06: Random Variables

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October 4, 2019

The fun never stops with hash tables

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

1. E = bucket 1 has $\geq$ 1 string hashed into it.
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it.
3. E = each of of buckets 1 to k has $\geq$ 1 string hashed into it.

What is $P(E)$?

Probability of events

E or F
$P(E \cup F)$

De Morgan's

E and F
$P(EF)$

Mutually exclusive?

Independent?

Just add!

$P(E) + P(F)$

Inclusion-Exclusion Principle

$P(E) + P(F) - P(E \cap F)$

Just multiply!

$P(E)P(F)$

Chain Rule

$P(E)P(F|E)$
$P(F)P(E|F)$

The fun never stops with hash tables

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

1. E = bucket 1 has ≥ 1 string hashed into it.
2. E = at least 1 of buckets 1 to k has ≥ 1 string hashed into it.
3. E = each of of buckets 1 to k has ≥ 1 string hashed into it.

What is $P(E)$?

Define F_i = bucket i has at least one string in it

WTF: $P(E) = P(F_1 F_2 \cdots F_k)$

$= 1 - P\left((F_1 F_2 \cdots F_k)^C\right)$ Complement

$= 1 - P\left(F_1^C \cup F_2^C \cup \cdots \cup F_k^C\right)$ De Morgan's Law

$= 1 - P\left(\bigcup_{i=1}^{k} F_i^c\right) = 1 - \sum_{r=1}^{k}(-1)^{(r+1)} \sum_{i_1<\cdots<i_r} P\left(F_{i_1}^c F_{i_2}^c \ldots F_{i_r}^c\right)$

where $P\left(F_{i_1}^c F_{i_2}^c \ldots F_{i_r}^c\right) = (1 - p_{i_1} - p_{i_2} \ldots - p_{i_r})^m$

It is expected that this last example will need some review!

DNA paternity testing

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$ Bayes' Theorem

Child is born with (A, a) gene pair (event $B_{A,a}$)

- Mother has (A, A) gene pair.
- Two possible fathers: M_1: (a, a), where $P(M_1 \text{ is father}) = p$
  M_2: (a, A), where $P(M_2 \text{ is father}) = P\left(M_1^C\right) = 1 - p$

What is $P\left(M_1 | B_{A,a}\right)$?

1. Define events & state goal
2. Identify known probabilities
3. Solve

DNA paternity testing

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What is $P\left(M_1|B_{A,a}\right)$?

1. Define events & state goal
2. Identify known probabilities
3. Solve

$$P\left(M_1|B_{A,a}\right) = \frac{P\left(B_{A,a}|M_1\right)P(M_1)}{P\left(B_{A,a}|M_1\right)P(M_1) + P\left(B_{A,a}|M_2\right)P(M_2)}$$

$$= \frac{1 \cdot p}{1 \cdot p + \frac{1}{2}(1-p)} = \frac{2p}{1+p} = \frac{2}{1+p}p > p$$

M_1 more likely to be father than he was before, since $P\left(M_1|B_{A,a}\right) > P(M_1)$

