13: Independent RVs

Lisa Yan

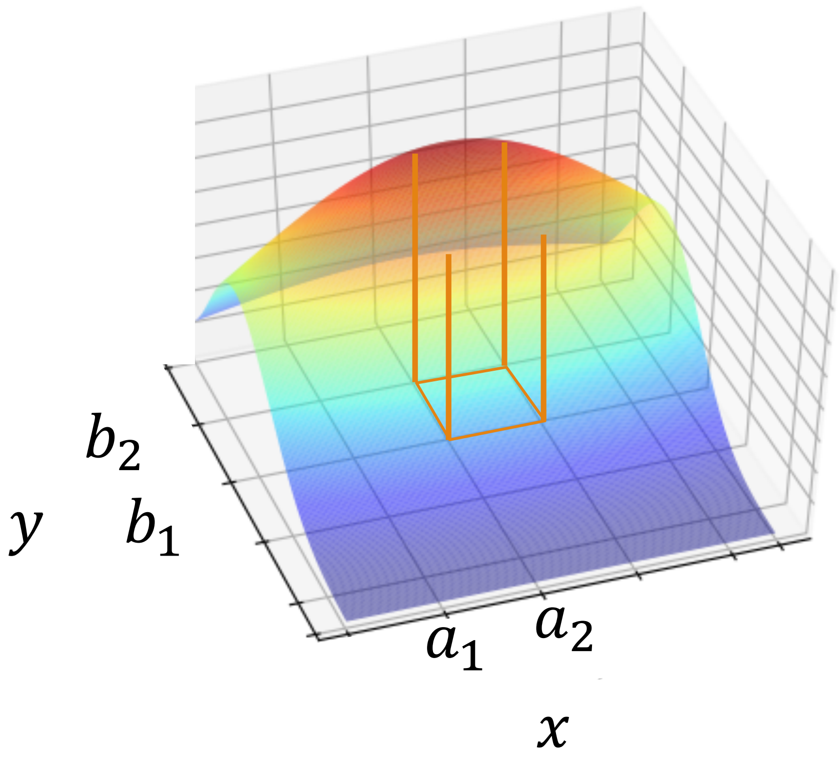
October 21, 2019

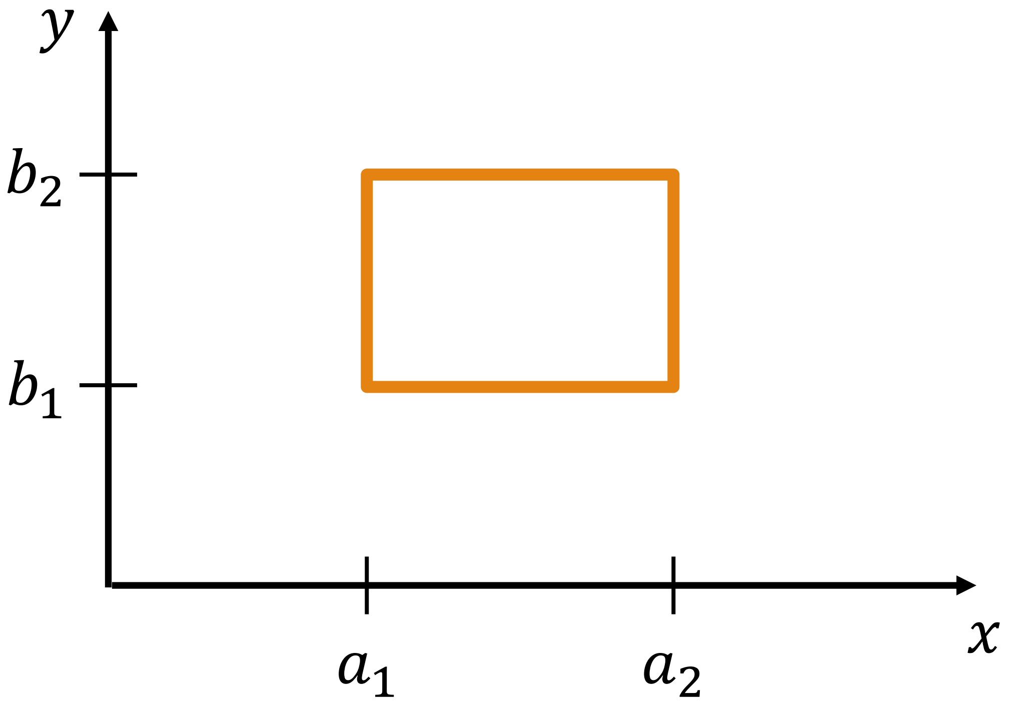
Probabilities from joint CDFs

Review

Joint CDF: $P(X \leq x, Y \leq y) = F_{X,Y}(x, y)$

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =$$
$$F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$





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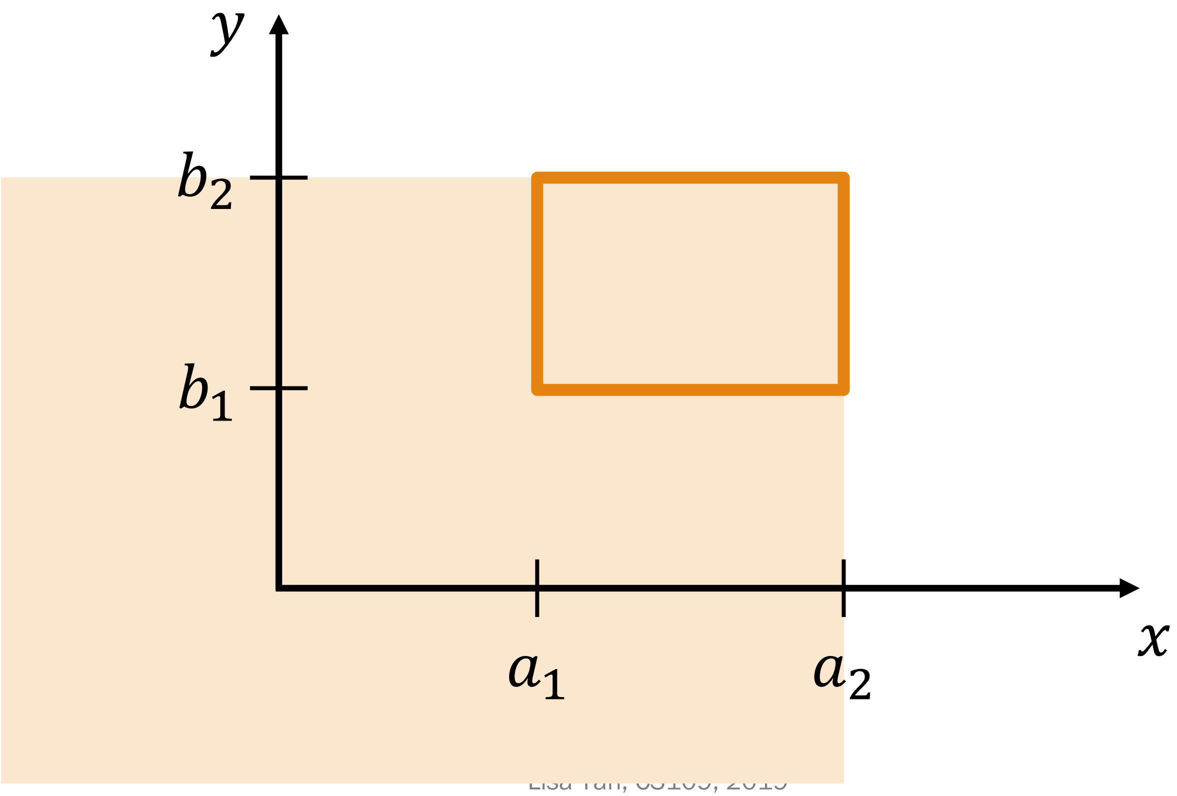

