there is candy
M = there is music
E = no one wears your costume
W = you wear a costume

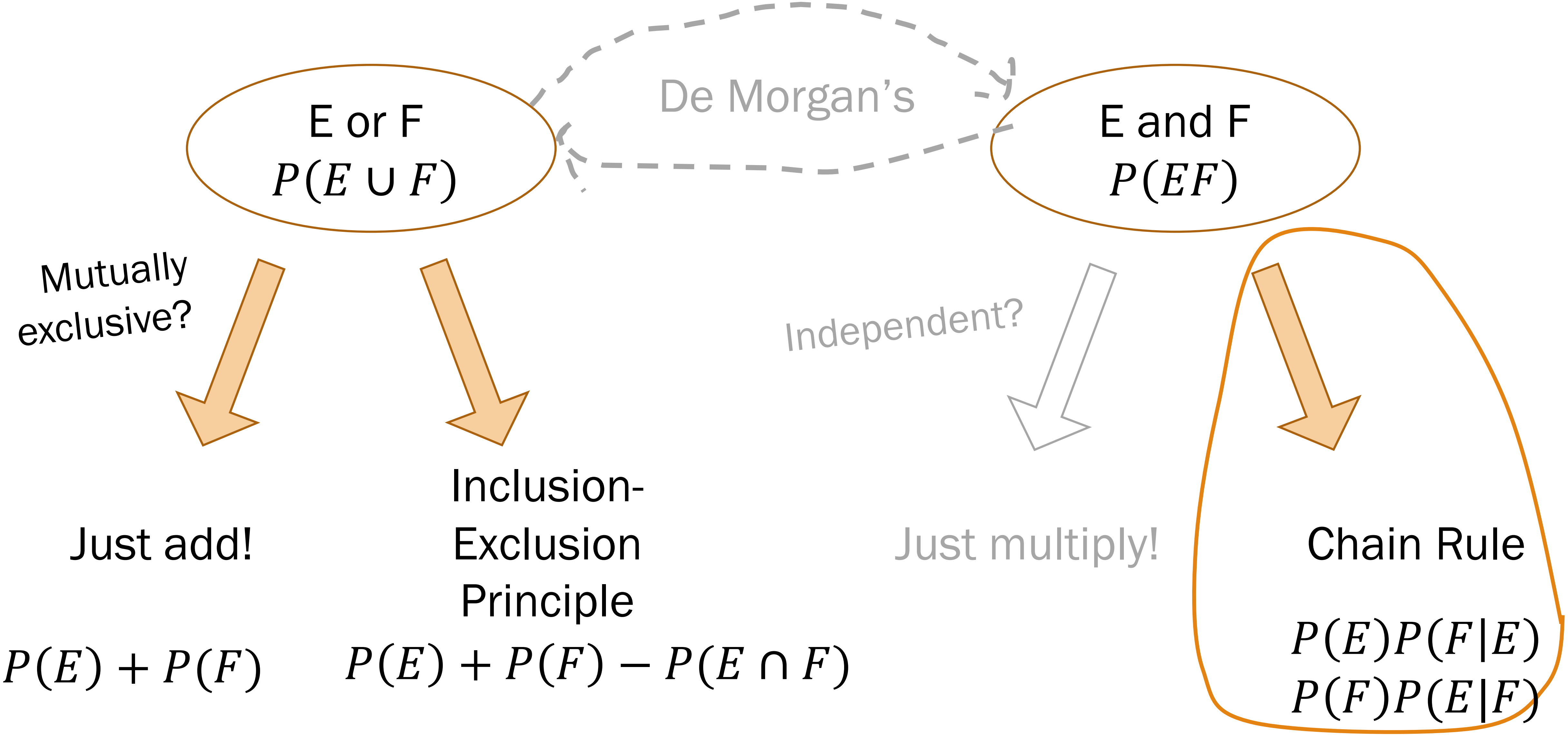
An awesome party means that all of these events must occur.

What is $P(\text{awesome party}) = P(CMEW)$?

A. $P(C)P(M|C)P(E|CM)P(W|CME)$
B. $P(M)P(C|M)P(E|MC)P(W|MCE)$
C. $P(W)P(E|W)P(CM|EW)$
[✓ D.] A, B, and C
E. None/other

👉 Chain Rule is a way of introducing "order" and "procedure" into probability. 🤔

Probability of events



E or F
$P(E \cup F)$

De Morgan's

E and F
$P(EF)$

Mutually exclusive?

Just add!

