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CS109

Lecture Notes #15
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Covariance

Based on a chapter by Chris Piech

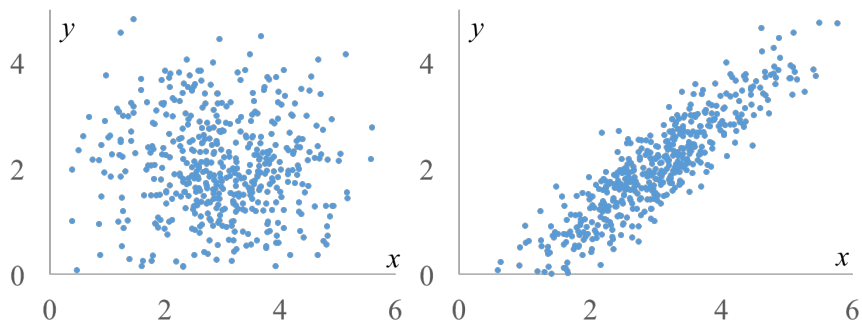
Expectations of Products Lemma

We know that the expectation of the sum of two random variables is equal to the sum of the expectations of the two variables. However, the expectation of the product of two random variables only has a nice decomposition in the case where the random variables are *independent* of one another.

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)] \qquad \textit{if } X \text{ and } Y \text{ are independent}$$

1 Covariance and Correlation

Consider the two multivariate distributions shown bellow. In both images I have plotted one thousand samples drawn from the underlying joint distribution. Clearly the two distributions are different. However, the mean and variance are the same in both the x and the y dimension. What is different?



Covariance is a quantitative measure of the extent to which the deviation of one variable from its mean matches the deviation of the other from its mean. It is a mathematical relationship that is defined as:

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$

That is a little hard to wrap your mind around (but worth pushing on a bit). The outer expectation will be a weighted sum of the inner function evaluated at a particular (x, y) weighted by the probability of (x, y). If x and y are both above their respective means, or if x and y are both below their respective means, that term will be positive. If one is above its mean and the other is below, the term is negative. If the weighted sum of terms is positive, the two random variables will have a positive correlation. We can rewrite the above equation to get an equivalent equation:

$$\text{Cov}(X, Y) = E[XY] - E[Y]E[X]$$

Using this equation (and the product lemma) is it easy to see that if two random variables are independent their covariance is 0. The reverse is *not* true in general.

Properties of Covariance

Say that X and Y are arbitrary random variables:

$$\begin{aligned}
&\text{Cov}(X, Y) = \text{Cov}(Y, X) \\
&\text{Cov}(X, X) = E[X^2] - E[X]E[X] = \text{Var}(X) \\
&\text{Cov}(aX + b, Y) = a\text{Cov}(X, Y)
\end{aligned}$$

Let $X = X_1 + X_2 + \cdots + X_n$ and let $Y = Y_1 + Y_2 + \cdots + Y_m$. The covariance of X and Y is:

$$\begin{aligned}
\text{Cov}(X, Y) &= \sum_{i=1}^{n} \sum_{j=1}^{m} \text{Cov}(X_i, Y_j) \\
\text{Cov}(X, X) = \text{Var}(X) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov}(X_i, X_j) \\
&= \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \text{Cov}(X_i, X_j)
\end{aligned}$$

That last property gives us a third way to calculate variance. You could use this definition to calculate the variance of the binomial.

Correlation

Covariance is interesting because it is a quantitative measurement of the relationship between two variables. Correlation between two random variables, $\rho(X, Y)$ is the covariance of the two variables normalized by the variance of each variable. This normalization cancels the units out and normalizes the measure so that it is always in the range [0, 1]:

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)Var(Y)}}$$

Correlation measures linearity between X and Y.

$$\begin{aligned}
&\rho(X, Y) = 1 && Y = aX + b \text{ where } a = \sigma_y/\sigma_x \\
&\rho(X, Y) = -1 && Y = aX + b \text{ where } a = -\sigma_y/\sigma_x \\
&\rho(X, Y) = 0 && \text{absence of linear relationship}
\end{aligned}$$

If $\rho(X, Y) = 0$ we say that X and Y are “uncorrelated." If two varaibles are independent, then their correlation will be 0. However, it doesn’t go the other way. A correlation of 0 does not imply independence.

When people use the term correlation, they are actually referring to a specific type of correlation called “Pearson" correlation. It measures the degree to which there is a linear relationship between the two variables. An alternative measure is “Spearman" correlation which has a formula almost identical to your regular correlation score, with the exception that the underlying random variables are first transformed into their rank. “Spearman" correlation is outside the scope of CS109.