02: Combinatorics

David Varodayan
January 8, 2020
Adapted from slides by Lisa Yan

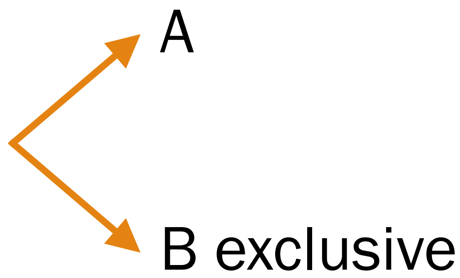
Takeaways from last time

Inclusion-Exclusion Principle (generalized Sum Rule)

If the outcome of an experiment can be either from Set A **or** set B, where A and B may overlap, then the total number of outcomes of the experiment is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

One-step experiment

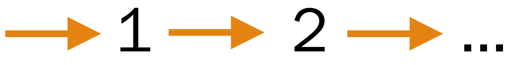


General Principle of Counting (generalized Product Rule)

If an experiment has r **steps**, such that step i has n_i outcomes for all $i = 1, \ldots, r$, then the total number of outcomes of the experiment is

$$n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i .$$

Multi-step experiment



Essential information

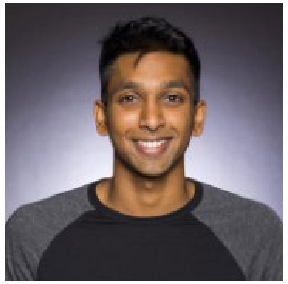
Website cs109.stanford.edu

Teaching Staff







Today's plan

Permutations (sort objects)

Combinations (choose objects)

Put objects into buckets

Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\,n_2!\cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct: r^n
  - Indistinct: $\frac{(n+r-1)!}{n!\,(r-1)!}$

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Sort n indistinct objects

Sort n distinct objects

Ayesha Tim Irina Joey Waddie

Permutations

A permutation is an ordered arrangement of distinct objects.

The number of unique orderings (permutations) of n distinct objects is

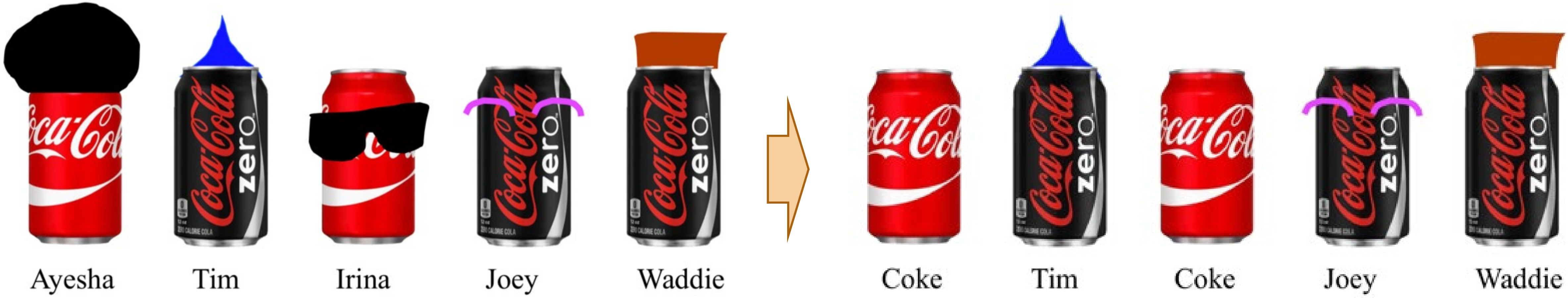
$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

Sort semi-distinct objects

Order n distinct objects $n!$

All distinct

Some indistinct



Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of <u>distinct</u> objects is a two-step process:

permutations of distinct objects $=$ permutations considering some objects are indistinct $\times$ Permutations of just the indistinct objects

Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of <u>distinct</u> objects is a two-step process:

$$\frac{\text{permutations of distinct objects}}{\text{Permutations of just the indistinct objects}} = \text{permutations considering some objects are indistinct}$$

General approach to counting permutations

When there are n objects such that

n_1 are the same (indistinguishable or indistinct), and

n_2 are the same, and

…

n_r are the same,

The number of unique orderings (permutations) is

For each group of indistinct objects,
Divide by the overcounted permutations

Sort semi-distinct objects

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many permutations?

Strings

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many orderings of letters are possible for the following strings?

1. BOBA

2. MISSISSIPPI

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Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

Consider the following generative process...