$P(E) + P(F)$

Inclusion-Exclusion Principle

$P(E) + P(F) - P(E \cap F)$

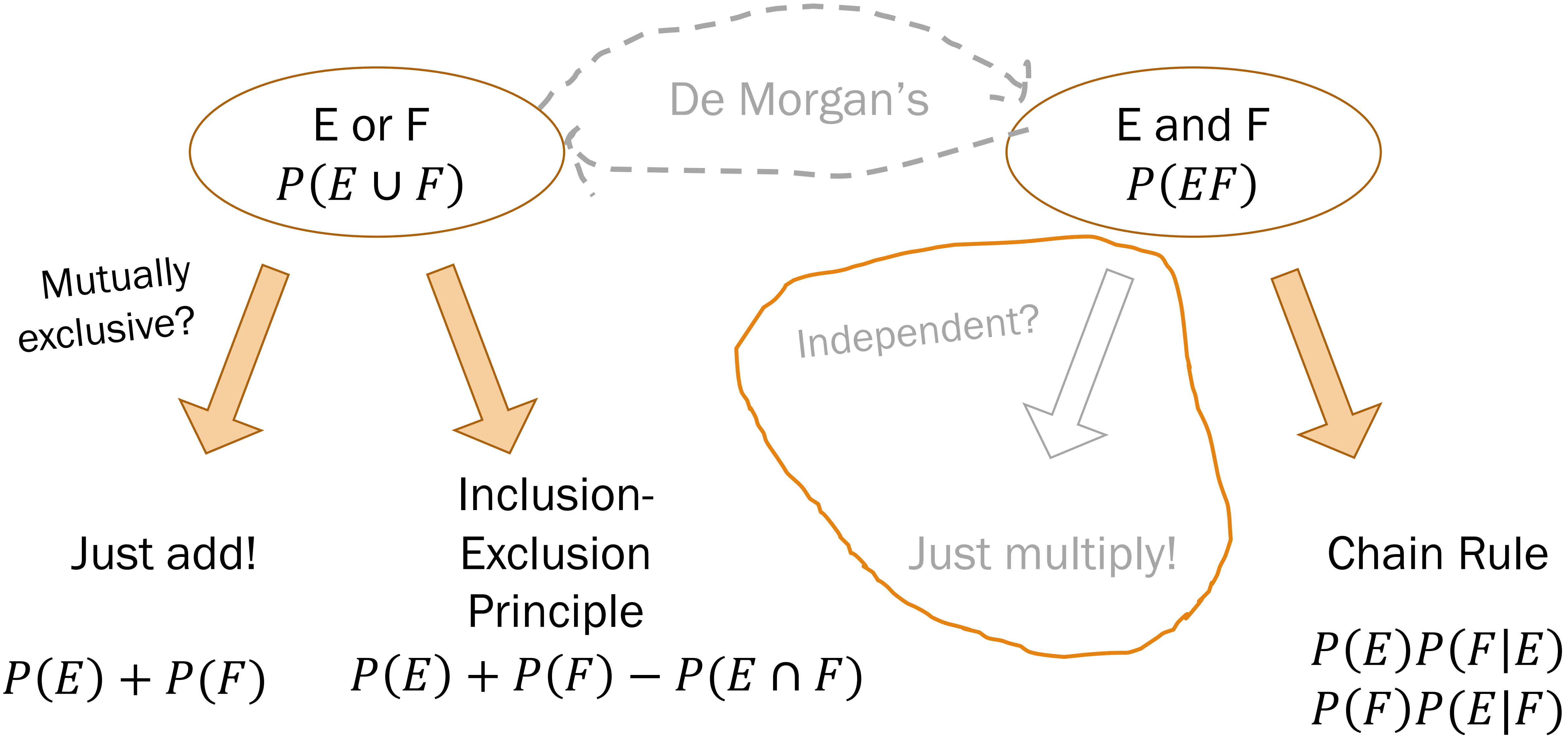
Independent?

Just multiply!

Chain Rule

$P(E)P(F|E)$
$P(F)P(E|F)$

Probability of events



E or F

$P(E \cup F)$

De Morgan's

E and F

$P(EF)$

Mutually exclusive?

Just add!

$P(E) + P(F)$

Inclusion-Exclusion Principle

$P(E) + P(F) - P(E \cap F)$

Independent?

Just multiply!

Chain Rule

$P(E)P(F|E)$

$P(F)P(E|F)$

Today's plan

- ➡ Independence
- Independent trials
- De Morgan's Laws
- Conditional independence (if time)

In CONGRESS, July 4, 1776.

A DECLARATION

By the REPRESENTATIVES of the

UNITED STATES OF AMERICA,

In GENERAL CONGRESS ASSEMBLED.

On this day in 1958, Guinea declared independence from France.

Independence

Two events E and F are defined as independent if:

$$P(EF) = P(E)P(F)$$

Otherwise E and F are called dependent events.

An equivalent definition:

$$P(E|F) = P(E)$$

Intuition through proof

Independent events E and F ⟺ $P(EF) = P(E)P(F)$

$P(E|F) = P(E)$

Statement:

If E and F are independent, then $P(E|F) = P(E)$.

Proof:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Definition of conditional probability

$$= \frac{P(E)P(F)}{P(F)}$$

Independence of E and F

$$= P(E)$$

Taking the bus to cancellation city

👉 Knowing that F happened **does not change** our belief that E happened.

Dice, our misunderstood friends

Independent events E and F ⟺ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

- Roll two 6-sided dice, yielding values D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 5$ $G = \{(1,4),(2,3),(3,2),(4,1)\}$