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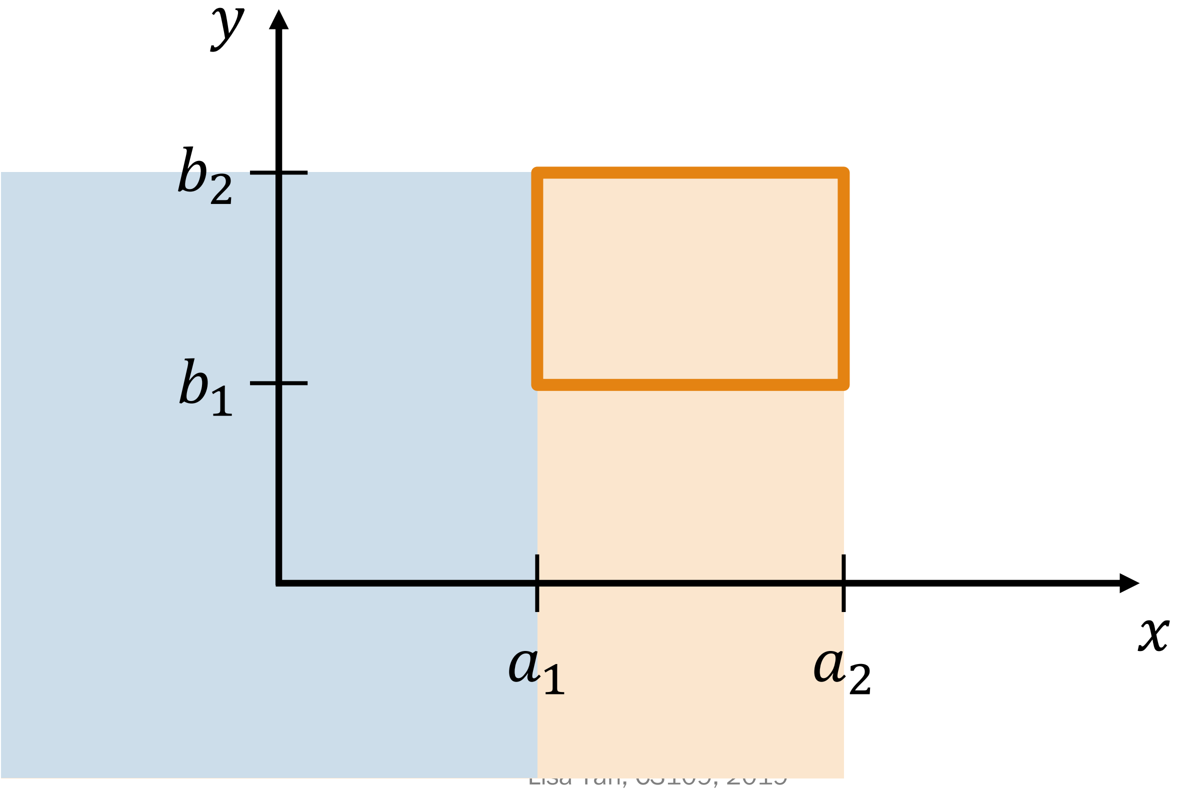

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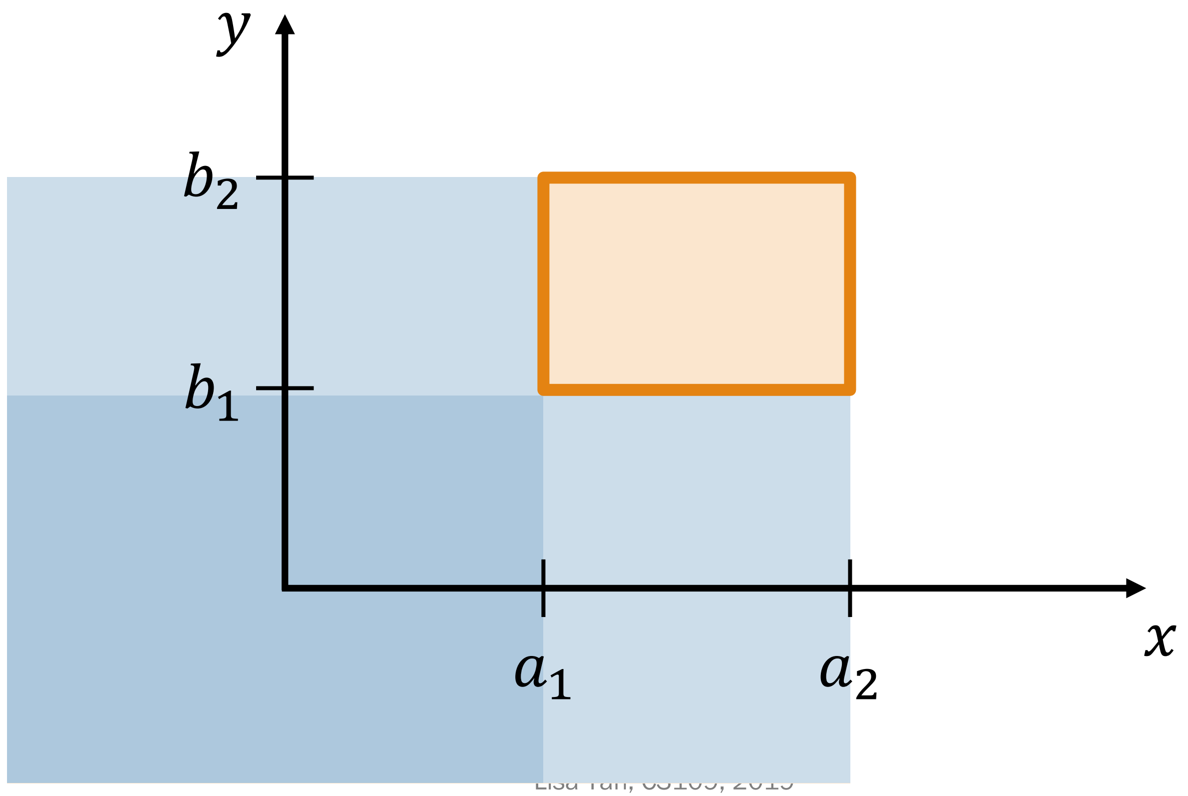