Today's plan

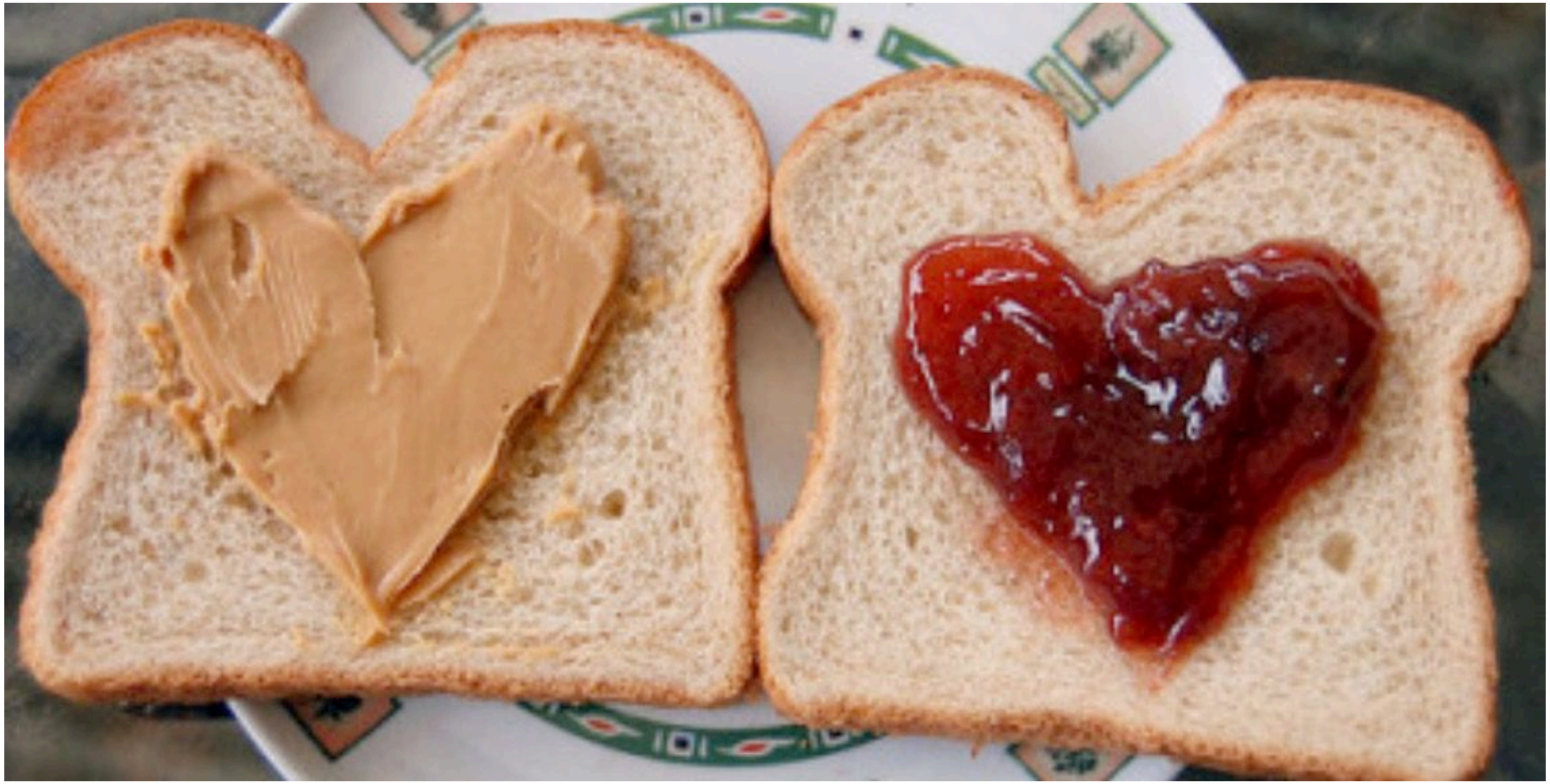
Conditional Independence

Random Variables

PMFs and CDFs

Expectation

Conditional Independence



Conditional Probability   Independence

Conditional Paradigm

For any events A, B, and E, you can condition consistently on E, and all formulas still hold:

Axiom 1	$0 \le P(A \mid E) \le 1$
Corollary 1 (complement)	$P(A \mid E) = 1 - P(A^C \mid E)$
Transitivity	$P(AB \mid E) = P(BA \mid E)$
Chain Rule	$P(AB \mid E) = P(B \mid E)P(A \mid BE)$
Bayes' Theorem	$P(A \mid BE) = \dfrac{P(B \mid AE)P(A \mid E)}{P(B \mid E)}$

BAE 's theorem?

Independence relationships can change with conditioning.

A and B independent

does NOT necessarily mean

A and B independent given E.

Conditional Independence

Independent events E and F ⟺ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

Two events A and B are defined as conditionally independent given E if:

$$P(AB|E) = P(A|E)P(B|E)$$

An equivalent definition:

A. $P(A|B) = P(A)$
B. $P(A|BE) = P(A)$
C. $P(A|BE) = P(A|E)$
D. $P(AB|E) = P(A|B)$

Conditional Independence

Independent events E and F ⟺ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

Two events A and B are defined as conditionally independent given E if:

$$P(AB|E) = P(A|E)P(B|E)$$

An equivalent definition:

A. $P(A|B) = P(A)$ Regular independence
B. $P(A|BE) = P(A)$
C. $P(A|BE) = P(A|E)$
D. $P(AB|E) = P(A|B)$

Netflix and Condition

Review

Let E = a user watches Life is Beautiful.

Let F = a user watches Amelie.

What is $P(E)$?

$$P(E) \approx \frac{\text{\# people who have watched movie}}{\text{\# people on Netflix}} = \frac{10{,}234{,}231}{50{,}923{,}123} \approx 0.20$$

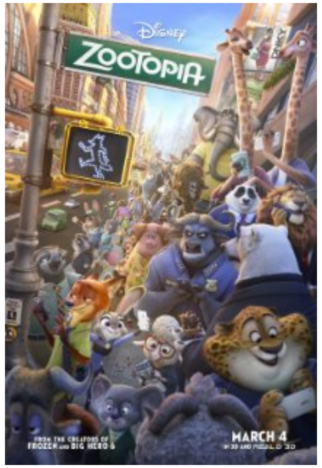
What is the probability that a user watches Life is Beautiful, given they watched Amelie?

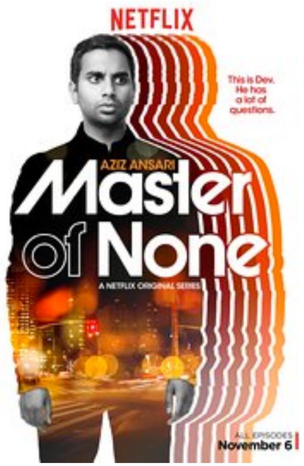
$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\text{\# people who have watched both}}{\text{\# people who have watched Amelie}} \approx 0.42$$

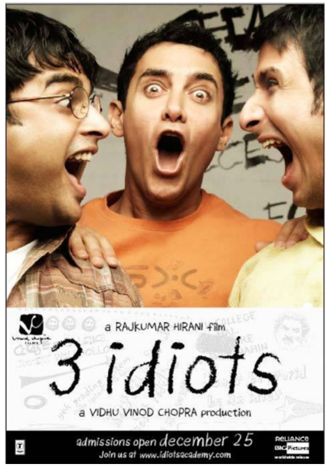
Netflix and Condition

Review

Let E be the event that a user watches the given movie.
Let F be the event that the same user watches Amelie.







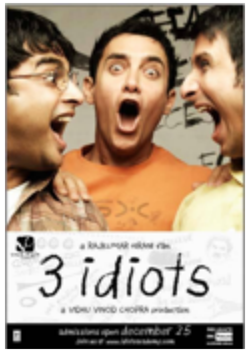
$P(E) = 0.19$	$P(E) = 0.32$	$P(E) = 0.20$	$P(E) = 0.09$	$P(E) = 0.20$
$P(E\|F) = 0.14$	$P(E\|F) = 0.35$	$P(E\|F) = 0.20$	$P(E\|F) = 0.72$	$P(E\|F) = 0.42$
		Independent!		

Netflix and Condition

Watched:

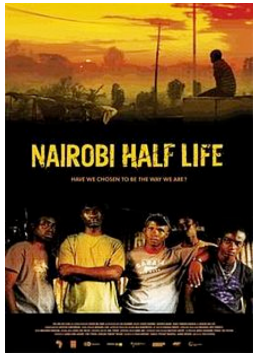
E_1

E_2



E_3

Will they watch?



