14: Conditional Distributions

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October 23, 2019

Sum of independent random variables

Review

$X \sim \text{Bin}(n_1, p), Y \sim \text{Bin}(n_2, p)$
X, Y independent
$\Rightarrow$ $X + Y \sim \text{Bin}(n_1 + n_2, p)$

$X \sim \text{Poi}(\lambda_1), Y \sim \text{Poi}(\lambda_2)$
X, Y independent
$\Rightarrow$ $X + Y \sim \text{Poi}(\lambda_1 + \lambda_2)$

$X \sim \mathcal{N}(\mu_1, \sigma_1^2)$
$Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$
X, Y independent
$\Rightarrow$ $X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Note: these also hold in the general case ($\geq$ 2 variables)

Quick questions

1. X and Y have the following joint PDF:
   Are X and Y independent?

$$f_{X,Y}(x,y) = \frac{8}{3}x^3y$$

where $0 < x < 1, 1 < y < 2$

2. Let $X \sim \text{Bin}(30, 0.01)$ and $Y \sim \text{Bin}(50, 0.02)$ be independent RVs. Can we use sum of independent Poisson RVs to approximate $P(X + Y = 1)$?

Quick questions

Review

1. X and Y have the following joint PDF: Are X and Y independent?

$$f_{X,Y}(x, y) = \frac{8}{3}x^3 y$$
where $0 < x < 1, 1 < y < 2$

Separable functions

$g(x) = C_1 x^3$
$h(y) = C_2 y$
, where C_1, C_2 are constants

2. Let $A \sim \text{Bin}(30, 0.01)$ and $B \sim \text{Bin}(50, 0.02)$ be independent RVs. Can we use sum of independent Poisson RVs to approximate $P(A + B = 2)$?

Quick questions

2. Let $A \sim \text{Bin}(30, 0.01)$ and $B \sim \text{Bin}(50, 0.02)$be independent RVs. Can we use sum of independent Poisson RVs to approximate $P(A + B = 2)$?

Sol 1: Approximate as sum of Poissons

$$A \approx X \sim \text{Poi}(\lambda_1 = 30 \cdot 0.01 = 0.3)$$
$$B \approx Y \sim \text{Poi}(\lambda_2 = 50 \cdot 0.02 = 1)$$
$$X + Y \sim \text{Poi}(\lambda = \lambda_1 + \lambda_2 = 1.3)$$

$$P(A + B = 2) \approx P(X + Y = 2) = \frac{\lambda^2}{2!} e^{-\lambda}$$
$$\approx 0.2302$$

Sol 2: No approximation

$$P(A + B = 2) = \sum_{k=0}^{2} P(A = k) P(B = 2 - k)$$
$$= \sum_{k=0}^{2} \binom{30}{k} 0.01^k (0.99)^{30-k} \binom{50}{2-k} 0.02^{2-k} 0.98^{48+k} \approx 0.2327$$

Today's plan

Sum of two Uniform independent RVs

Expectation of sum of two RVs

midterm content up to here

Discrete conditional distributions

Ratio of probabilities interlude

Continuous conditional distributions

Dance, Dance, Convolution Extreme

Recall for independent discrete random variables X and Y:

$$P(X+Y=n)=\sum_{k} P(X=k)P(Y=n-k)$$

the convolution of p_X and p_Y

Dance, Dance, Convolution Extreme

Recall for independent discrete random variables X and Y:

$$P(X+Y=n)=\sum_k P(X=k)P(Y=n-k)$$

the convolution of p_X and p_Y

For independent continuous random variables X and Y:

$$f_{X+Y}(\alpha)=\int_{-\infty}^{\infty} f_X(x)f_Y(\alpha-x)dx$$

the convolution of f_X and f_Y

Sum of independent Uniforms

X and Y independent + continuous $\quad f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x)\, dx$

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent random variables.

What is the distribution of $X + Y$, f_{X+Y}?

$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} \underbrace{f_X(k) f_Y(\alpha - k)}\, dk$$

$f_X(k) f_Y(\alpha - k) = 1$ when: (select one)

A. always
B. $0 \le k \le 1$
C. $0 \le \alpha - k \le 1$
D. $0 \le \alpha \le 2$
E. Other

Sum of independent Uniforms

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$f_X(k) f_Y(\alpha - k) = 1$ when:

$0 \leq \alpha \leq 2$ and
$0 \leq k \leq 1$ and
$0 \leq \alpha - k \leq 1 \Rightarrow \alpha - 1 \leq k \leq \alpha$

The precise integration bounds on k depend on α.

What are the bounds on k when:

1. $\alpha = 1/2$?

2. $\alpha = 3/2$?

Sum of independent Uniforms

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The precise integration bounds on k depend on α.

What are the bounds on k when:

1. $\alpha = 1/2$?

   $0 \le k \le \alpha$

   $\int_{k=0}^{\alpha} 1 dk \quad = \alpha \quad = 1/2$

2. $\alpha = 3/2$?

   $\alpha - 1 \le k \le 1$

   $\int_{k=\alpha-1}^{1} 1 dk = 2 - \alpha = 1/2$

Sum of independent Uniforms

X and Y independent + continuous $\quad f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x)\, dx$

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent random variables.

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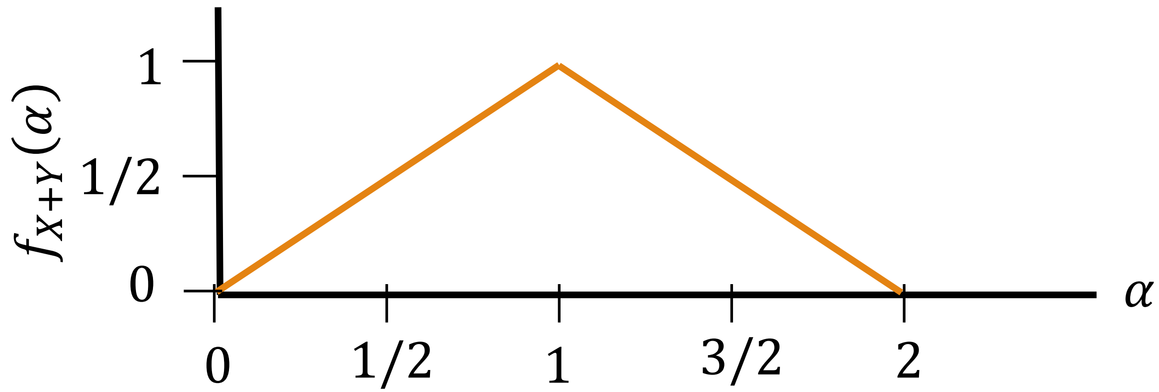