1. Are E and F independent?

$$P(E) = 1/6$$
$$P(F) = 1/6$$
$$P(EF) = 1/36$$

independent

2. Are E and G independent?

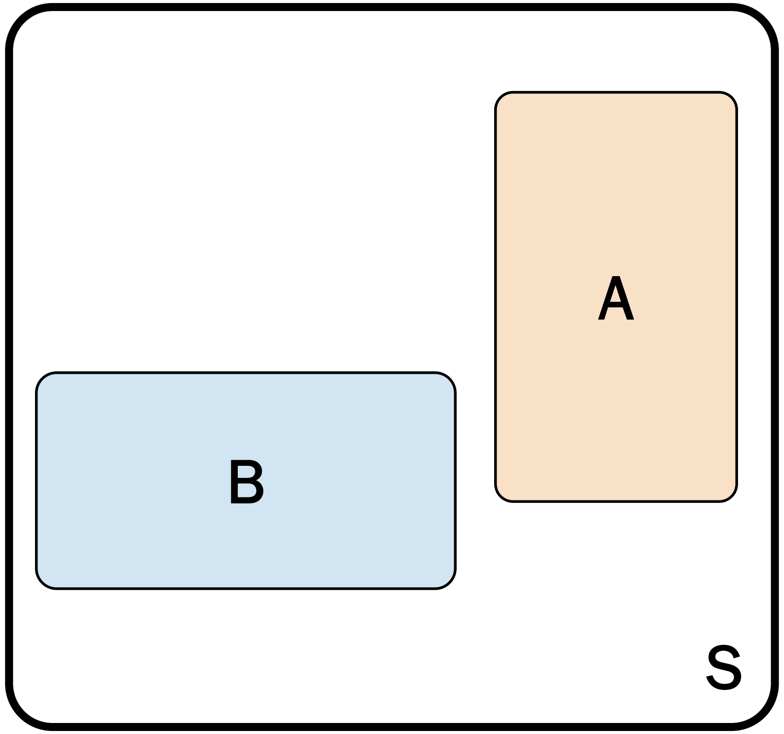
$$P(E) = 1/6$$
$$P(G) = 4/36 = 1/9$$
$$P(EG) = 1/36 \neq P(E)P(G)$$

dependent

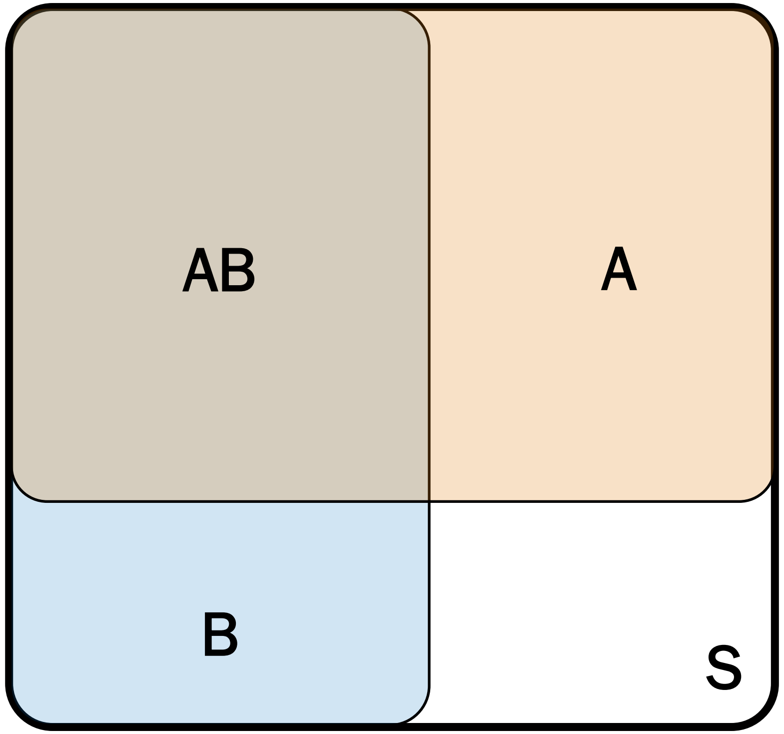
Independence?

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

A.



B.

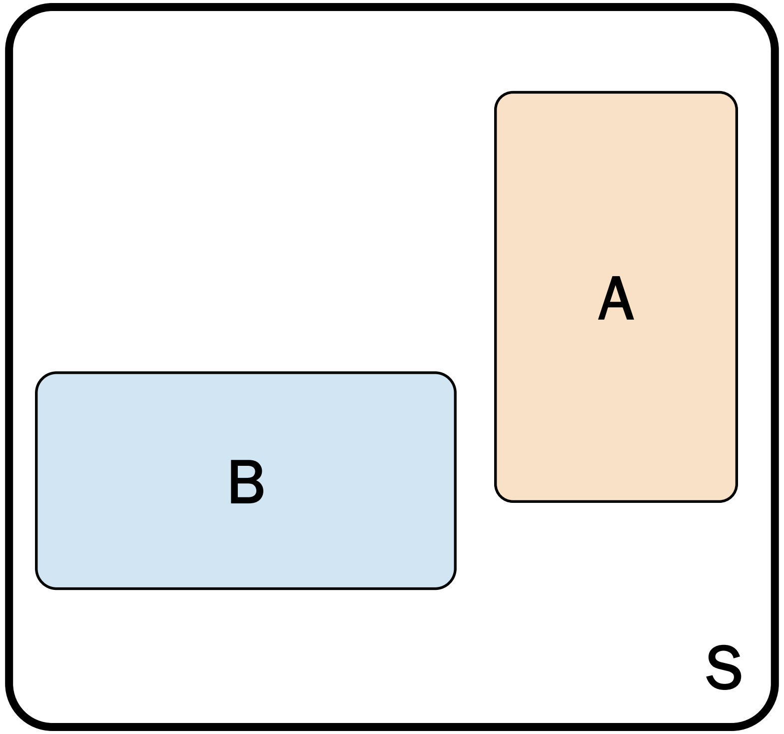


Independence?