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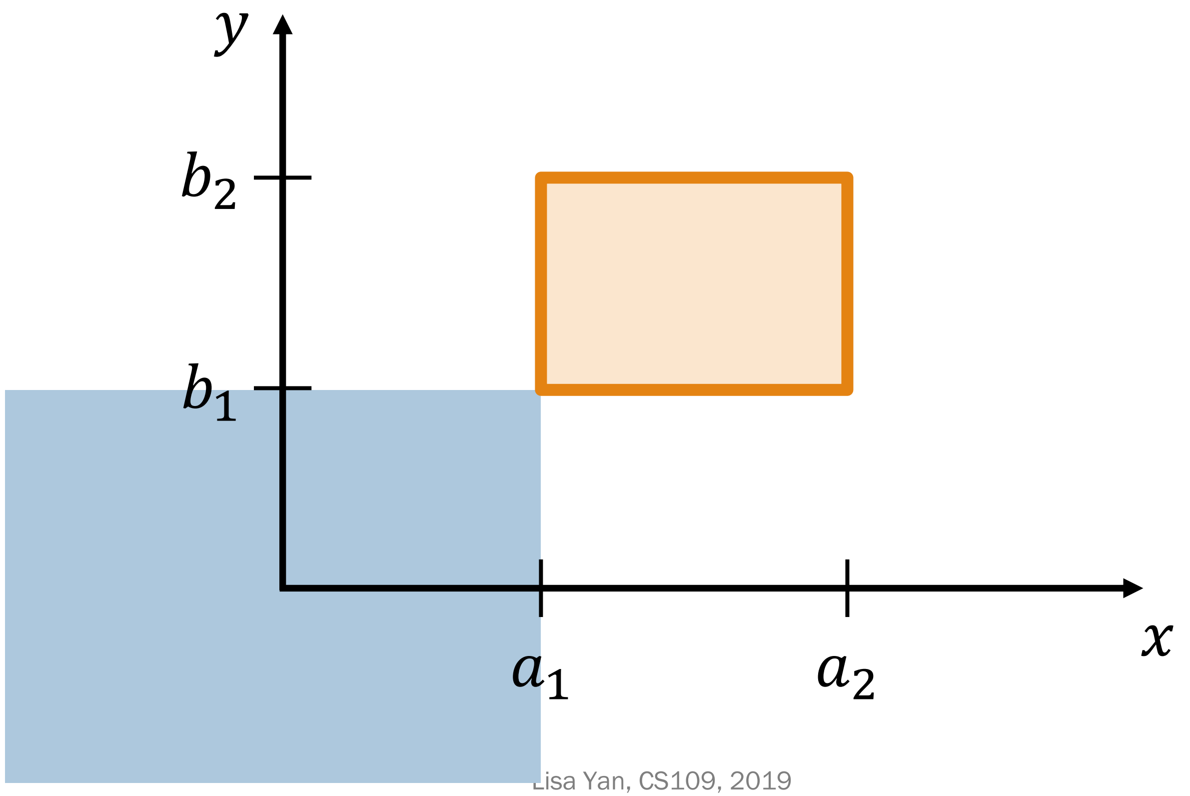

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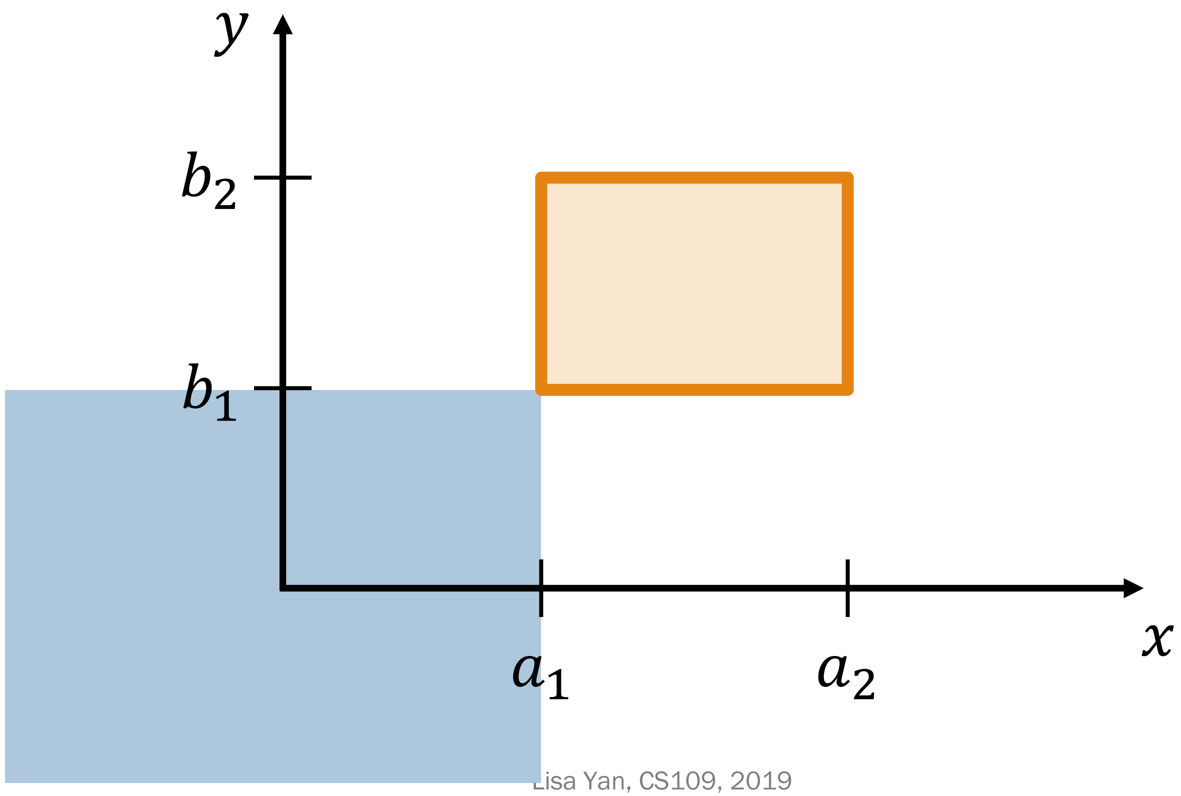

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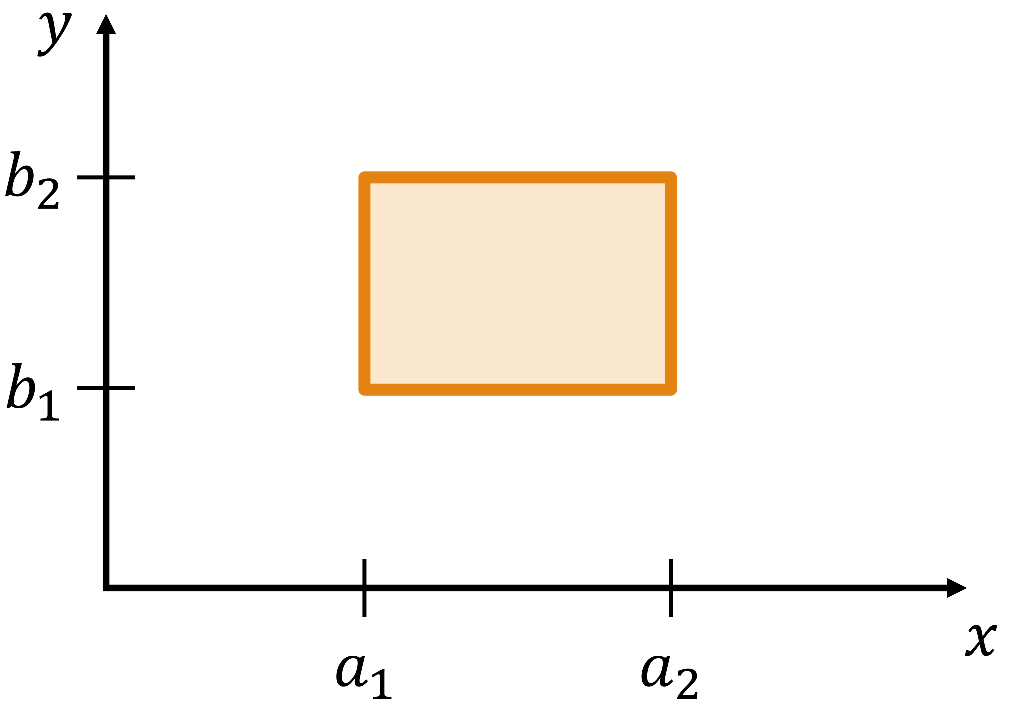


Probability with Instagram!

$$P(a_1 < X \le a_2, b_1 < Y \le b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$

In image processing, a Gaussian blur is the result of blurring an image by a Gaussian function. It is a widely used effect in graphics software, typically to reduce image noise.