$f_X(k) f_Y(\alpha - k) = 1$ when:

$0 \le \alpha \le 2$ and
$0 \le k \le 1$ and
$0 \le \alpha - k \le 1 \Rightarrow \alpha - 1 \le k \le \alpha$

The precise integration bounds on k depend on α.

$$f_{X+Y}(\alpha) = \begin{cases} a & 0 \le a \le 1 \\ 2 - a & 1 \le a \le 2 \\ 0 & \text{otherwise} \end{cases}$$

whew

Today's plan

Sum of two Uniform independent RVs

Expectation of sum of two RVs

Discrete conditional distributions

Continuous conditional distributions

Properties of Expectation, extended to two RVs

1. Linearity:

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

2. Expectation of a sum = sum of expectation:

$$E[X + Y] = E[X] + E[Y]$$

(we've seen this; we'll prove this next)

3. Unconscious statistician:

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$$

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

Proof of expectation of a sum of RVs

$E[X+Y] = E[X] + E[Y]$

$$E[X+Y] = \sum_x \sum_y (x+y) p_{X,Y}(x,y)$$

LOTUS,
$g(X,Y) = X + Y$

$$= \sum_x \sum_y x p_{X,Y}(x,y) + \sum_x \sum_y y p_{X,Y}(x,y)$$

$$= \sum_x x \sum_y p_{X,Y}(x,y) + \sum_y y \sum_x p_{X,Y}(x,y)$$

Linearity of summations
(cont. case: linearity of integrals)

$$= \sum_x x p_X(x) + \sum_y y p_Y(y)$$

Marginal PMFs for X and Y

$$= E[X] + E[Y]$$

Even if the **joint distribution is unknown**, you can calculate the **expectation of sum as sum of expectations**.

Example: $E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$ despite dependent trials X_i

Expectations of common RVs

Review

$$X \sim \text{Bin}(n, p) \quad E[X] = np$$

$$X = \sum_{i=1}^{n} X_i$$

Let $X_i = i$th trial is heads
$X_i \sim \text{Ber}(p), E[X_i] = p$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np$$

Expectations of common RVs

$X \sim \text{Bin}(n, p) \quad E[X] = np$

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$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np$$

$Y \sim \text{NegBin}(r, p) \quad E[Y] = \frac{r}{p}$

Suppose:

$$Y = \sum_{i=1}^{?} Y_i$$

How should we define Y_i?

A. Y_i = ith trial is heads. $Y_i \sim \text{Ber}(p), i = 1, \dots, n$

B. Y_i = # trials to get ith success (after $(i-1)$th success)
$Y_i \sim \text{Geo}(p), i = 1, \dots, r$

C. Y_i = # successes in n trials
$Y_i \sim \text{Bin}(n, p), i = 1, \dots, r$, we look for $P(Y_i = 1)$

Expectations of common RVs

$X \sim \text{Bin}(n,p) \quad E[X] = np$

$$X = \sum_{i=1}^{n} X_i$$

Let X_i = ith trial is heads
$X_i \sim \text{Ber}(p), E[X_i] = p$

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Expectations of common RVs

$X \sim \text{Bin}(n, p) \quad E[X] = np$

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$X_i \sim \text{Ber}(p), E[X_i] = p$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np$$

$Y \sim \text{NegBin}(r, p) \quad E[Y] = \frac{r}{p}$

$Y = \sum_{i=1}^{r} Y_i$ Let Y_i = # trials to get ith success (after $(i-1)$th success)
$Y_i \sim \text{Geo}(p), E[Y_i] = \frac{1}{p}$

$$E[Y] = E\left[\sum_{i=1}^{r} Y_i\right] = \sum_{i=1}^{r} E[Y_i] = \sum_{i=1}^{r} \frac{1}{p} = \frac{r}{p}$$

(beginning of non-midterm content)

Break for jokes/ announcements

Announcements

Midterm study tips

Easy to do: Charts/equations (essential, but not hard)

Harder to do: Glean common strategies from practice exams/section handouts/psets (top priority: reflect and form links)

No matter what: Start early. Take breaks. Stay hydrated. Sleep.

Concept checks

Week 5's: Wednesday 10/30 1pm
Includes mid-quarter feedback

Today's plan

Sum of two Uniform independent RVs

Expectation of sum of two RVs

Discrete conditional distributions

Continuous conditional distributions

CS109 roadmap

Multiple events:

intersection	conditional probability	independence
$P(E \cap F) = P(EF)$	$P(E\|F) = \frac{P(EF)}{P(F)}$	$P(EF) = P(E)P(F)$

Joint (Multivariate) distributions

joint PMF/PDF	conditional distributions?	independent RVs
$p_{X,Y}(x,y)$ $f_{X,Y}(x,y)$	today	sum of independent RVs

Model ALL the things!

Discrete conditional distributions

Recall the definition of the conditional probability of event E given event F:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

For discrete random variables X and Y, the **conditional PMF** of X given Y is

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Different notation, same idea:

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

Quick check

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Number or function?

1. $P(X = 2|Y = 5)$

   number

2. $P(X = x|Y = 5)$

   1-D function

3. $P(X = 2|Y = y)$

   1-D function

4. $P(X = x|Y = y)$

   2-D function

True or false?

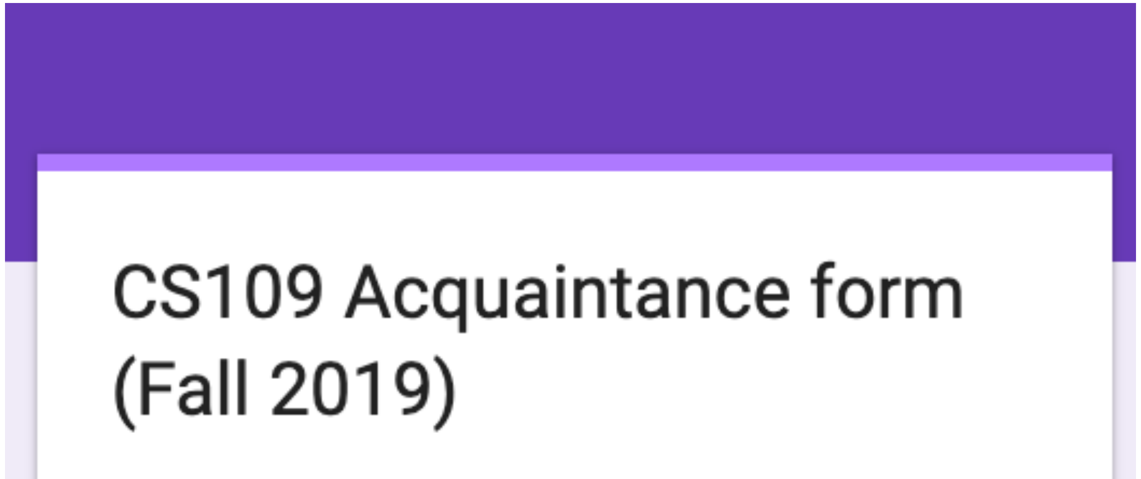
5. $\sum_x P(X = x|Y = 5) = 1$ true

6. $\sum_y P(X = 2|Y = y) = 1$ false

7. $\sum_x \left(\sum_y P(X = x|Y = y)P(Y = y) \right) = 1$

   true

Discrete probabilities of CS109


CS109 Acquaintance form
(Fall 2019)