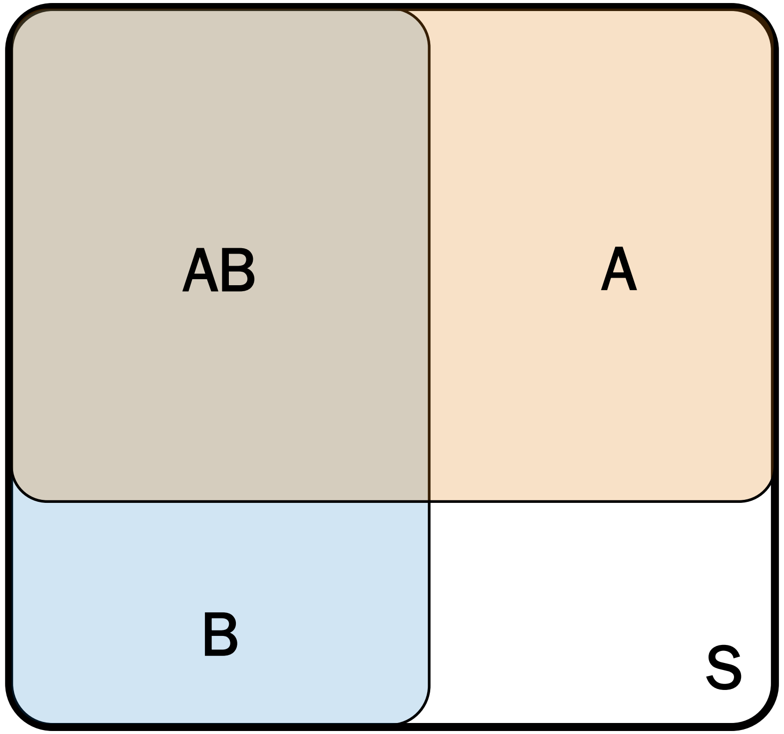
Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$

$P(E|F) = P(E)$

A.



B.



(Def. 1, assuming equally likely outcomes)

$$P(AB) = P(A)P(B)$$

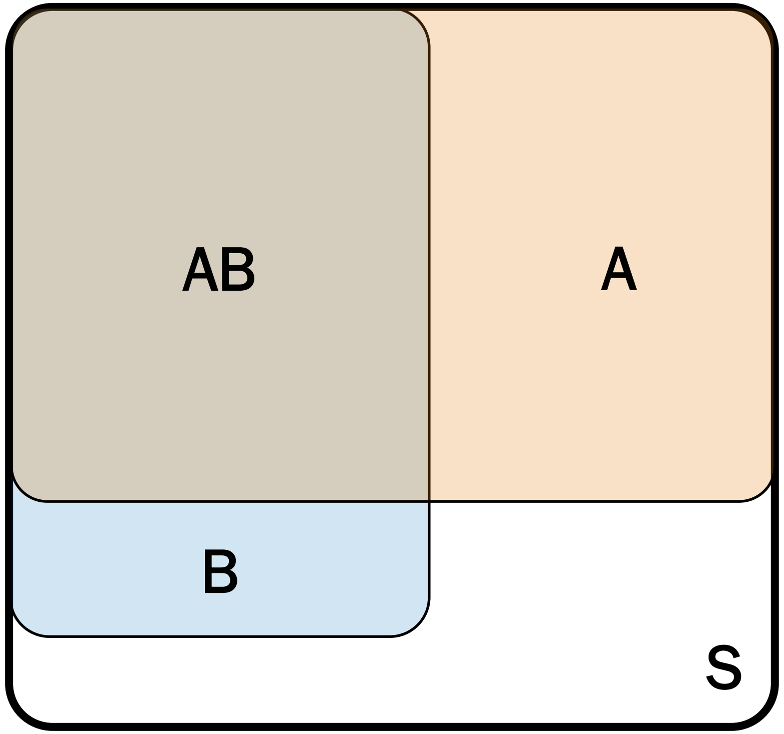
$$\cancelto{0}{\frac{|AB|}{|S|}} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

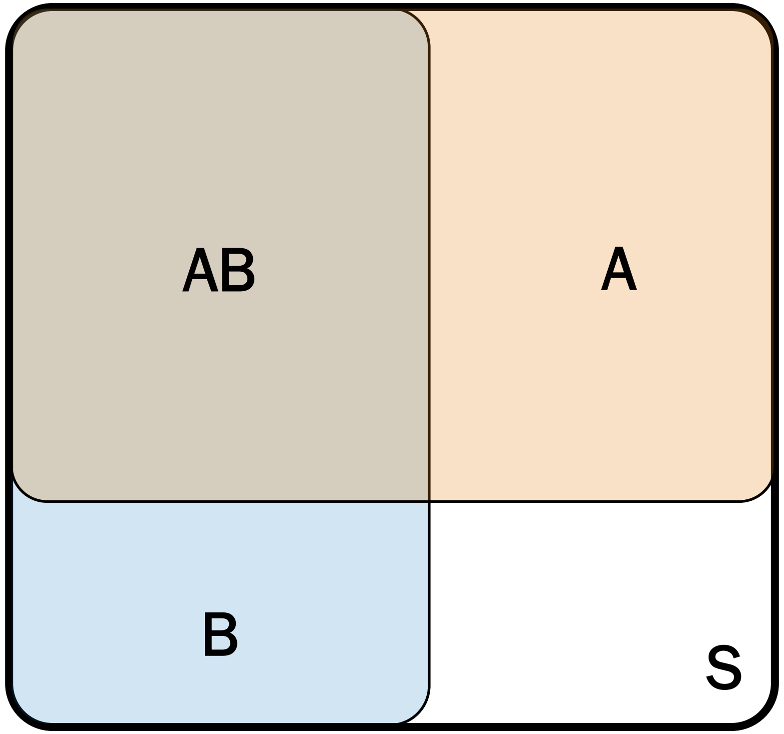
Independence?