Gaussian blur

$$P(a_1 < X \le a_2, b_1 < Y \le b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$

In a Gaussian blur, for every pixel:

- Weight each pixel by the probability that X and Y are both within the pixel bounds
- The weighting function is a Gaussian joint PDF with a standard deviation parameter σ.

Gaussian blurring with $\sigma = 3$

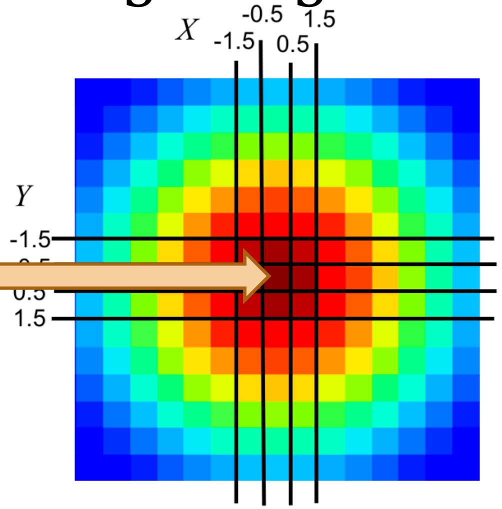
Joint PDF:

$$f_{X,Y}(x, y) = \frac{1}{2\pi \cdot 3^2} e^{-(x^2+y^2)/2\cdot 3^2}$$

Joint CDF:

$$F_{X,Y}(x, y) = \Phi\left(\frac{x}{3}\right)\Phi\left(\frac{y}{3}\right)$$

Weight matrix:



Center pixel: (0, 0)

Pixel bounds:

$-0.5 < x \le 0.5$

$-0.5 < y \le 0.5$

Gaussian blur

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$

In a Gaussian blur:

- Weight each pixel by the probability that X and Y are both within the pixel bounds

What is the weight of the center pixel?

$$P(-0.5 < X \leq 0.5, -0.5 < Y \leq 0.5)$$
$$= F_{X,Y}(0.5, 0.5) - F_{X,Y}(-0.5, 0.5)$$
$$- F_{X,Y}(0.5, -0.5) + F_{X,Y}(-0.5, -0.5)$$
$$= \Phi\left(\frac{0.5}{3}\right)\Phi\left(\frac{0.5}{3}\right) - 2 \cdot \Phi\left(\frac{-0.5}{3}\right)\Phi\left(\frac{0.5}{3}\right)$$
$$+ \Phi\left(\frac{-0.5}{3}\right)\Phi\left(\frac{-0.5}{3}\right)$$
$$\approx 0.5662^2 - 2 \cdot 0.5662 \cdot 0.4338 + 0.4338^2$$
$$\approx 0.206$$

Gaussian blurring with $\sigma = 3$

Joint PDF:

$$f_{X,Y}(x, y) = \frac{1}{2\pi \cdot 3^2} e^{-(x^2+y^2)/2\cdot 3^2}$$

Joint CDF:

$$F_{X,Y}(x, y) = \Phi\left(\frac{x}{3}\right)\Phi\left(\frac{y}{3}\right)$$

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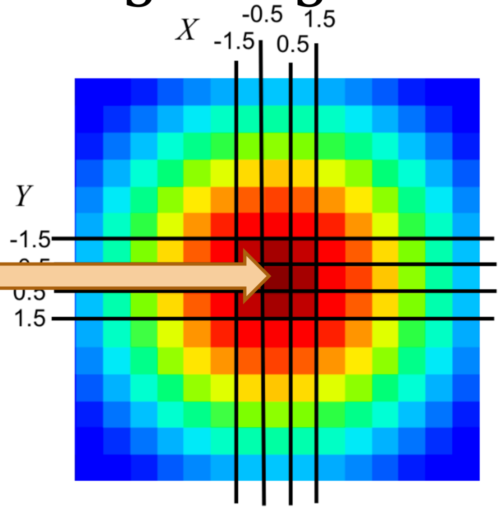


Center pixel: (0, 0)

Pixel bounds:

$$-0.5 < x \leq 0.5$$
$$-0.5 < y \leq 0.5$$

CS109 roadmap

Multiple events:

intersection	conditional probability	independence
$P(E \cap F) = P(EF)$	$P(E\|F) = \frac{P(EF)}{P(F)}$	$P(EF) = P(E)P(F)$

Joint (Multivariate) distributions

joint PMF/PDF	conditional distributions?	independent RVs?
$p_{X,Y}(x,y)$ $f_{X,Y}(x,y)$	Yes! (Wednesday)	Yes! (today)

Model ALL the things!

Today's plan (covered on midterm)

Independent RVs

Sum of independent RVs

- ✅ Binomial
- Convolution
- ✅ Poisson
- ✅ Normal
- ⚠️ Uniform

Expectation of sum of RVs (next class)

Independent discrete RVs

Recall the definition of independent events E and F:

$$P(EF) = P(E)P(F)$$

Two discrete random variables X and Y are **independent** if:

⚠️ for all x, y: ⚠️

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

Different notation, same idea:

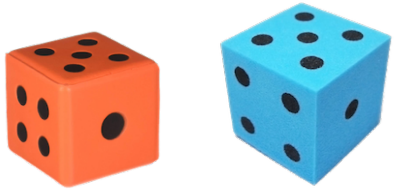
$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$

Intuitively: knowing value of X tells us nothing about the distribution of Y (and vice versa)

If two variables are not independent, they are called **dependent**.

Dice (after all this time, still our friends)

Let: D_1 and D_2 be the outcomes of two rolls
$S = D_1 + D_2$, the sum of two rolls



- Each roll of a 6-sided die is an independent trial.
- D_1 and D_2 are independent.

Are S and D_1 independent?

1. $P(D_1 = 1, S = 7)$?
2. $P(D_1 = 1, S = 5)$?

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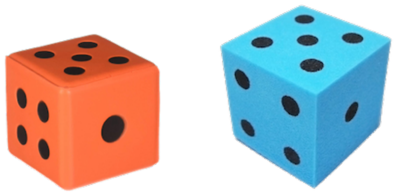

- Each roll of a 6-sided die is an independent trial.
- D_1 and D_2 are independent.

Are S and D_1 independent? ✕

1. $P(D_1 = 1, S = 7)$?

Event $(S = 7)$: $\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$$P(D_1 = 1)P(S = 7) = (1/6)(1/6) = 1/36 = P(D_1 = 1, S = 7)$$

2. $P(D_1 = 1, S = 5)$?

Event $(S = 5)$: $\{(1,4), (2,3), (3,2), (4,1)\}$

$$P(D_1 = 1)P(S = 5) = (1/6)(4/36) \neq 1/36 = P(D_1 = 1, S = 5)$$

Independent events $(D_1 = 1)$,$(S = 7)$
Dependent events $(D_1 = 1)$,$(S = 5)$

All events $(X = x, Y = y)$ must be independent for X, Y to be independent random variables.

Coin flips

Flip a coin with probability p of "heads" a total of $n+m$ times.

Let $X =$ number of heads in first n flips. $X \sim \text{Bin}(n, p)$
$Y =$ number of heads in next m flips. $Y \sim \text{Bin}(m, p)$
$Z =$ total number of heads in $n+m$ flips.

1. Are X and Z independent? ❌

Counterexample: What if $Z = 0$?

🤔

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1. Are X and Z independent? ❌ Counterexample: What if $Z = 0$?
2. Are X and Y independent?

Strategy:

A. No, proof by counterexample
B. Yes, proof by counting
C. None/other

🤔

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Z = total number of heads in $n + m$ flips.