E_4

What if $E_1 E_2 E_3 E_4$ are not independent? (e.g., all international emotional comedies)

$$P(E_4 | E_1 E_2 E_3) = \frac{P(E_1 E_2 E_3 E_4)}{P(E_1 E_2 E_3)}$$

$P(E_1 E_2 E_3 E_4)$ → $\frac{\text{\# people who have watched all 4}}{\text{\# people who watch any 4 movies}}$

$P(E_1 E_2 E_3)$ → $\frac{\text{\# people who have watched those 3}}{\text{\# people who watch any 3 movies}}$

Big numbers → tiny probabilities → underflow!

Netflix and Condition

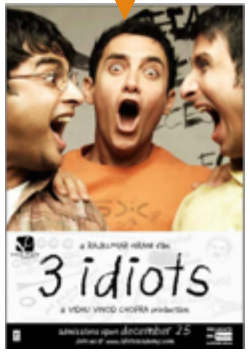
Cond. independent E and F given G	⟺	$P(EF\|G) = P(E\|G)P(F\|G)$ $P(E\|FG) = P(E\|G)$

K: likes international emotional comedies

Watched:

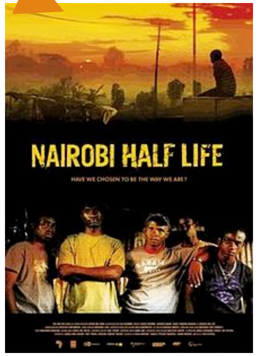
E_1

E_2



E_3

Will they watch?



E_4

What if $E_1E_2E_3E_4$ are *conditionally independent* given K?

$$P(E_4|E_1E_2E_3) = \frac{P(E_1E_2E_3E_4)}{P(E_1E_2E_3)}$$

$$P(E_4|E_1E_2E_3K) = P(E_4|K)$$

👉 An easier probability to store and compute!

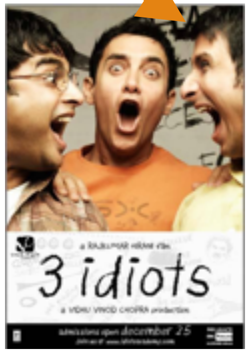
Netflix and Condition

Cond. independent E and F given G ⟺ $P(EF|G) = P(E|G)P(F|G)$
$P(E|FG) = P(E|G)$

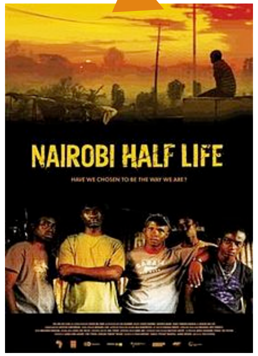
K: likes international emotional comedies

E_1

E_2



E_3



E_4

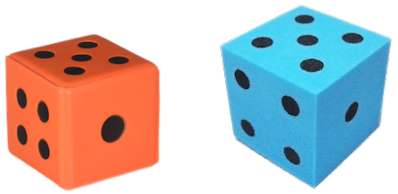
$E_1 E_2 E_3 E_4$ are dependent ⇒ $E_1 E_2 E_3 E_4$ are conditionally independent given K

Dependent events can become conditionally independent.

Not-so-independent dice

Cond. independent E and F given G	⟺	$P(EF|G) = P(E|G)P(F|G)$ $P(E|FG) = P(E|G)$

Roll two 6-sided dice, yielding values D_1 and D_2.



Let event E: $D_1 = 1$
event F: $D_2 = 6$
event G: $D_1 + D_2 = 7$

$G = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

1. Are E and F independent?

$P(E) = 1/6$ $\quad P(F) = 1/6$ $\quad P(EF) = 1/36$

2. Are E and F independent given G?

$P(E|G) = 1/6$ $\quad P(F|G) = 1/6$ $\quad P(EF|G) = 1/6$

$P(EF|G) \neq P(E|G)P(F|G)$

$\rightarrow E|G, F|G$ dependent

Independent events can become conditionally dependent.

The beauty of conditional independence

Generalized Chain Rule:

$$P(E_1 E_2 E_3 \ldots E_n F) =$$
$$P(F)P(E_1|F)P(E_2|E_1 F)P(E_3|E_1 E_2 F) \ldots P(E_n|E_1 E_2 \ldots E_{n-1} F)$$

If $E_1, E_2, \ldots, E_n$ are all conditionally independent given F:

$$P(E_1 E_2 E_3 \ldots E_n F) = P(F)P(E_1|F)P(E_2|F) \cdots P(E_n|F)$$

More on this in a future lecture!

Conditional independence is a Big Deal

Conditional independence is a practical, real-world way of decomposing hard probability questions.

"Exploiting conditional independence to generate fast probabilistic computations is one of the main contributions CS has made to probability theory."

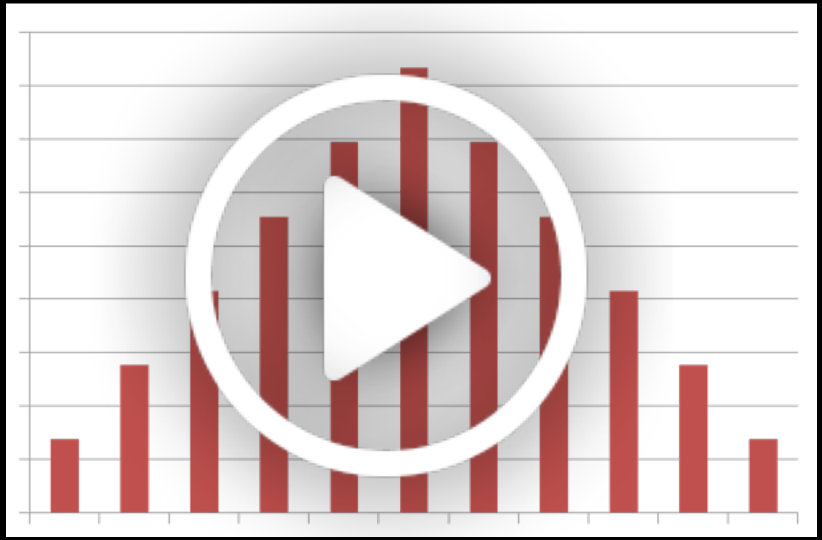
–Judea Pearl wins 2011 Turing Award,
"For fundamental contributions to artificial intelligence through the development of a calculus for probabilistic and causal reasoning"

Independence relationships can change with conditioning.