Each student responds with (major X, year Y, `bool` pokemon master M)

$M = 1$ (yes) or $M = 0$ (no)

Discrete probabilities of CS109

Each student responds with (major X, year Y, `bool` pokemon master M):

Joint PMF of X, Y, M

		CS $X=1$	SymSys/ MCS/EE $X=2$	Other Eng/Sci /Math $X=3$	Hum/ SocSci/ Ling $X=4$	Double major $X=5$	Undec. $X=6$
$M=0$	$Y=1$	.006	.000	.000	.000	.000	.000
	$Y=2$	.155	.069	.034	.006	.023	.029
	$Y=3$	.092	.063	.023	.006	.006	.000
	$Y=4$	.017	.029	.011	.006	.000	.000
	$Y\geq 5$	.029	.006	.011	.006	.000	.000
$M=1$	$Y=1$	.000	.000	.000	.000	.000	.000
	$Y=2$	.126	.040	.017	.017	.000	.017
	$Y=3$	.046	.040	.006	.011	.000	.006
	$Y=4$	.006	.006	.000	.000	.000	.000
	$Y\geq 5$	.006	.000	.017	.011	.000	.000

$P(Y \geq 5, X = 1, M = 1)$

Joint PMF of X, M

	CS $X=1$	SymSys/ MCS/EE $X=2$	Other Eng/Sci/ Math $X=3$	Hum/ SocSci/ Ling $X=4$	Double major $X=5$	Undec. $X=6$
$M=0$	.299	.167	.080	.023	.029	.029
$M=1$	.184	.086	.040	.040	.000	.023

$P(X = 1, M = 1) = 0.18$

Discrete probabilities of CS109

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

The below tables are conditional probability tables for the conditional PMFs $P(M = m|X = x)$ and $P(X = x|M = m)$.

1. Which table is which?
2. Fill in the missing probability.

Joint PMF of X, M

	CS $X = 1$	SymSys/ MCS/EE $X = 2$	Other Eng/Sci/ Math $X = 3$	Hum/ SocSci/ Ling $X = 4$	Double major $X = 5$	Undec. $X = 6$
$M = 0$	.299	.167	.080	.023	.029	.029
$M = 1$	.184	.086	.040	.040	.000	.023

$$P(X = 1, M = 1) = 0.18$$

	CS $X = 1$	SymSys/ MCS/EE $X = 2$	Other Eng/Sci/ Math $X = 3$	Hum/ SocSci/ Ling $X = 4$	Double major $X = 5$	Undec. $X = 6$
$M = 0$	.619	.659	.667	.364	1.000	.556
$M = 1$	.381	.341	.333	.636	.000	.444

	CS $X = 1$	SymSys/ MCS/EE $X = 2$	Other Eng/Sci/ Math $X = 3$	Hum/ SocSci/ Ling $X = 4$	Double major $X = 5$	Undec. $X = 6$
$M = 0$	.477	.266	.128	.037	.046	.046
$M = 1$		.231	.108	.108	.000	.062

Discrete probabilities of CS109

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

The below tables are conditional probability tables for the conditional PMFs $P(M = m|X = x)$ and $P(X = x|M = m)$.

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Joint PMF of X, M

	CS $X = 1$	SymSys/ MCS/EE $X = 2$	Other Eng/Sci/ Math $X = 3$	Hum/ SocSci/ Ling $X = 4$	Double major $X = 5$	Undec. $X = 6$
$M = 0$	0.30	0.17	0.08	0.02	0.03	0.03
$M = 1$	0.18	0.09	0.04	0.04	0.00	0.02

$P(X = 1, M = 1) = 0.18$

Conditional PMF $P(M = m|X = x)$

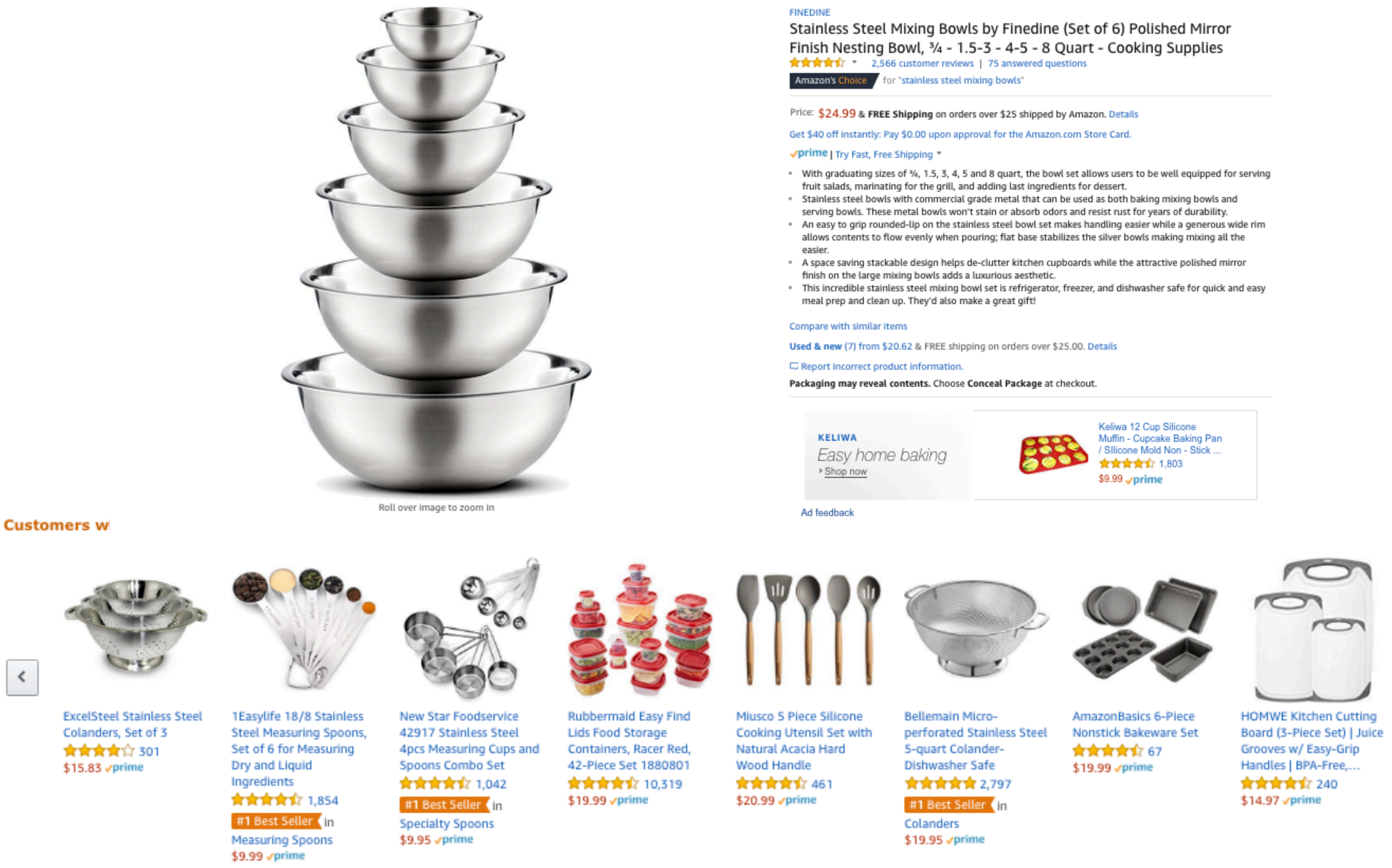
	CS $X = 1$	SymSys/ MCS/EE $X = 2$	Other Eng/Sci/ Math $X = 3$	Hum/ SocSci/ Ling $X = 4$	Double major $X = 5$	Undec. $X = 6$
$M = 0$	.619	.659	.667	.364	1.000	.556
$M = 1$	.381	.341	.333	.636	.000	.444