1. Are X and Z independent? ❌ Counterexample: What if $Z = 0$?
2. Are X and Y independent? ✅

$$P(X = x, Y = y) = P\left(\begin{array}{c}\text{first } n \text{ flips have } x \text{ heads} \\ \text{and next } m \text{ flips have } y \text{ heads}\end{array}\right)$$

# of mutually exclusive outcomes in event: $\binom{n}{x}\binom{m}{y}$

$P(\text{each outcome})$
$= p^x(1-p)^{n-x}p^y(1-p)^{m-y}$

$$= \binom{n}{x} p^x (1-p)^{n-x} \binom{m}{y} p^y (1-p)^{m-y}$$

$$= P(X = x)P(Y = y)$$

🤔

Independent continuous RVs

Two continuous random variables X and Y are independent if:

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

Equivalently:

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$
$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

More generally, X and Y are independent if joint density factors separately:

$$f_{X,Y}(x, y) = h(x)g(y), \text{ where } -\infty < x, y < \infty$$

Is the Gaussian blur distribution independent?

Gaussian blurring with $\sigma = 3$

Joint PDF:

$$f_{X,Y}(x,y) = \frac{1}{2\pi \cdot 3^2} e^{-(x^2+y^2)/2\cdot 3^2}$$

Joint CDF:

$$F_{X,Y}(x,y) = \Phi\left(\frac{x}{3}\right)\Phi\left(\frac{y}{3}\right)$$ ✅

Weight matrix:

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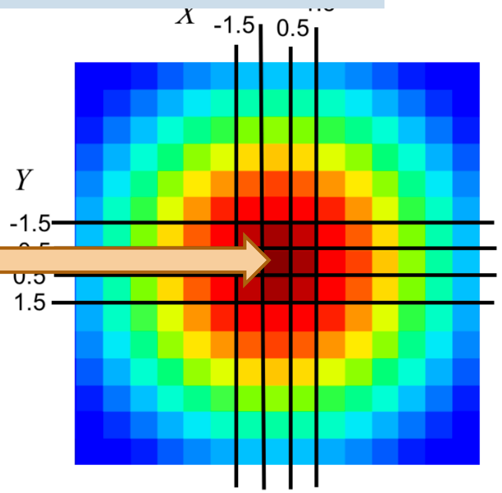

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Equivalently:

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$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

More generally, X and Y are independent if joint density factors separately:

👉 $f_{X,Y}(x,y) = g(x)h(y)$, where $-\infty < x, y < \infty$

Pop quiz! (just kidding)

$f_{X,Y}(x,y) = g(x)h(y)$, where $-\infty < x, y < \infty$ → independent X and Y

Are X and Y independent in the following cases?

1. $f_{X,Y}(x,y) = 6e^{-3x}e^{-2y}$
   where $0 < x, y < \infty$

2. $f_{X,Y}(x,y) = 4xy$
   where $0 < x, y < 1$

3. $f_{X,Y}(x,y) = 4xy$
   where $0 < x + y < 1$

Pop quiz! (just kidding)

$f_{X,Y}(x,y) = g(x)h(y)$, where $-\infty < x, y < \infty$ ➡ independent X and Y

Are X and Y independent in the following cases?

✅ 1. $f_{X,Y}(x,y) = 6e^{-3x}e^{-2y}$ where $0 < x, y < \infty$

Separable functions: $g(x) = 3e^{-3x}$, $h(y) = 2e^{-2y}$

2. $f_{X,Y}(x,y) = 4xy$ where $0 < x, y < 1$

Separable functions: $g(x) = 2x$, $h(y) = 2y$

3. $f_{X,Y}(x,y) = 4xy$ where $0 < x + y < 1$

Cannot capture constraint on $x + y$ into factorization!

👉 If you can factor densities over all of the support, you have independence. 🤔

Break for jokes/ announcements

Announcements

Midterm exam

When: Tuesday, October 29th, 7:00pm-9:00pm
Where: Hewlett 200

Covers: Up to (and including) week 4 + Lecture Notes #13 (updated 10/21)

Practice: http://web.stanford.edu/class/cs109/exams/midterm.html
Review session: Saturday, 10am-12pm, Shiram 104
not recorded; materials will be posted though

Problem Set 4

Out: later today
Due: Wednesday 11/6
Midterm coverage: First half (marked)

Concept checks

Week 4's: Tuesday 10/22 1pm
Week 5's: Wednesday 10/31 1pm

Today's plan

Independent RVs

Sum of independent RVs

- ✅ Binomial
- Convolution
- ✅ Poisson
- ✅ Normal
- ⚠️ Uniform

Expectation of sum of RVs (next class)

Sum of independent Binomials

$X \sim \text{Bin}(n_1, p)$
$Y \sim \text{Bin}(n_2, p)$
X, Y independent

$$\Rightarrow \quad X + Y \sim \text{Bin}(n_1 + n_2, p)$$

Intuition:

- Each trial in X and Y is independent and has same success probability p
- Define $Z = n_1 + n_2$ independent trials, each with success probability p
  $Z \sim \text{Bin}(n_1 + n_2, p)$, and also $Z = X + Y$

Holds in general case:

$X_i \sim \text{Bin}(n_i, p)$
X_i independent for $i = 1, \ldots, n$

$$\Rightarrow \quad \sum_{i=1}^{n} X_i \sim \text{Bin}\left(\sum_{i=1}^{n} n_i, p\right)$$

If only it were always so simple...

Convolution: Sum of independent random variables

For any discrete random variables X and Y:

$$P(X + Y = n) = \sum_k P(X = k, Y = n - k)$$

In particular, for independent discrete random variables X and Y:

$$\underbrace{P(X + Y = n) = \sum_k P(X = k)P(Y = n - k)}_{\text{the convolution of } p_X \text{ and } p_Y}$$

Insight into convolution

For independent discrete random variables X and Y:

$$P(X+Y=n)=\sum_k P(X=k)P(Y=n-k)$$

the convolution of p_X and p_Y

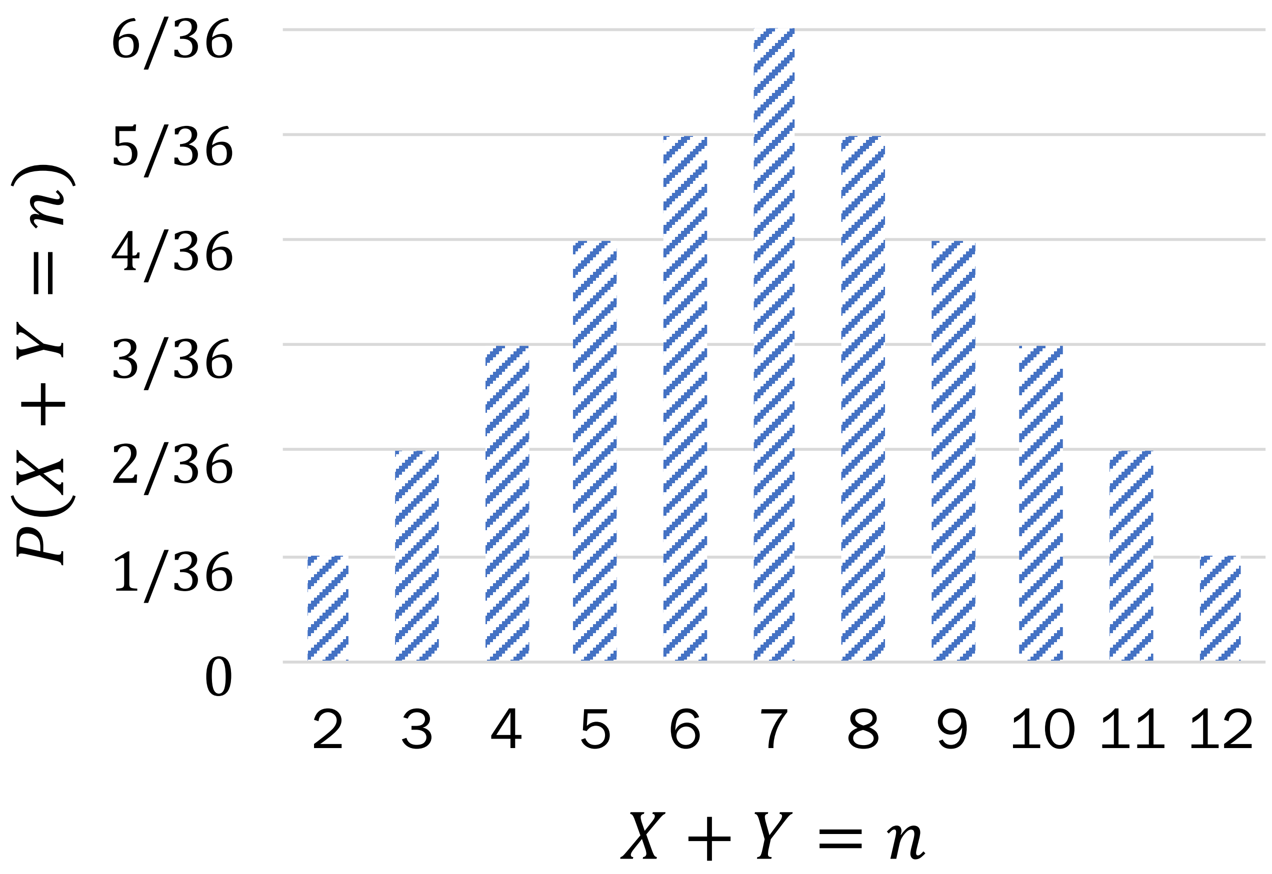
Suppose X and Y are independent, both with support $\{0, 1, \ldots\}$:

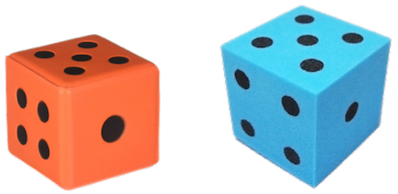
$X=k$	$Y=n-k$	Probability
0	n	$P(X=0)P(Y=n)$
1	$n-1$	$P(X=1)P(Y=n-1)$
2	$n-2$	$P(X=2)P(Y=n-2)$
$\cdots$	$\cdots$	$\cdots$
n	0	$P(X=n)P(Y=0)$
$n+1$	$-$	0

Sum of mutually exclusive events

Sum of dice rolls

X and Y independent + discrete $\quad P(X+Y=n)=\sum_k P(X=k)P(Y=n-k)$





The distribution of a sum of dice rolls is a convolution.

Note for $k, n-k$ in the support,

$$\begin{aligned} P(X=k, Y=n-k) \\ &= P(X=k)P(Y=n-k) \\ &= 1/36 \end{aligned}$$

Today's plan

Independent RVs

Sum of independent RVs

- ✅ Binomial
- Convolution
- ✅ Poisson
- ✅ Normal
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Expectation of sum of RVs (next class)

Sum of independent Poissons

$X \sim \text{Poi}(\lambda_1), Y \sim \text{Poi}(\lambda_2)$
X, Y independent

$\Rightarrow$ $X + Y \sim \text{Poi}(\lambda_1 + \lambda_2)$

Proof (just for reference):

$$P(X+Y=n) = \sum_k P(X=k)P(Y=n-k)$$

X and Y independent, convolution

$$= \sum_{k=0}^{n} e^{-\lambda_1} \frac{\lambda_1^k}{k!} e^{-\lambda_2} \frac{\lambda_2^{n-k}}{(n-k)!} = e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!\,(n-k)!}$$

PMF of Poisson RVs

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \lambda_1^k \lambda_2^{n-k} = \underbrace{\frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1+\lambda_2)^n}_{\text{Poi}(\lambda_1+\lambda_2)}$$

Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

General sum of independent Poissons

Holds in general case:

$X_i \sim \text{Poi}(\lambda_i)$
X_i independent for $i = 1, \dots, n$

$$\sum_{i=1}^{n} X_i \sim \text{Poi}\left(\sum_{i=1}^{n} \lambda_i\right)$$

Sum of independent Gaussians

$X \sim \mathcal{N}(\mu_1, \sigma_1^2)$,
$Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$
X, Y independent

$$X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

(proof left to Wikipedia)

Holds in general case:

$X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$
X_i independent for $i = 1, \ldots, n$

$$\sum_{i=1}^{n} X_i \sim \mathcal{N}\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

Virus infections

Suppose you are working with the WHO to plan a response to the initial conditions of a virus. There are two exposed groups:

- G1: 200 people, each independently infected with $p_1 = 0.1$
- G2: 100 people, each independently infected with $p_2 = 0.4$

What is $P(\text{people infected} \geq 55)$?

1. Define RVs & state goal

Let $A =$ # infected in G1.

$A \sim \text{Bin}(200, 0.1)$

$B =$ # infected in G2.

$B \sim \text{Bin}(100, 0.4)$

Want: $P(A + B \geq 55)$

Strategy:

A. Dance, Dance, Convolution
B. Sum of indep. Binomials
C. (approximate) Sum of indep. Poissons
D. (approximate) Sum of indep. Normals
E. None/other

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What is $P(\text{people infected} \geq 55)$?

1. Define RVs & state goal

Let $A =$ # infected in G1.

$A \sim \text{Bin}(200, 0.1)$

$B =$ # infected in G2.

$B \sim \text{Bin}(100, 0.4)$

Want: $P(A + B \geq 55)$

Strategy:

A. Dance, Dance, Convolution
B. Sum of indep. Binomials
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Virus infections

Suppose you are working with the WHO to plan a response to the initial conditions of a virus. There are two exposed groups:

- G1: 200 people, each independently infected with $p_1 = 0.1$
- G2: 100 people, each independently infected with $p_2 = 0.4$

What is $P(\text{people infected} \geq 55)$?

1. Define RVs & state goal

Let A = # infected in G1.
$A \sim \text{Bin}(200, 0.1)$
B = # infected in G2.
$B \sim \text{Bin}(100, 0.4)$

Want: $P(A + B \geq 55)$

2. Approximate as sum of Normals

$A \approx X \sim \mathcal{N}(20, 18)$ $\quad B \approx Y \sim \mathcal{N}(40, 24)$

$P(A + B \geq 55) \approx P(X + Y \geq 54.5)$ continuity correction

3. Solve

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Let $W = X + Y \sim \mathcal{N}(20 + 40 = 60, 18 + 24 = 42)$

$$P(W \geq 54.5) = 1 - \Phi\left(\frac{54.5 - 60}{\sqrt{42}}\right) \approx 1 - \Phi(-0.85)$$

$$\approx 0.8023$$

Linear transforms vs. independence

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = X + X$. What is the distribution of Y?

- Are both approaches valid?

Independent RVs approach

Let $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$
be independent.
Then $Y = X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Linear transform approach

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.
If $Y = aX + b$,
then $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

Linear transforms vs. independence ⚠️

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$Y = X + X$

$X + X \sim \mathcal{N}(\mu + \mu, \sigma^2 + \sigma^2)$

$Y \sim \mathcal{N}(2\mu, 2\sigma^2)$?

X is NOT independent of X!