A and B independent | does NOT necessarily mean | A and B independent given E.

Next Episode Playing in 5 seconds

$P(X = k)$



$E[X]$

Back to Browse

More Episodes

Today's plan

Conditional Independence

Random Variables

PMFs and CDFs

Expectation

Random variables are like typed variables

type name value

```
int a = 5;
```

A is the number of Pokemon we bring to our *future* battle.

$$A \in \{1, 2, \dots, 6\}$$

```
double b = 4.2;
```

B is the amount of money we get *after* we win a battle.

$$B \in \mathbb{R}^+$$

```
bit c = 1;
```

C is 1 *if* we successfully beat the Elite Four. 0 otherwise.

$$C \in \{0,1\}$$

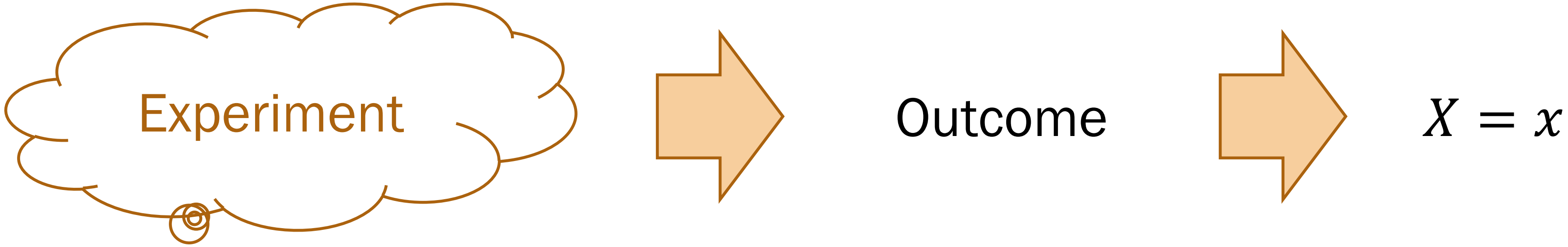
CS variables

Random variables

Random variables are like typed variables (with uncertainty)

Random Variable

A random variable is a real-valued function defined on a sample space.



Example:

3 coins are flipped.

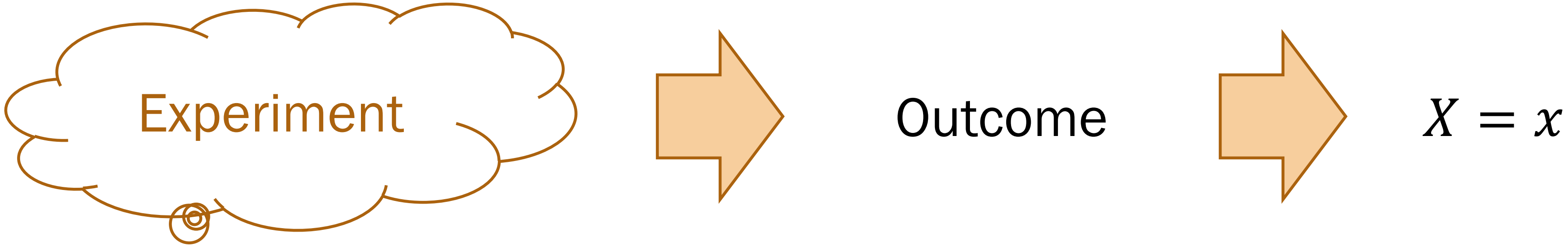
Let X = # of heads.

X is a random variable.

1. What is the value of X for the outcomes:
   - (T,T,T)?
   - (H,H,T)?
2. What is the event (set of outcomes) where $X = 2$?
3. What is $P(X = 2)$?

Random Variable

A random variable is a real-valued function defined on a sample space.



Example:

3 coins are flipped.
Let X = # of heads.
X is a random variable.

1. What is the value of X for the outcomes:
   - (T,T,T)? 0
   - (H,H,T)? 2
2. What is the event (set of outcomes) where $X = 2$?

   {(H, H, T), (H, T, H), (T, H, H)}
3. What is $P(X = 2)$? 3/8

Random variables are NOT events!

It is confusing that random variables and events use the same notation.

- **Random variables $\neq$ events.**
- We can define an event to be a particular assignment of a random variable.

Example:

3 coins are flipped.
Let X = # of heads.
X is a random variable.

$X = x$	$P(X = x)$	Set of outcomes	Possible event E
$X = \mathbf{0}$	1/8	{(T, T, T)}	Flip 0 heads
$X = \mathbf{1}$	3/8	{(H, T, T), (T, H, T), (T, T, H)}	Flip exactly 1 head
$X = \mathbf{2}$	3/8	{(H, H, T), (H, T, H), (T, H, H)}	The event where $X = \mathbf{2}$
$X = \mathbf{3}$	1/8	{(H, H, H)}	Flip 0 tails
$X \geq 4$	0	{ }	Flip 4 or more heads

Example random variable

Consider 5 flips of a coin which comes up heads with probability p.