Conditional PMF $P(X = x|M = m)$

	CS $X = 1$	SymSys/ MCS/EE $X = 2$	Other Eng/Sci/ Math $X = 3$	Hum/ SocSci/ Ling $X = 4$	Double major $X = 5$	Undec. $X = 6$
$M = 0$	.477	.266	.128	.037	.046	.046
$M = 1$	.492	.231	.108	.108	.000	.062

Be clear about which probabilities should sum to one.

Extended to Amazon



P(bought item X | bought item Y)

Today's plan

Sum of two Uniform independent RVs

Expectation of sum of two RVs

Discrete conditional distributions

Ratio of probabilities interlude

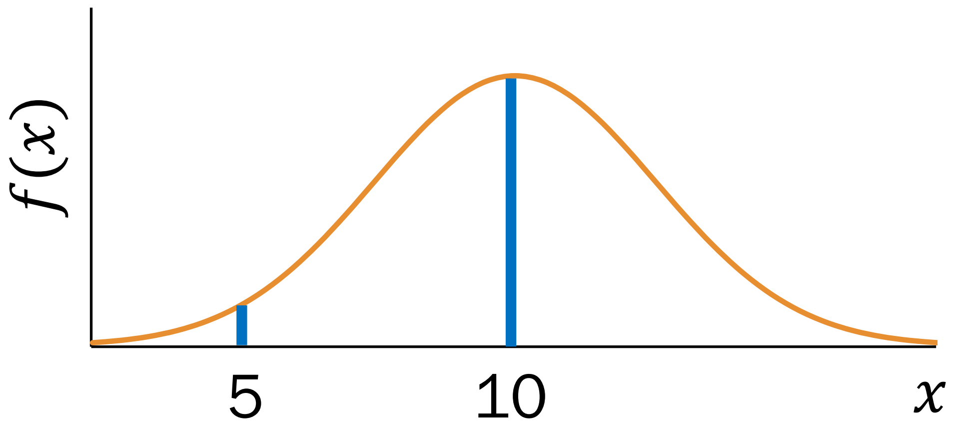
Continuous conditional distributions

Relative probabilities of continuous random variables

Let X = time to finish problem set 2.

Suppose $X \sim \mathcal{N}(10, 2)$.

How much *more likely* are you to complete in 10 hours than 5 hours?



$$\frac{P(X=10)}{P(X=5)} =$$

A. 0/0 = undefined

B. $\frac{f(10)}{f(5)}$

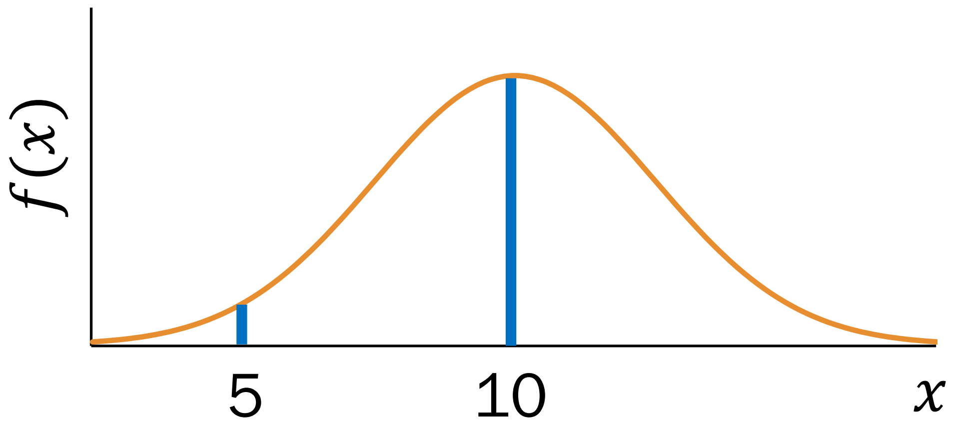
C. stay healthy

Relative probabilities of continuous random variables

Let X = time to finish problem set 2.

Suppose $X \sim \mathcal{N}(10, 2)$.