Linear transform approach ✅

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.
If $Y = aX + b$,
then $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

$Y = 2X$

$Y \sim \mathcal{N}(2\mu, 4\sigma^2)$

🤔

Motivating idea: Zero sum games

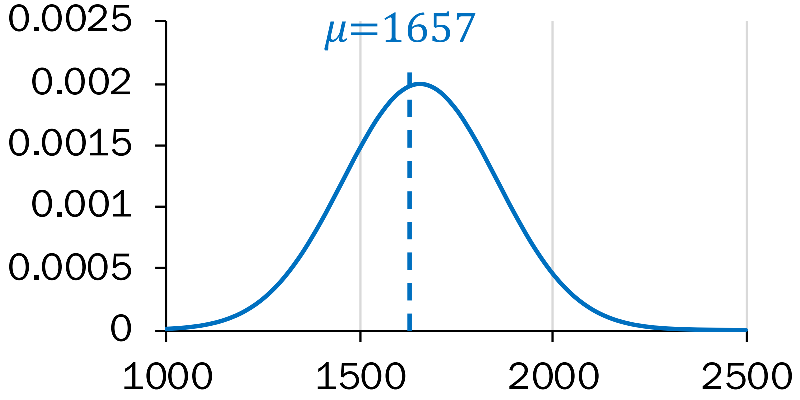
Want: $P(\text{Warriors win}) = P(A_W > A_B)$
$= P(A_W - A_B > 0)$

Assume A_W, A_B are independent.
Let $D = A_W - A_B$.

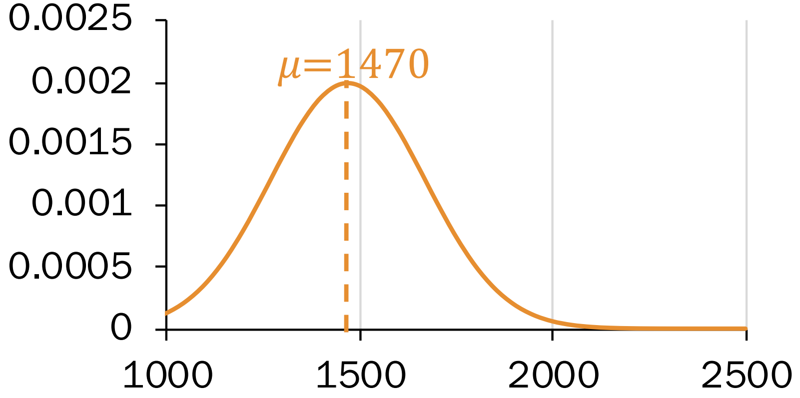
What is the distribution of D?

A. $D \sim \mathcal{N}(1657 - 1470,\ 200^2 - 200^2)$
B. $D \sim \mathcal{N}(1657 - 1470,\ 200^2 + 200^2)$
C. $D \sim \mathcal{N}(1657 + 1470,\ 200^2 + 200^2)$
D. Dance, Dance, Convolution
E. None/other

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponents $A_B \sim \mathcal{N}(S = 1470, 200^2)$



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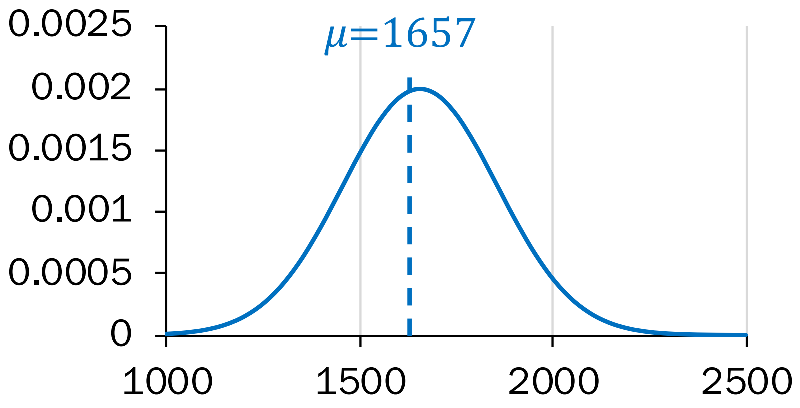

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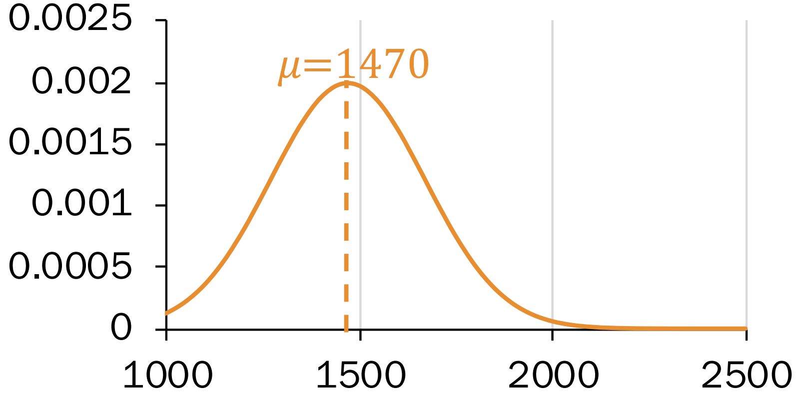


If $X \sim \mathcal{N}(\mu_1, \sigma^2)$,
then $(-X) \sim \mathcal{N}(-\mu, (-1)^2\sigma^2 = \sigma^2)$

Motivating idea: Zero sum games

Want: $P(\text{Warriors win}) = P(A_W > A_B)$
$= P(A_W - A_B > 0)$

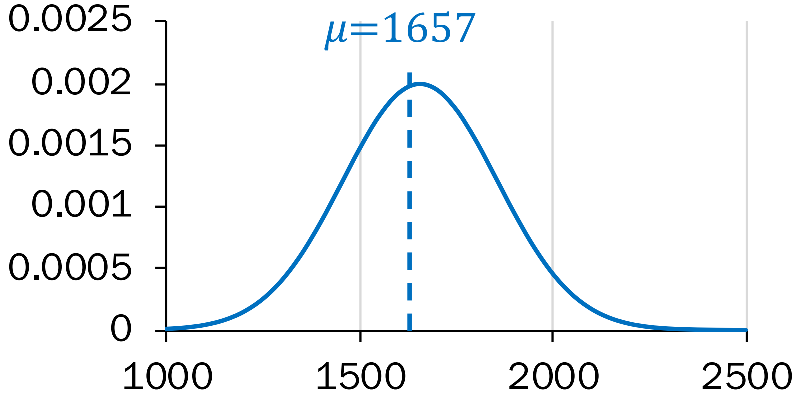
Assume A_W, A_B are independent.
Let $D = A_W - A_B$.

$$D \sim \mathcal{N}(1657 - 1470,\ \ 200^2 + 200^2)$$
$$\sim \mathcal{N}(187, 2 \cdot 200^2) \qquad \sigma \approx 283$$

$$P(D > 0) = 1 - F_D(0) = 1 - \Phi\left(\frac{0 - 187}{283}\right)$$
$$\approx 0.7454$$

Compare with 0.7488, calculated by sampling!