- Each coin flip is an independent trial.
- Recall $P(2 \text{ heads}) = \binom{5}{2} p^2 (1-p)^3$, $P(3 \text{ heads}) = \binom{5}{3} p^3 (1-p)^2$

Let Y = # of heads on 5 flips.

1. What is the range of Y?
   In other words, what are the values that Y can take on with non-zero probability?

2. What is $P(Y = k)$, where k is in the range of Y?

Example random variable

Consider 5 flips of a coin which comes up heads with probability p.

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Let Y = # of heads on 5 flips.

1. What is the range of Y?
   In other words, what are the values that Y can take on with non-zero probability?

   $\{0, 1, 2, 3, 4, 5\}$

2. What is $P(Y = k)$, where k is in the range of Y?

   $P(Y = k) = \binom{5}{k} p^k (1-p)^{5-k}$

Today's plan

Conditional Independence

Random Variables

PMFs and CDFs

Expectation

Probability Mass Function

Y

random variable (e.g., # of heads in 5 coin flips, unbiased coin)

$Y = 2$

event

number

$P(Y = 2)$

probability (number b/t 0 and 1)

variable

$P(Y = k)$

function on k with range 0 and 1

Probability Mass Function

$$P(Y = k)$$

$k = 5$

Input k:
a value of Y

0.03125

output:
probability of the event $Y = k$

Y

random variable (e.g., # of heads in 5 coin flips, unbiased coin)

```
N = 5
P = 0.5

def eventProbability(k):
  n_ways = scipy.special.binom(N, k)
  p_way = np.power(P, k) * np.power(1 - P, N-k)
  return n_ways * p_way
```

Discrete RVs and Probability Mass Functions

A random variable X is discrete if its range has countably many values.

- $X = x$, where $x \in \{x_1, x_2, x_3, \dots\}$

The probability mass function (PMF) of a discrete random variable is

$$P(X = x) = p(x) = p_X(x)$$

shorthand notation

- Probabilities must sum to 1: $$\sum_{i=1}^{\infty} p(x_i) = 1$$

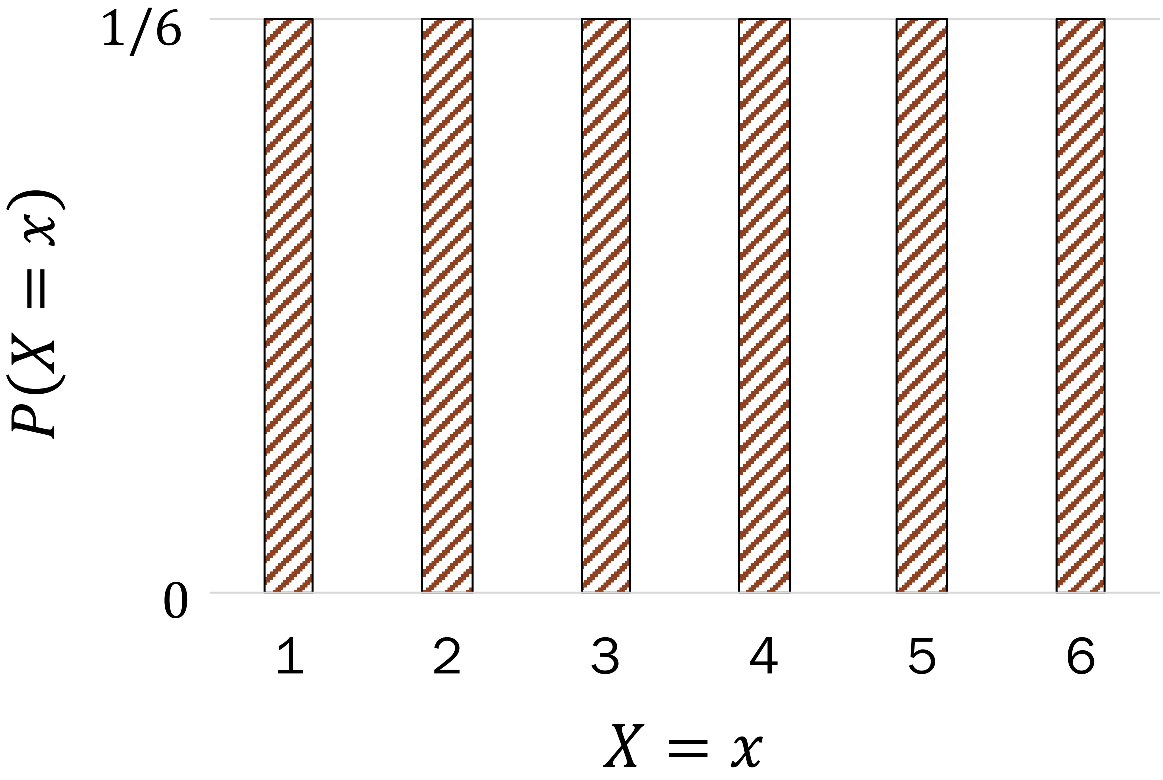
This last bullet is a good way to verify any PMF you create.

PMF for a single 6-sided die

Let X be a random variable that represents the result of a single dice roll.