Independent events E and F ⟺ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

A.



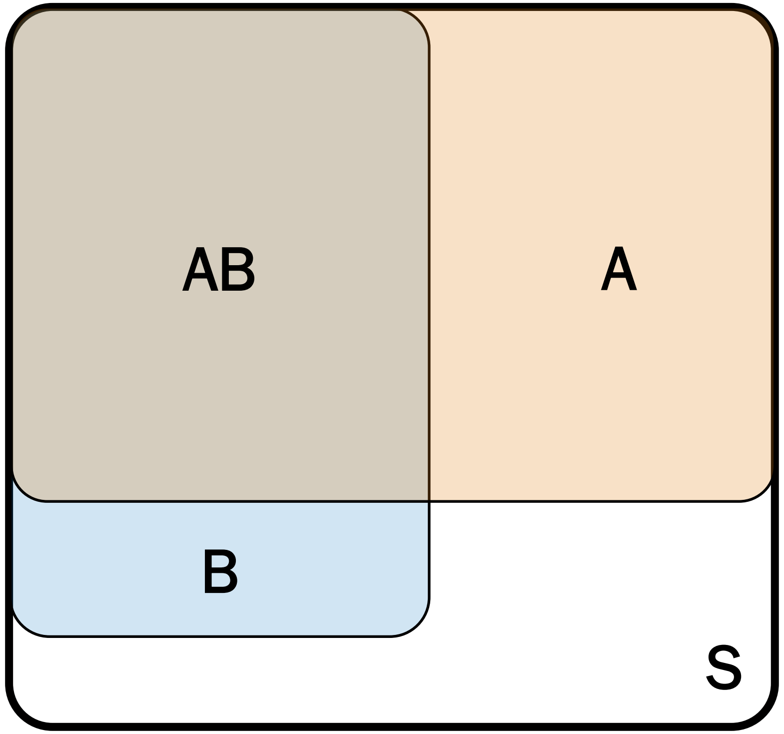
B.



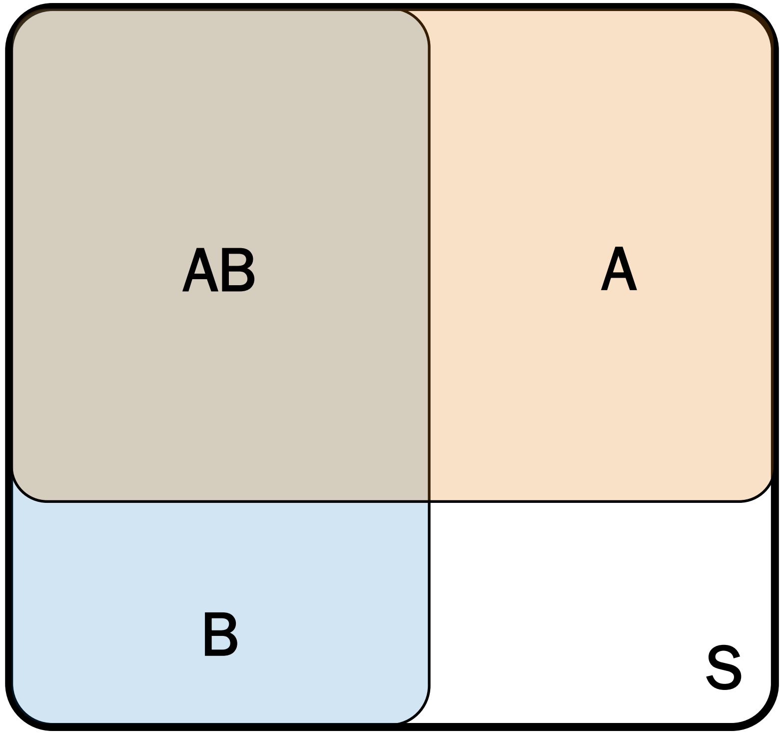
Independence?

Independent events E and F ⟺ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

A.



B.



(Def. 2, assuming equally likely outcomes)

$$P(A|B) = P(A)$$

$$\frac{|AB|}{|B|} \neq \frac{|A|}{|S|}$$

$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

🤔

Independence of complements

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

Statement:

If E and F are independent, then E and F^C are independent.

Proof:

$P(EF^C) = P(E) - P(EF)$	Intersection
$= P(E) - P(E)P(F)$	Independence of E and F
$= P(E)[1 - P(F)]$	Factoring
$= P(E)P(F^C)$	Complement
E and F^C are independent	Definition of independence

👉 Knowing that F **didn't happen** does not change our belief that E happened.

Today's plan

Independence

➡ Independent trials

De Morgan's Laws

Conditional independence (if time)

Generalizing independence

Three events E, F, and G are independent if:

$$\begin{cases} P(EFG) = P(E)P(F)P(G), \text{ and} \\ P(EF) = P(E)P(F), \text{ and} \\ P(EG) = P(E)P(G), \text{ and} \\ P(FG) = P(F)P(G) \end{cases}$$

n events $E_1, E_2, \dots, E_n$ are independent if:

```
for $r = 1, \dots, n$:
    for every subset $E_1, E_2, \dots, E_r$:
        $P(E_1, E_2, \dots, E_r) = P(E_1)P(E_2) \cdots P(E_r)$
```

Independent trials:

Outcomes of n separate flips of a coin are all independent of one another.