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



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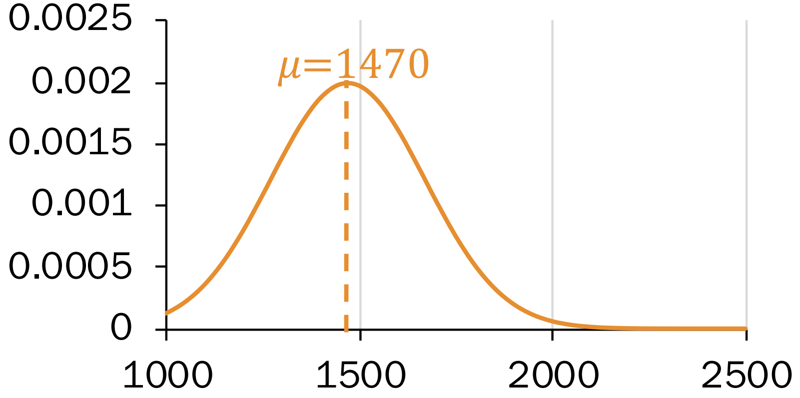

Today's plan

Independent RVs

Sum of independent RVs

- ✅ Binomial
- Convolution
- ✅ Poisson
- ✅ Normal
- ⚠️ Uniform

Expectation of sum of RVs (next class)

Dance, Dance, Convolution Extreme

For independent discrete random variables X and Y:

$$P(X+Y=n)=\sum_k P(X=k)P(Y=n-k)$$

the convolution of p_X and p_Y

For independent continuous random variables X and Y:

$$f_{X+Y}(\alpha)=\int_{-\infty}^{\infty} f_X(x)f_Y(\alpha-x)dx$$

the convolution of f_X and f_Y

Sum of independent Uniforms

X and Y independent + continuous $\quad f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x)\, dx$

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent random variables.

What is the distribution of $X + Y$, f_{X+Y}?

$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} \underbrace{f_X(k) f_Y(\alpha - k)}\, dk$$

$f_X(k) f_Y(\alpha - k) = 1$ when: (select one)

A. between 0 and 1
B. $0 \leq k \leq 1$
C. $0 \leq \alpha - k \leq 1$
D. $0 \leq \alpha \leq 2$
E. Other

Sum of independent Uniforms

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$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(k) f_Y(\alpha - k) dk$$

$f_X(k) f_Y(\alpha - k) = 1$:

$0 \le \alpha \le 2$

$0 \le k \le 1$

$0 \le \alpha - k \le 1$

$\alpha - 1 \le k \le \alpha$

The precise integration bounds on k depend on α.

What are the bounds on k when:

1. $\alpha = 1/2$? $\quad 0 \le k \le \alpha$ $\quad \int_{k=0}^{\alpha} 1 dk = \alpha = 1/2$
2. $\alpha = 3/2$? $\quad \alpha - 1 \le k \le 1$ $\quad \int_{k=\alpha-1}^{1} 1 dk = \alpha - 1 = 1/2$
3. $\alpha = 1$? $\quad 0 \le k \le \alpha$ $\quad \int_{k=0}^{\alpha} 1 dk = \alpha = 1$

(the other bound works too)

Sum of independent Uniforms

X and Y independent + continuous $\quad f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x)\, dx$

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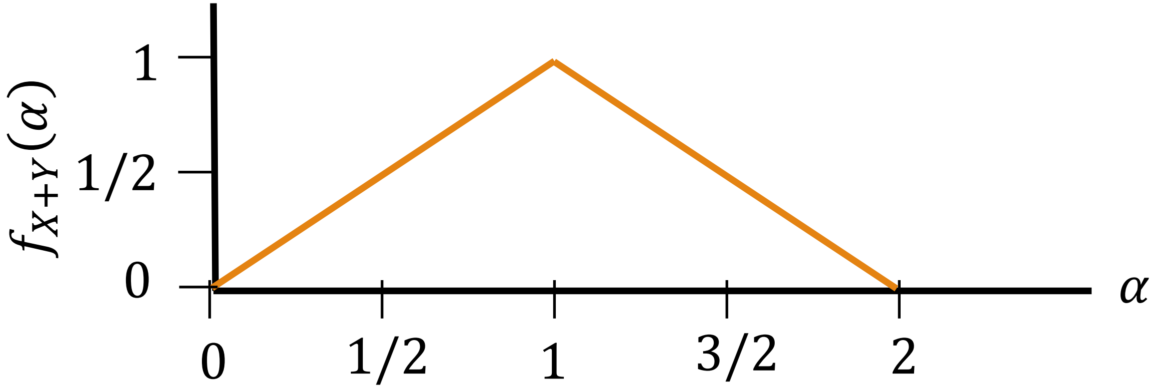
$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(k) f_Y(\alpha - k) dk$$

$f_X(k) f_Y(\alpha - k) = 1$ when:

$$0 \le \alpha \le 2$$
$$0 \le k \le 1$$
$$0 \le \alpha - k \le 1$$
$$\alpha - 1 \le k \le \alpha$$

The precise integration bounds on k depend on α.



$$f_{X+Y}(\alpha) = \begin{cases} \alpha & 0 \le a \le 1 \\ 2 - a & 1 \le a \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Today's plan

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Expectation of sum of RVs (next class)

Properties of Expectation, extended to two RVs

1. Linearity:

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

2. Expectation of a sum = sum of expectation:

$$E[X + Y] = E[X] + E[Y]$$

(we've seen this; we'll prove this next)

3. Unconscious statistician:

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$$

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

Proof of expectation of a sum of RVs

$E[X+Y] = E[X] + E[Y]$

$$E[X+Y] = E[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y) = \sum_x \sum_y (x+y) p_{X,Y}(x,y)$$

LOTUS,
$g(X,Y) = X + Y$

$$= \sum_x \sum_y x p_{X,Y}(x,y) + \sum_x \sum_y y p_{X,Y}(x,y)$$

$$= \sum_x x \sum_y p_{X,Y}(x,y) + \sum_y y \sum_x p_{X,Y}(x,y)$$

Linearity of summations
(cont. case: linearity of integrals)

$$= \sum_x x p_X(x) + \sum_y y p_Y(y)$$

Marginal PMFs for X and Y

$$= E[X] + E[Y]$$

Even if the **joint distribution is unknown**, you can calculate the **expectation of sum as sum of expectations**.

Example: $E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$ despite dependent trials X_i

Expectations of common RVs

Review

$$X \sim \text{Bin}(n, p) \quad E[X] = np$$

$$X = \sum_{i=1}^{n} X_i$$

Let $X_i = i$th trial is heads
$X_i \sim \text{Ber}(p), E[X_i] = p$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np$$

Expectations of common RVs

$X \sim \text{Bin}(n, p) \quad E[X] = np$

$X = \sum_{i=1}^{n} X_i$ Let $X_i = i$th trial is heads
$X_i \sim \text{Ber}(p), E[X_i] = p$

$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np$

$Y \sim \text{NegBin}(r, p) \quad E[Y] = \frac{r}{p}$

Suppose:

$$Y = \sum_{i=1}^{?} Y_i$$

How should we define Y_i?

A. $Y_i = i$th trial is heads. $Y_i \sim \text{Ber}(p), i = 1, \ldots, n$

B. $Y_i =$ # trials to get ith success (after $(i-1)$th success)
$Y_i \sim \text{Geo}(p), i = 1, \ldots, r$

C. $Y_i =$ # successes in n trials
$Y_i \sim \text{Bin}(n, p), i = 1, \ldots, r$, we look for $P(Y_i = 1)$

🤔

Expectations of common RVs

$X \sim \text{Bin}(n, p) \quad E[X] = np$

$X = \sum_{i=1}^{n} X_i$ Let X_i = ith trial is heads
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$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np$$

$Y \sim \text{NegBin}(r, p) \quad E[Y] = \frac{r}{p}$

$Y = \sum_{i=1}^{r} Y_i$ Let Y_i = # trials to get ith success (after $(i-1)$th success)
$Y_i \sim \text{Geo}(p), E[Y_i] = \frac{1}{p}$

$$E[Y] = E\left[\sum_{i=1}^{r} Y_i\right] = \sum_{i=1}^{r} E[Y_i] = \sum_{i=1}^{r} \frac{1}{p} = \frac{r}{p}$$