How much *more likely* are you to complete in 10 hours than 5 hours?



$$\frac{P(X = 10)}{P(X = 5)} =$$

A. 0/0 = undefined

B. $\dfrac{f(10)}{f(5)}$

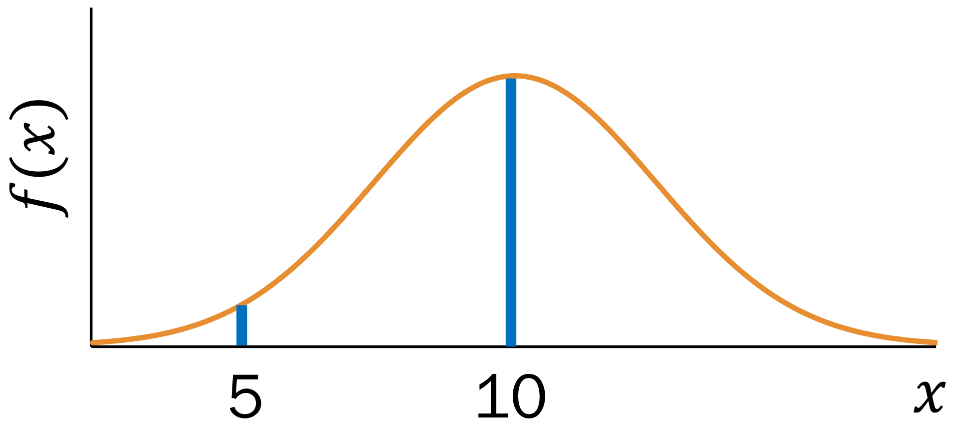
C. stay healthy

Relative probabilities of continuous random variables

Let X = time to finish problem set 2.

Suppose $X \sim \mathcal{N}(10,2)$.

How much *more likely* are you to complete in 10 hours than 5 hours?



$$\frac{P(X=10)}{P(X=5)} = \frac{f(10)}{f(5)}$$

$$P(X=a) = P\left(a - \frac{\varepsilon}{2} \le X \le a + \frac{\varepsilon}{2}\right) = \int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} f(x)dx \approx \varepsilon f(a)$$

Therefore $\dfrac{P(X=a)}{P(X=b)} = \dfrac{\varepsilon f(a)}{\varepsilon f(b)} = \dfrac{f(a)}{f(b)}$

$$= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(10-\mu)^2}{2\sigma^2}}}{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(5-\mu)^2}{2\sigma^2}}}$$

$$= \frac{e^{-\frac{(10-10)^2}{2\cdot 2}}}{e^{-\frac{(5-10)^2}{2\cdot 2}}} = \frac{e^0}{e^{-\frac{25}{4}}} = 518$$

👉 Ratios of PDFs are meaningful

Today's plan

Sum of two Uniform independent RVs

Expectation of sum of two RVs

Discrete conditional distributions

Ratio of probabilities interlude

Continuous conditional distributions

Continuous conditional distributions

For continuous RVs X and Y, the **conditional PDF** of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Intuition: $P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$ ⟺ $f_{X|Y}(x|y)\varepsilon_X = \frac{f_{X,Y}(x,y)\varepsilon_x\varepsilon_y}{f_Y(y)\varepsilon_y}$

Note that conditional PDF $f_{X|Y}$ is a true density:

$$\int_{-\infty}^{\infty} f_x(x|y)dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}dx = \frac{f_Y(y)}{f_Y(y)} = 1$$

Bayes' Theorem with Continuous RVs

For continuous RVs X and Y,

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y) f_Y(y)}{f_X(x)}$$

Intuition:

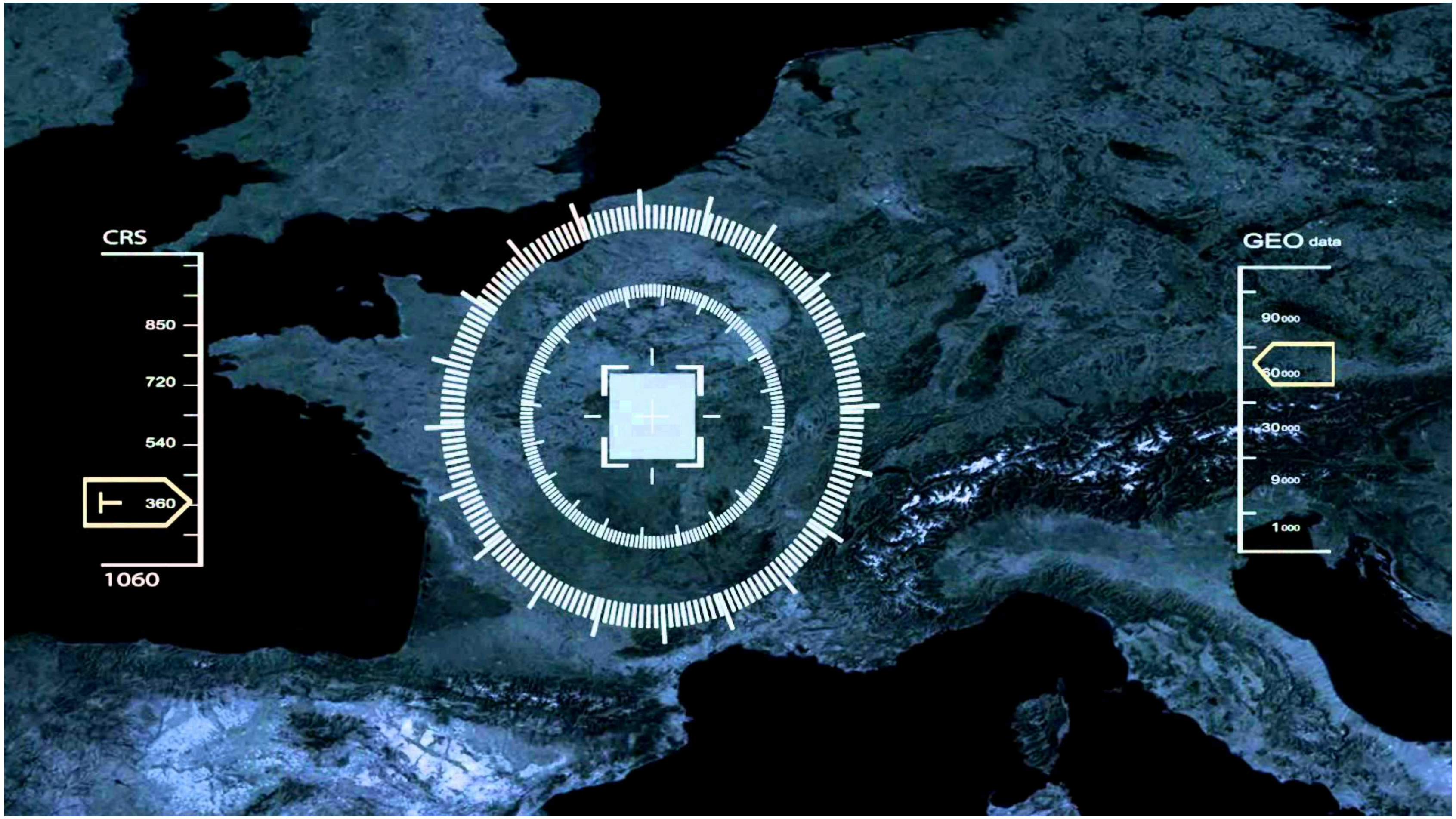
$$P(Y = y|X = x) = \frac{P(X = x|Y = y) P(Y = y)}{P(X = x)}$$

$$\Updownarrow$$

$$f_{Y|X}(y|x)\varepsilon_Y = \frac{\left(f_{X|Y}(x|y)\varepsilon_X\right)\left(f_Y(y)\varepsilon_y\right)}{f_X(x)\varepsilon_X}$$


This example is hard!
Don’t feel discouraged!