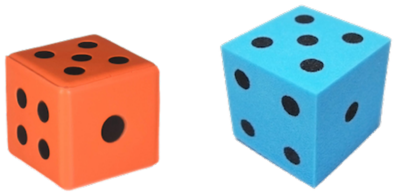
- Range of X : $\{1, 2, 3, 4, 5, 6\}$
- Therefore X is a discrete random variable.
- PMF of X:

$$p(x) = \begin{cases} 1/6 & x \in \{1, \dots, 6\} \\ 0 & \text{otherwise} \end{cases}$$



PMF for the sum of two dice

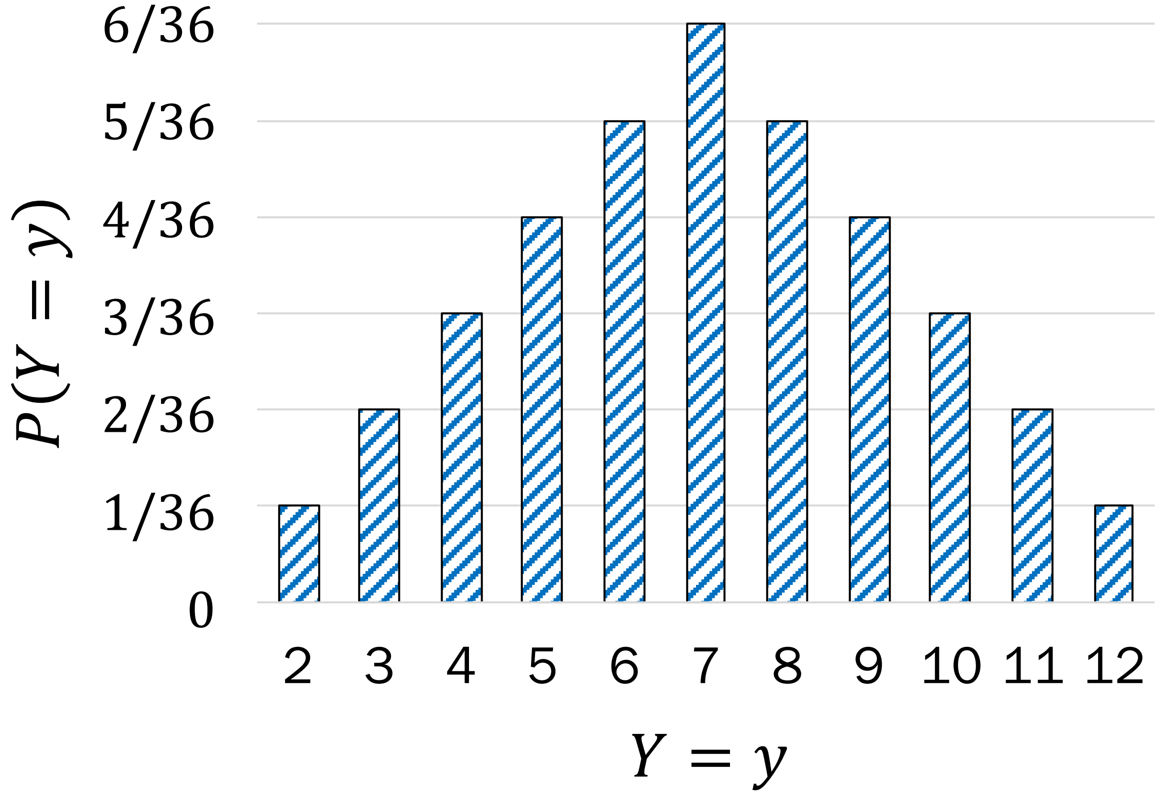
Let Y be a random variable that represents the sum of two independent dice rolls.



Range of Y: $\{2, 3, \dots, 11, 12\}$

$$p(y) = \begin{cases} \frac{y-1}{36} & y \in \mathbb{Z}, 2 \le y \le 6 \\ \frac{13-y}{36} & y \in \mathbb{Z}, 7 \le y \le 12 \\ 0 & \text{otherwise} \end{cases}$$

Sanity check: $$\sum_{y=2}^{12} p(y) = 1$$



Break for Friday/ announcements

Announcements

Problem Set 1

Due:	an hour ago
On-time grades:	next Friday
Solutions:	next Friday

Problem Set 2

Out:	today
Due:	Monday 10/14
Covers:	through today

Concept checks

Due date: every Tuesday 1:00pm

You can edit your response, so don't be afraid of submitting multiple times.

Optional readings:

Lecture notes:	website
Textbook sections:	(scroll down)

Cumulative Distribution Functions

For a random variable X, the cumulative distribution function (CDF) is defined as

$$F(a) = F_X(a) = P(X \leq a), \text{where } -\infty < a < \infty$$

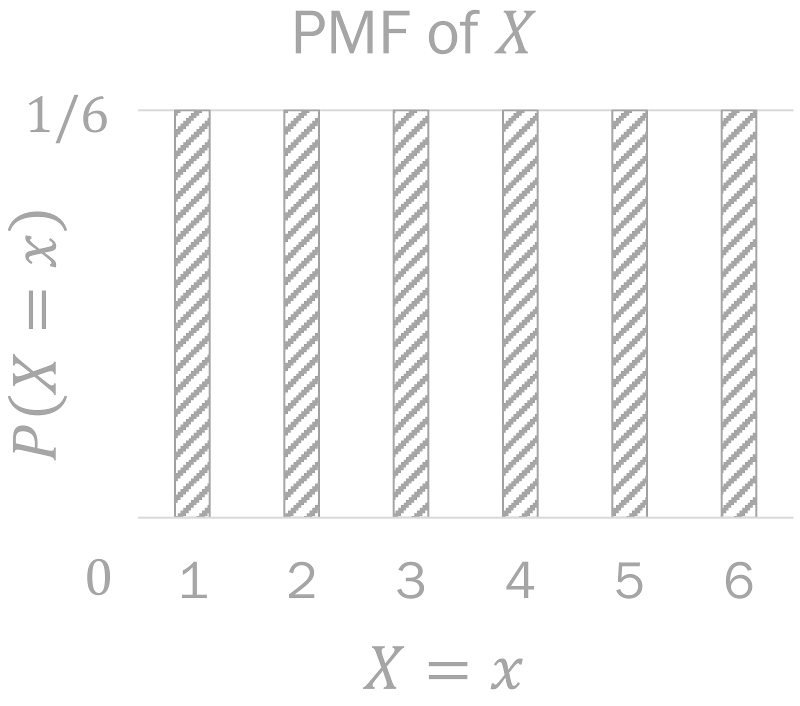
For a discrete RV X, the CDF is:

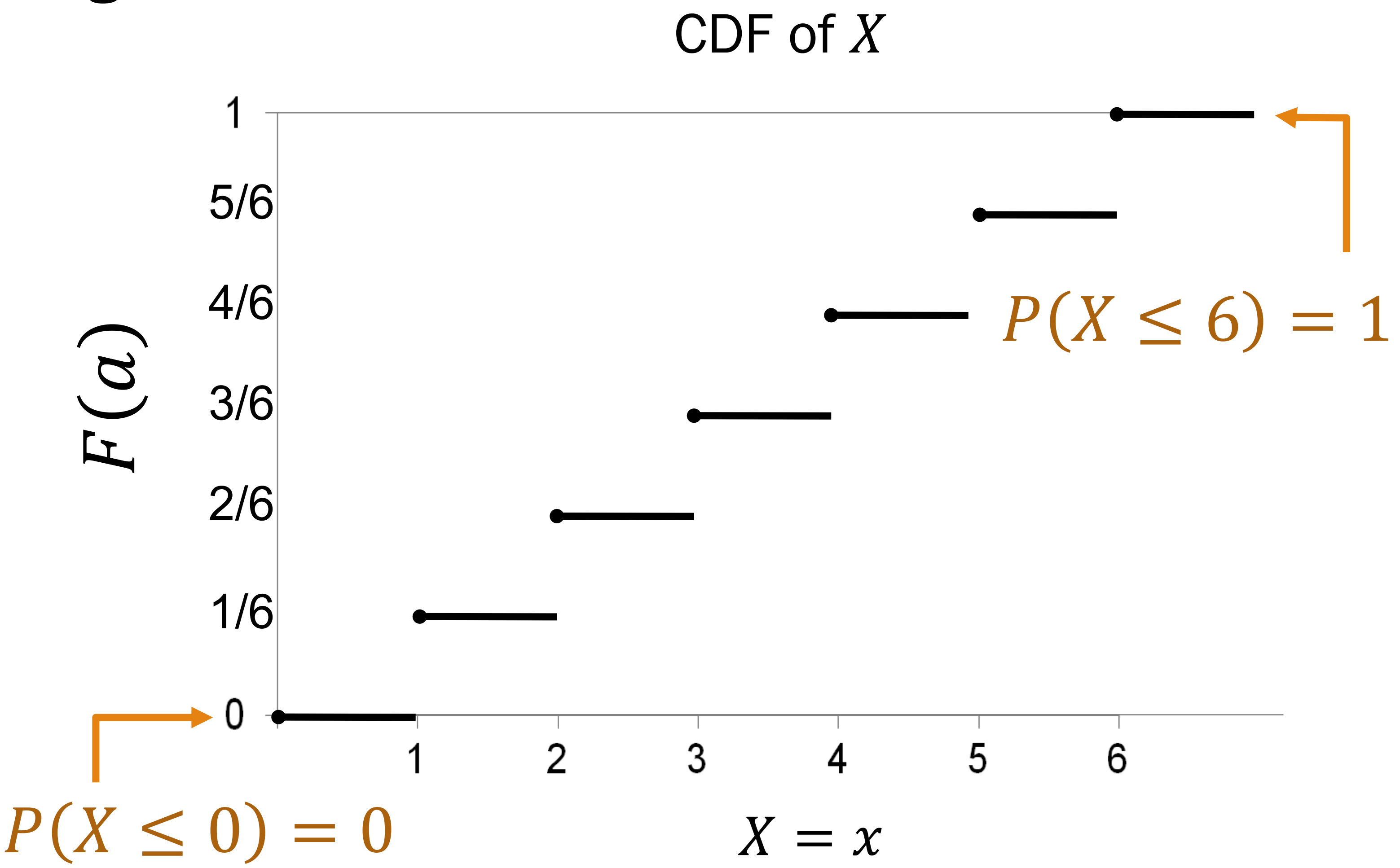
$$F(a) = P(X \leq a) = \sum_{\text{all } x \leq a} p(x)$$

CDFs as graphs

CDF (cumulative distribution function) $F(a) = P(X \le a)$

Let X be a random variable that represents the result of a single dice roll.





Today's plan

Conditional Independence

Random Variables

PMFs and CDFs

Expectation

Expectation

The expectation of a discrete random variable X is defined as:

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

- Note: sum over all values of $X = x$ that have non-zero probability.
- Other names: mean, expected value, weighted average, center of mass, first moment

Expectation of a die roll

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$ Expectation of X

What is the expected value of a 6-sided die roll?

1. Define random variables

X = RV for value of roll

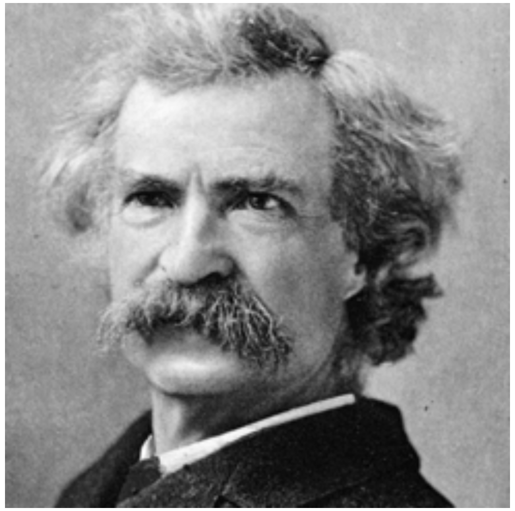
$$P(X = x) = \begin{cases} 1/6 & x \in \{1, \dots, 6\} \\ 0 & \text{otherwise} \end{cases}$$

2. Solve

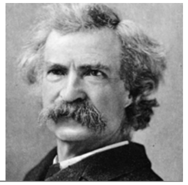
$$E[X] = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = \frac{7}{2}$$

Lying with statistics

"There are three kinds of lies:
lies, damned lies, and statistics"
–popularized by Mark Twain, 1906



Lying with statistics



A school has 3 classes with 5, 10, and 150 students.
What is the average class size?