Each flip in this case is a trial of the experiment.

Dice, increasingly misunderstood (still our friends)

- Each roll of a 6-sided die is an independent trial.
- Two rolls: D_1 and D_2.
- Let event E: $D_1 = 1$

  event F: $D_2 = 6$

  event G: $D_1 + D_2 = 7$ $\quad G = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

1. Are E and F independent? ✅
2. Are E and G independent?
3. Are F and G independent?
4. Are E, F, G independent?

$$P(E) = 1/6$$
$$P(F) = 1/6$$
$$P(EF) = 1/36$$

🤔

Dice, increasingly misunderstood (still our friends)

- Each roll of a 6-sided die is an independent trial.
- Two rolls: D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 7$

$$G = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

1. Are E and F independent?

$$P(E) = 1/6$$
$$P(F) = 1/6$$
$$P(EF) = 1/36$$

2. Are E and G independent?

$$P(E) = 1/6$$
$$P(G) = 1/6$$
$$P(EG) = 1/36$$

3. Are F and G independent?

$$P(F) = 1/6$$
$$P(G) = 1/6$$
$$P(FG) = 1/36$$

4. Are E, F, G independent?

$$P(EFG) = 1/36$$
$$\neq \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$$

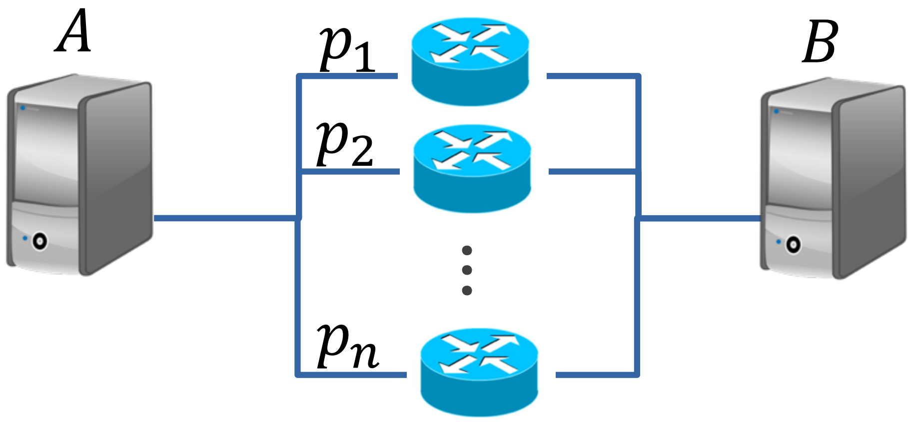
👉 Pairwise independence is not sufficient to prove independence of >2 events!

Network reliability

Consider the following parallel network:

- n independent routers, each with probability p_i of functioning (where $1 \leq i \leq n$)
- E = functional path from A to B exists.

What is $P(E)$?



$$\begin{aligned} P(E) &= P(\geq 1 \text{ one router works}) \\ &= 1 - P(\text{all routers fail}) \\ &= 1 - (1 - p_1)(1 - p_2) \cdots (1 - p_n) \\ &= 1 - \prod_{i=1}^{n} (1 - p_i) \end{aligned}$$

👉 ≥ 1 : take complement

(biased) Coin Flips

Suppose we flip a coin n times.

- A coin comes up heads with probability p.
- Each coin flip is an independent trial.

1. $P(n$ heads on n coin flips$)$ p^n
2. $P(n$ tails on n coin flips$)$ $(1-p)^n$

(biased) Coin Flips

Suppose we flip a coin n times.

- A coin comes up heads with probability p.
- Each coin flip is an independent trial.

1. $P(n$ heads on n coin flips) p^n
2. $P(n$ tails on n coin flips) $(1-p)^n$
3. $P($first k heads, then $n-k$ tails)

4. $P($exactly k heads on n coin flips)

🤔

(biased) Coin Flips

Suppose we flip a coin n times.

- A coin comes up heads with probability p.
- Each coin flip is an independent trial.

1. $P(n \text{ heads on } n \text{ coin flips})$ $\qquad p^n$
2. $P(n \text{ tails on } n \text{ coin flips})$ $\qquad (1-p)^n$
3. $P(\text{first } k \text{ heads, then } n-k \text{ tails})$ $\qquad p^k(1-p)^{n-k}$

4. $P(\text{exactly } k \text{ heads on } n \text{ coin flips})$ $\qquad \binom{n}{k} p^k(1-p)^{n-k}$

# of mutually exclusive outcomes

P(a particular outcome's k heads on n coin flips)

Break for jokes/ announcements

Announcements

Section

Starts: today
Late signups/changes: by end of day
Solutions: end of week

Problem Set 1

Gradescope: entry code M7B45K
Assignment portal: available

Concept checks

Due date: every Tuesday 1:00pm

You can edit your response, so don't be afraid of submitting multiple times.

This quarter

Beginning: fast-paced
Later: deep into concepts
Counting: the hardest part!