Tracking in 2-D space?



You want to know the 2-D location of an object.

Your satellite ping gives you a noisy 1-D measurement of the distance of the object from the satellite (0,0).

Tracking in 2-D space

- You have a prior belief about the 2-D location of an object, (X, Y).
- You observe a noisy distance measurement, $D = 4$.
- What is your updated (posterior) belief of the 2-D location of the object after observing the measurement?

Recall Bayes terminology:

posterior belief

likelihood (of evidence)

prior belief

$$f_{X,Y|D}(x, y|d) = \frac{f_{D|X,Y}(d|x, y) f_{X,Y}(x, y)}{f_D(d)}$$

normalization constant

Tracking in 2-D space

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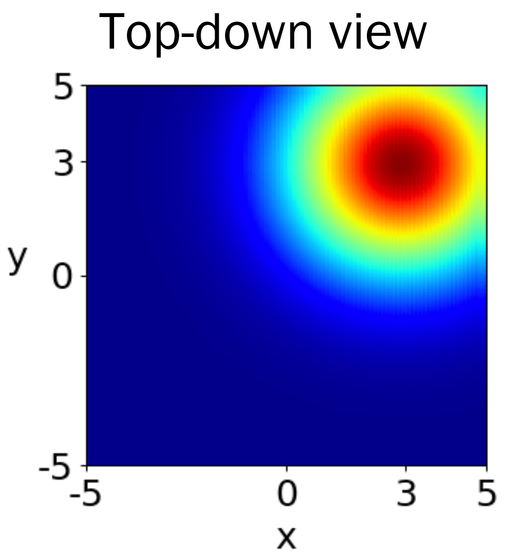
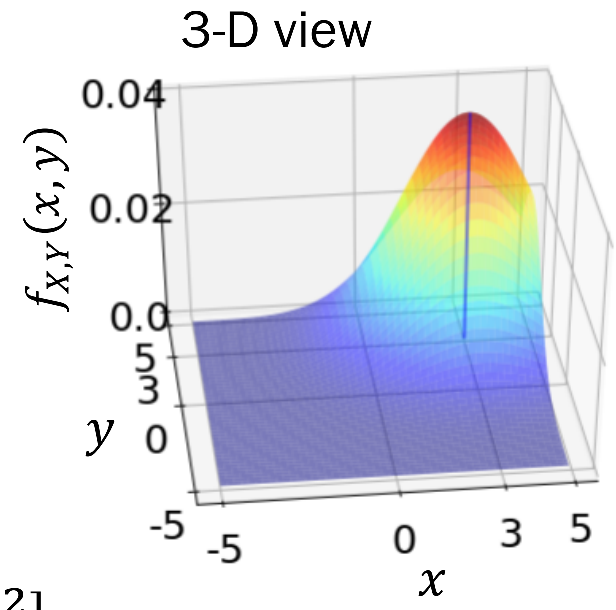
Let $(X, Y) =$ object's 2-D location.
(your satellite is at (0,0)

Suppose the prior distribution is a symmetric bivariate normal distribution:





$$f_{X,Y}(x,y) = \frac{1}{2\pi 2^2} e^{-\frac{[(x-3)^2+(y-3)^2]}{2(2^2)}} = K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}$$

normalizing constant

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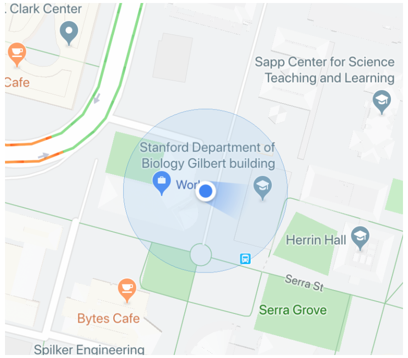
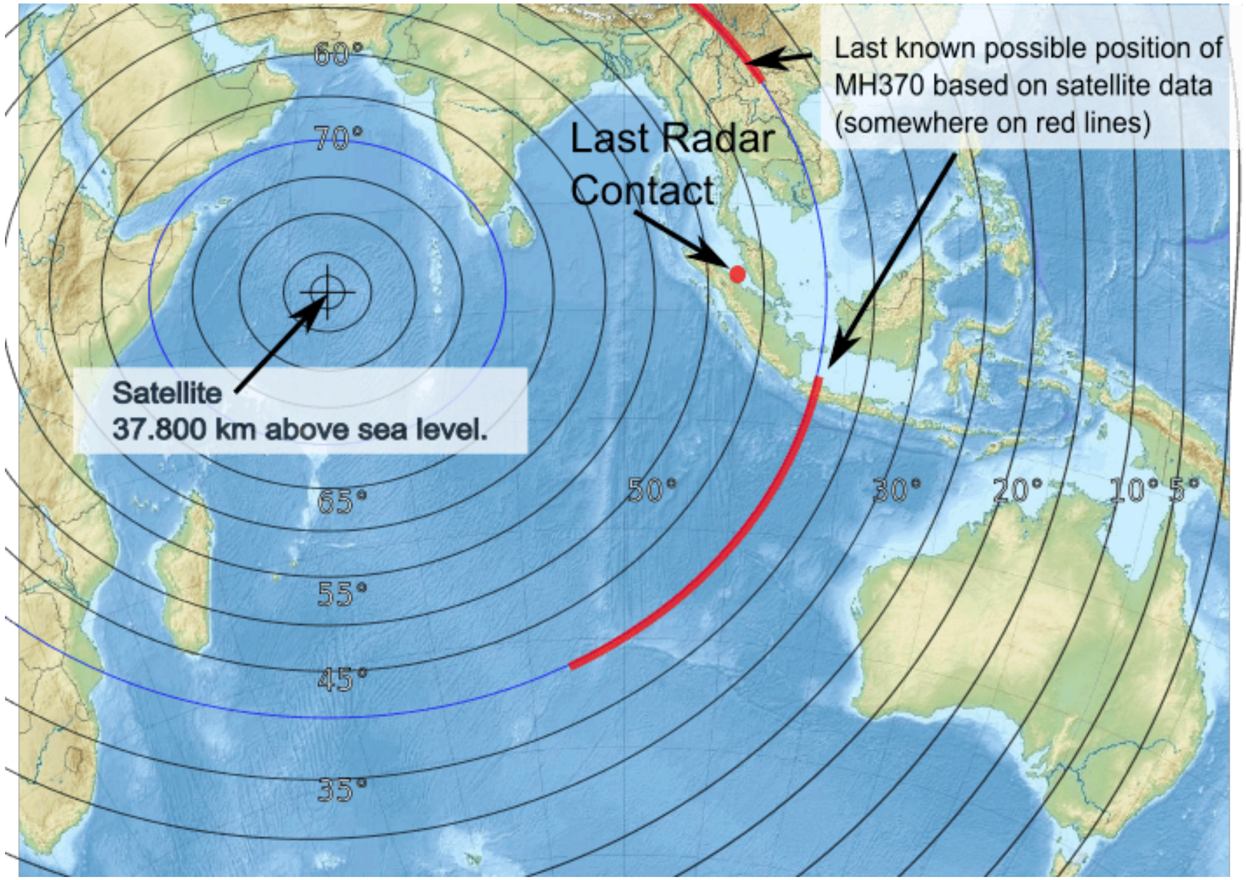