1. Interpretation #1

- Randomly choose a <u>class</u> with equal probability.
- X = size of chosen class

$$E[X] = 5\left(\frac{1}{3}\right) + 10\left(\frac{1}{3}\right) + 150\left(\frac{1}{3}\right)$$

$$= \frac{165}{3} = 55$$

What universities usually report

2. Interpretation #2

- Randomly choose a <u>student</u> with equal probability.
- Y = size of chosen class

$$E[Y] = 5\left(\frac{5}{165}\right) + 10\left(\frac{10}{165}\right) + 150\left(\frac{150}{165}\right)$$

$$= \frac{22635}{165} \approx 137$$

Average student perception of class size

Important properties of expectation

1. Linearity:

$$E[aX + b] = aE[X] + b$$

- Let X = 6-sided dice roll, $Y = 2X - 1$.
- $E[X] = 3.5$
- $E[Y] = 6$

2. Expectation of a sum = sum of expectation:

$$E[X + Y] = \mathrm{E}[X] + E[Y]$$

Sum of two dice rolls:

- Let X = roll of die 1, Y = roll of die 2
- $E[X + Y] = 3.5 + 3.5 = 7$

3. Unconscious statistician:

$$E[g(X)] = \sum_x g(x)p(x)$$

Being a statistician unconsciously

$E[g(X)] = \sum_x g(x)p(x)$ Expectation of $g(X)$

Let X be a discrete random variable.

- $P(X = x) = \frac{1}{3}$ for $x \in \{-1, 0, 1\}$

Let $Y = |X|$. What is $E[Y]$?

A. $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot -1 \qquad = 0$

B. $E[Y] = E[0] \qquad = 0$

C. $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 \qquad = \frac{2}{3}$

D. $\frac{1}{3} \cdot |-1| + \frac{1}{3} \cdot |0| + \frac{1}{3} |1| = \frac{2}{3}$

E. C and D

Being a statistician unconsciously

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B. $E[Y] = E[0] \quad = 0$ $\qquad E[E[X]]$

C. $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 \quad = \frac{2}{3}$

D. $\frac{1}{3} \cdot |-1| + \frac{1}{3} \cdot |0| + \frac{1}{3} |1| = \frac{2}{3}$

E. C and D

1. Find PMF of Y: $p_Y(0) = \frac{1}{3}, p_Y(1) = \frac{2}{3}$
2. Compute $E[Y]$

Use LOTUS by using PMF of X:

1. $P(X = x) \cdot |x|$
2. Sum up

LOTUS proof

$$E[g(X)] = \sum_x g(x)p(x)$$ Expectation of $g(X)$

Let $Y = g(X)$, where g is a real-valued function.

$$E[g(X)] = E[Y] = \sum_j y_j p(y_j)$$

$$= \sum_j y_j \sum_{i:g(x_i)=y_j} p(x_i)$$

$$= \sum_j \sum_{i:g(x_i)=y_j} y_j \, p(x_i)$$

$$= \sum_j \sum_{i:g(x_i)=y_j} g(x_i) \, p(x_i)$$

$$= \sum_i g(x_i) \, p(x_i)$$

For you to review so that you can sleep at night

I want to play a game

$$E[g(x)] = \sum_x g(x)p(x)$$ Expectation of $g(X)$

St. Petersburg Paradox

$$E[g(x)] = \sum_x g(x)p(x)$$ Expectation of $g(X)$

- A fair coin (comes up “heads” with $p = 0.5$)
- Define Y = number of coin flips (“heads”) before first “tails”
- You win $\$2^Y$

How much would you pay to play? (How much you expect to win?)

A. $10000
B. $∞
C. $1
D. $0.50
E. $0 but let me play
F. I will not play

St. Petersburg Paradox

$$E[g(x)] = \sum_x g(x)p(x)$$ Expectation of $g(X)$

- A fair coin (comes up "heads" with $p = 0.5$)
- Define Y = number of coin flips ("heads") before first "tails"
- You win $\$2^Y$

How much would you pay to play? (How much you expect to win?)

1. Define random variables

For $i \geq 0$: $P(Y = i) = \left(\frac{1}{2}\right)^{i+1}$

Let W = your winnings, 2^Y.

2. Solve

$$E[W] = E[2^Y] = \left(\frac{1}{2}\right)^1 2^0 + \left(\frac{1}{2}\right)^2 2^1 + \left(\frac{1}{2}\right)^3 2^2 + \cdots$$

$$= \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{i+1} 2^i = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right) = \infty$$

St. Petersburg + Reality

$E[g(x)] = \sum_x g(x)p(x)$ Expectation of $g(X)$

What if Lisa has only \$65,536?

- Same game
- Define Y = # heads before first tails
- You win $W = \$2^Y$

- If you win over \$65,536, I leave the country

1. Define random variables

For $i \geq 0$: $P(Y = i) = \left(\frac{1}{2}\right)^{i+1}$

Let W = your winnings, 2^Y.

2. Solve

$$E[W] = \left(\frac{1}{2}\right)^1 2^0 + \left(\frac{1}{2}\right)^2 2^1 + \left(\frac{1}{2}\right)^3 2^2 + \cdots$$

$k = \log_2(65{,}536)$
$= 16$

$$= \sum_{i=0}^{k} \left(\frac{1}{2}\right)^{i+1} 2^i = \sum_{i=0}^{16} \left(\frac{1}{2}\right) = 8.5$$