Today's plan

Independence

Independent trials

➡ De Morgan's Laws

Conditional independence

Augustus De Morgan

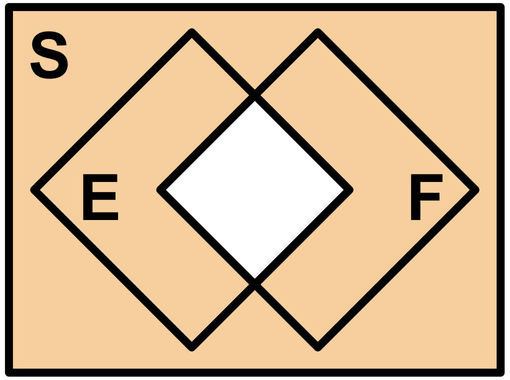
Augustus De Morgan (1806–1871):

British mathematician who wrote the book *Formal Logic* (1847).

He looked remarkably similar to Jason Alexander (George from Seinfeld)

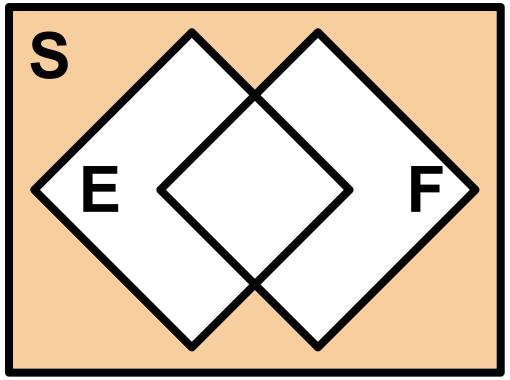
(but that's not important right now)

De Morgan's Laws



$$(E \cap F)^C = E^C \cup F^C$$

$$\left(\bigcap_{i=1}^{n} E_i\right)^C = \bigcup_{i=1}^{n} E_i^C$$



$$(E \cup F)^C = E^C \cap F^C$$

$$\left(\bigcup_{i=1}^{n} E_i\right)^C = \bigcap_{i=1}^{n} E_i^C$$

In probability:

$$P(E_1 E_2 \cdots E_n) = 1 - P\left(E_1^C \cup E_2^c \cup \cdots \cup E_n^c\right)$$

Great if E_i^C mutually exclusive!

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = 1 - P(E_1^c E_2^c \cdots E_n^c)$$

Great if E_i independent!

👉 De Morgan's: AND ⟷ OR

Hash table fun

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

1. E = bucket 1 has ≥ 1 string hashed into it.

What is $P(E)$?

Define S_i = string i is hashed into bucket 1
S_i^C = string i is not hashed into bucket 1

$$P(S_i) = p_1$$
$$P\left(S_i^C\right) = 1 - p_1$$

$$
\begin{aligned}
P(E) &= P(S_1 \cup S_2 \cup \cdots \cup S_m) \\
&= 1 - P\left((S_1 \cup S_2 \cup \cdots \cup S_m)^C\right) && \text{Complement} \\
&= 1 - P\left(S_1^C S_2^C \cdots S_m^C\right) && \text{De Morgan's Law} \\
&= 1 - P\left(S_1^C\right)P\left(S_2^C\right) \cdots P(S_m^C) = 1 - \left(P\left(S_1^C\right)\right)^m && S_i \text{ independent trials} \\
&= 1 - (1 - p_1)^m
\end{aligned}
$$

More hash table fun

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

1. E = bucket 1 has $\geq$ 1 string hashed into it.
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it.

What is $P(E)$?

Define F_i = bucket i has at least one string in it

WTF (not-real acronym for Want To Find)

$P(E) =$

A. $P(F_1 F_2 \ldots F_k)$
B. $1 - P(F_1^C)P(F_2^C)\cdots P(F_k^c)$
C. $P(F_1 \cup F_2 \cup \cdots \cup F_k)$
D. $P(F_1) + P(F_2) + \cdots + P(F_k)$
E. None/other

More hash table fun

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

1. E = bucket 1 has $\geq$ 1 string hashed into it.
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it.

What is $P(E)$?

WTF (not-real acronym for Want To Find)

Define F_i = bucket i has at least one string in it

$P(E) =$

A. $P(F_1 F_2 \dots F_k)$

B. $1 - P(F_1^C)P(F_2^C)\cdots P(F_k^C)$

[✓ C.] $P(F_1 \cup F_2 \cup \cdots \cup F_k)$

D. $P(F_1) + P(F_2) + \cdots + P(F_k)$

E. None/other

Bucket events F_i are not independent

🤔

👉 define well before complementing!

More hash table fun

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

1. E = bucket 1 has $\geq$ 1 string hashed into it.
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it.

What is $P(E)$?

Define F_i = bucket i has at least one string in it

WTF: $P(E) = P(F_1 \cup F_2 \cup \cdots \cup F_k)$

$$= 1 - P\left((F_1 \cup F_2 \cup \cdots \cup F_k)^C\right)$$

$$= 1 - P\left(F_1^C F_2^C \cdots F_k^C\right)$$

$$= P(\text{no strings hashed to buckets 1 to } k)$$

$$= \left(P(\text{string hashed outside bkts 1 to } k)\right)^m$$

$$= (1 - p_1 - p_2 \ldots - p_k)^m$$

$$P(E) = 1 - (1 - p_1 - p_2 \ldots - p_k)^m$$

The fun never stops with hash tables

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

1. E = bucket 1 has $\geq$ 1 string hashed into it.
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it.
3. E = each of of buckets 1 to k has $\geq$ 1 string hashed into it.

What is $P(E)$?