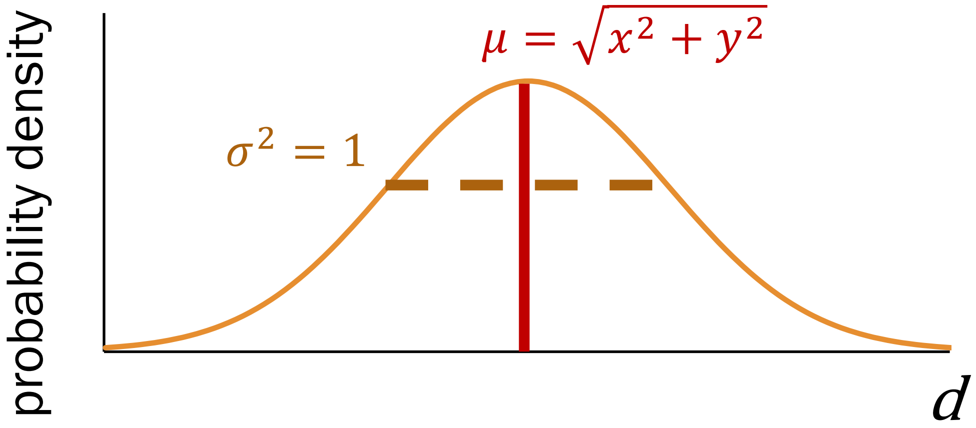
Let D = distance from the satellite (radially).
Suppose you knew your actual position: (x, y).

- D is still noisy! Suppose noise is unit variance: $\sigma^2 = 1$
- On average, D is your actual position: $\mu = \sqrt{x^2 + y^2}$

If you knew your actual location (x, y), you could say how likely a measurement $D = 4$ is!!

Tracking in 2-D space





Distance measurement of a ping is normal with respect to the true location.

Tracking in 2-D space

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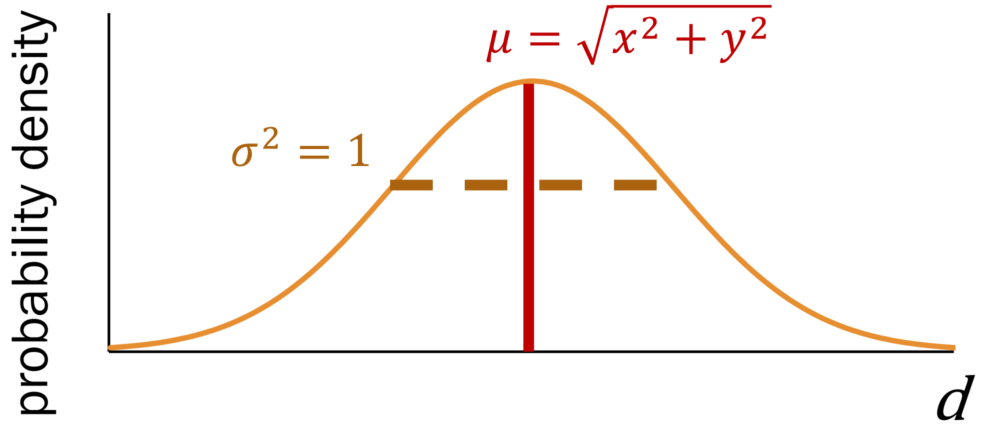

If noise is normal: $$D|X, Y \sim N\left(\mu = \sqrt{x^2 + y^2}, \sigma^2 = 1\right)$$

Distance measurement of a ping is normal with respect to the true location.

Tracking in 2-D space

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If you knew your actual location (x, y), you could say how likely a measurement $D = 4$ is!!

$$D|X,Y \sim \mathcal{N}\left(\mu = \sqrt{x^2 + y^2}, \sigma^2 = 1\right)$$

$$f_{D|X,Y}(D = d|X = x, Y = y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(d-\mu)^2}{2\sigma^2}}$$

substitute μ and σ^2

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-\left(d-\sqrt{x^2+y^2}\right)^2}{2}} = K_2 \cdot e^{\frac{-\left(d-\sqrt{x^2+y^2}\right)^2}{2}}$$

normalizing constant

Deep breath

Tracking in 2-D space

- You have a prior belief about the 2-D location of an object, (X, Y).
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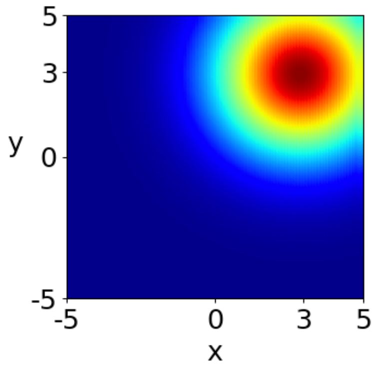
Prior belief



Top-down view

$$f_{X,Y}(x, y) = K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}$$

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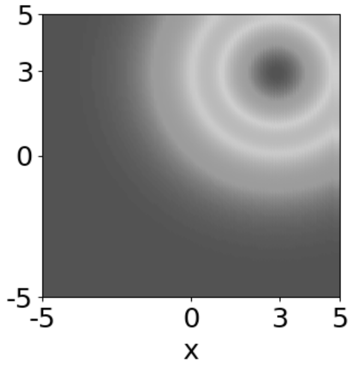