The fun never stops with hash tables

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

1. E = bucket 1 has ≥ 1 string hashed into it.
2. E = at least 1 of buckets 1 to k has ≥ 1 string hashed into it.
3. E = each of of buckets 1 to k has ≥ 1 string hashed into it.

What is $P(E)$?

Define F_i = bucket i has at least one string in it

WTF: $P(E) = P(F_1 F_2 \cdots F_k)$

$= 1 - P\left((F_1 F_2 \cdots F_k)^C\right)$ Complement

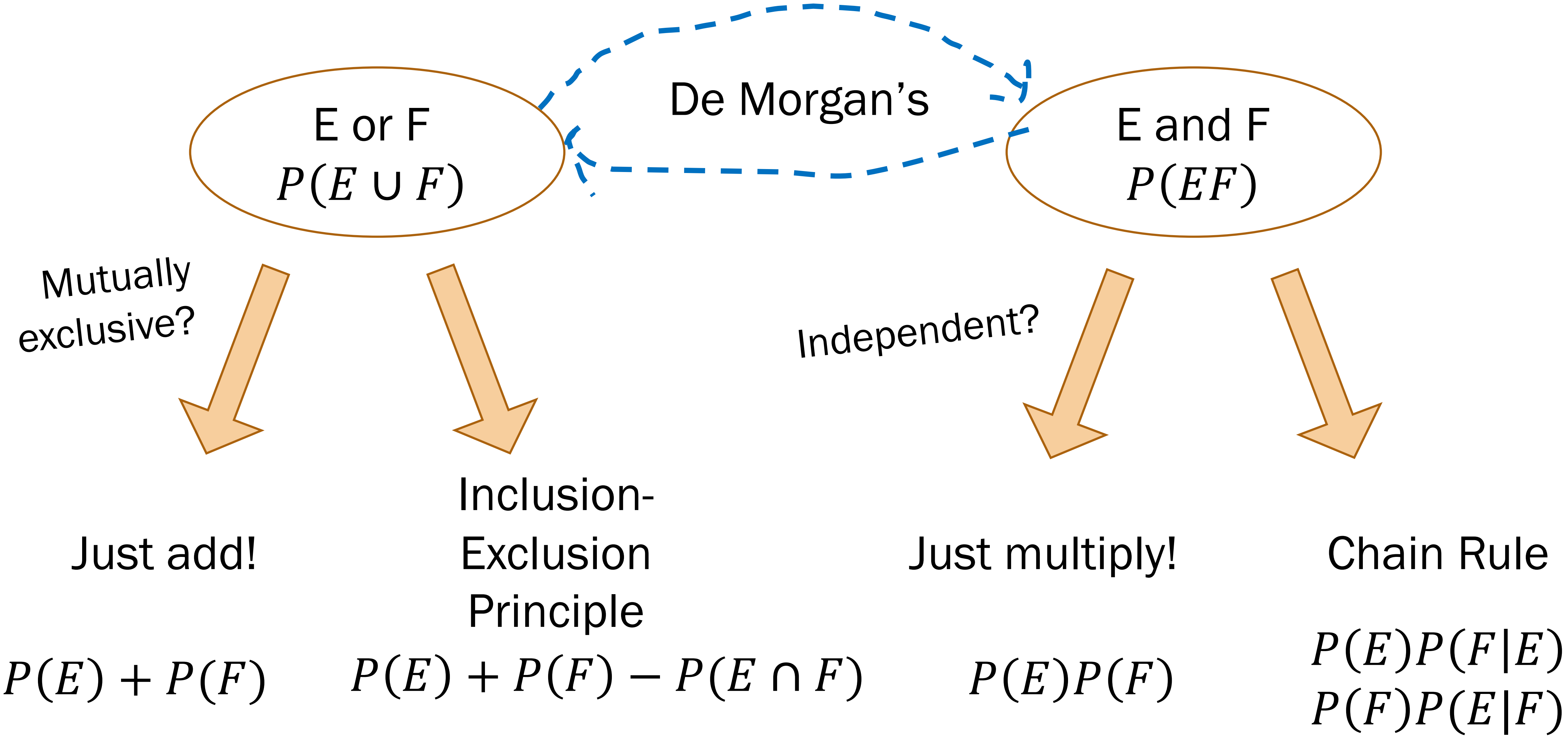
$= 1 - P\left(F_1^C \cup F_2^C \cup \cdots \cup F_k^C\right)$ De Morgan's Law

$$= 1 - P\left(\bigcup_{i=1}^{k} F_i^c\right) = 1 - \sum_{r=1}^{k}(-1)^{(r+1)} \sum_{i_1<\cdots<i_r} P\left(F_{i_1}^c F_{i_2}^c \dots F_{i_r}^c\right)$$

where $P\left(F_{i_1}^c F_{i_2}^c \dots F_{i_r}^c\right) = (1 - p_{i_1} - p_{i_2} \dots - p_{i_r})^m$

It is expected that this last example will need some review!

Probability of events



E or F
$P(E \cup F)$

De Morgan's

E and F
$P(EF)$

Mutually exclusive?

Just add!

$P(E) + P(F)$

Inclusion-Exclusion Principle

$P(E) + P(F) - P(E \cap F)$

Independent?

Just multiply!

$P(E)P(F)$

Chain Rule

$P(E)P(F|E)$
$P(F)P(E|F)$

Today’s plan

Independence

Independent trials

De Morgan’s Laws

➡ Conditional independence (if time)