Top-down view

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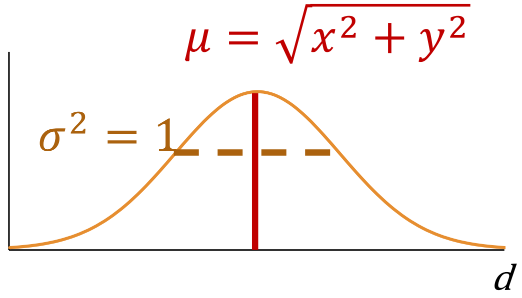
Observation likelihood



$$f_{D|X,Y}(d|x, y) = K_2 \cdot e^{\frac{-\left(d-\sqrt{x^2+y^2}\right)^2}{2}}$$

Tracking in 2-D space

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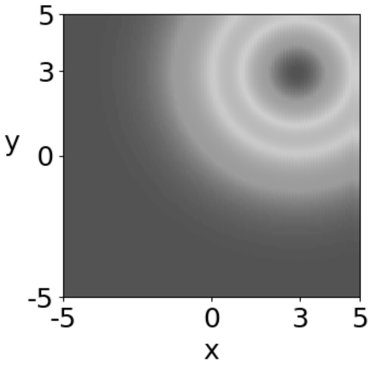
Prior belief



Top-down view

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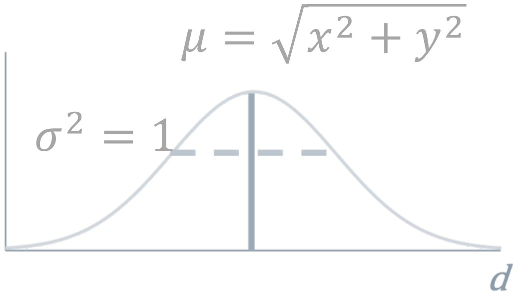
Observation likelihood



$$f_{D|X,Y}(d|x,y) = K_2 \cdot e^{\frac{-\left(d-\sqrt{x^2+y^2}\right)^2}{2}}$$

Posterior belief

$$f_{X,Y|D}(x,y|4) = f_{X,Y|D}(X=x, Y=y|D=4)$$

Tracking in 2-D space

What is your updated (posterior) belief of the 2-D location of the object after observing the measurement?

likelihood of $D = 4$ prior belief

$$f_{X,Y|D}(X = x, Y = y|D = 4) = \frac{f_{D|X,Y}(D = 4|X = x, Y = y) f_{X,Y}(x, y)}{f(D = 4)}$$

Bayes' Theorem

$$= \frac{K_2 \cdot e^{-\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}} \cdot K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}}{f(D = 4)}$$

$$= \frac{K_3 \cdot e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2} + \frac{[(x-3)^2+(y-3)^2]}{8}\right]}}{f(D = 4)}$$

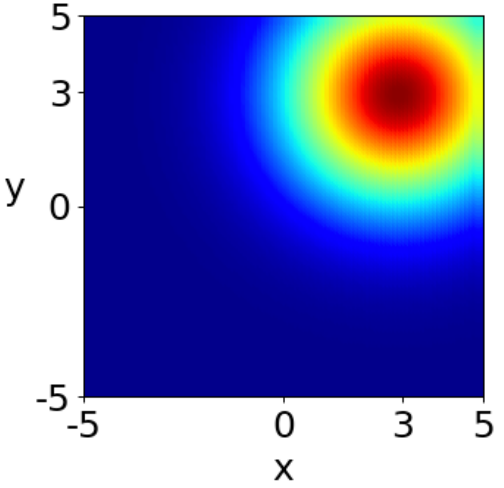
$$= K_4 \cdot e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2} + \frac{[(x-3)^2+(y-3)^2]}{8}\right]}$$

For your notes...

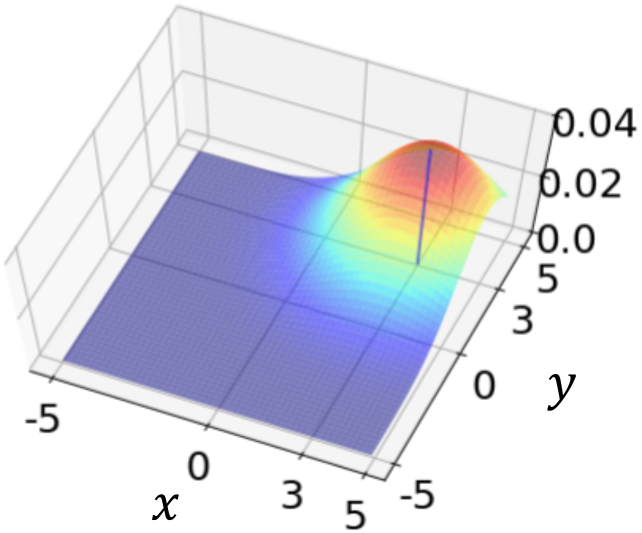
Tracking in 2-D space: Posterior belief

Prior belief

Top-down view



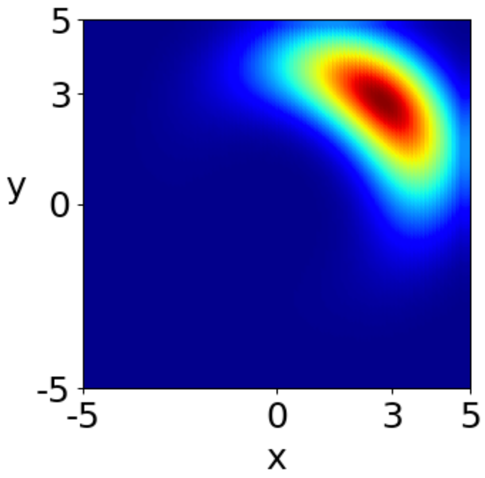
3-D view



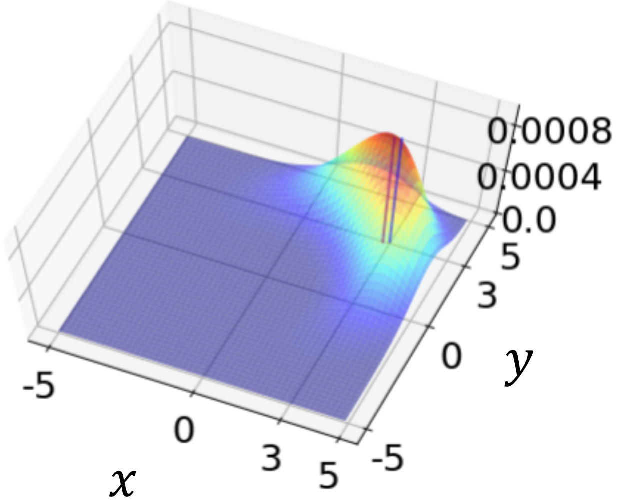
$$f_{X,Y}(x,y) = K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}$$

Posterior belief

Top-down view



3-D view



$$f_{X,Y|D}(x,y|4) =$$

$$K_4 \cdot e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}+\frac{[(x-3)^2+(y-3)^2]}{8}\right]}$$

Good job today