Combinations with cake

There are $n = 20$ people.
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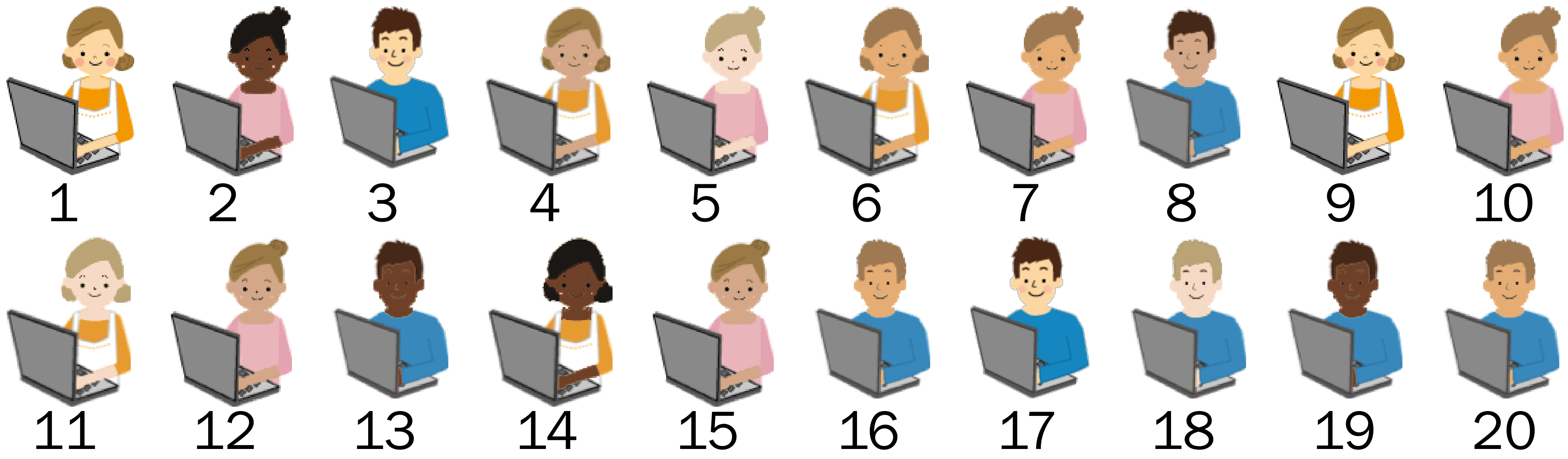
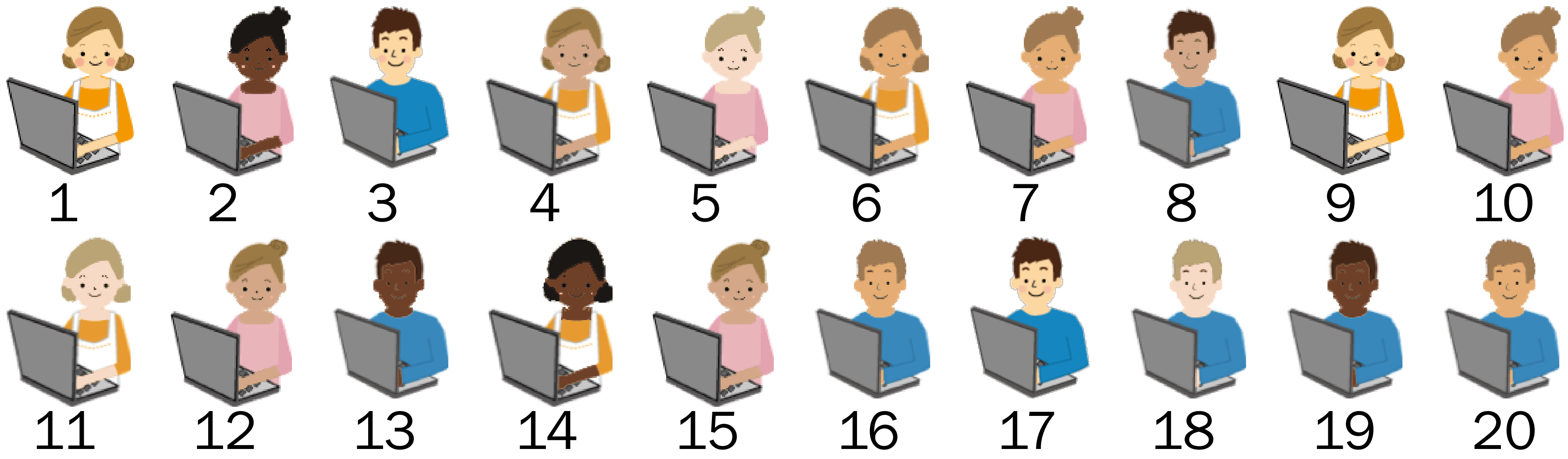


1. n people get in line

$n!$ ways

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

Combinations with cake

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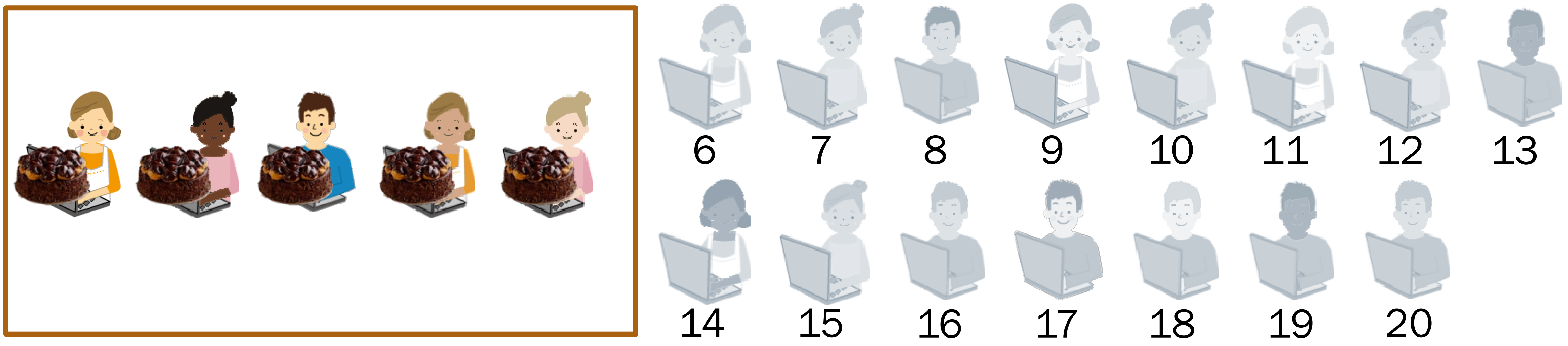

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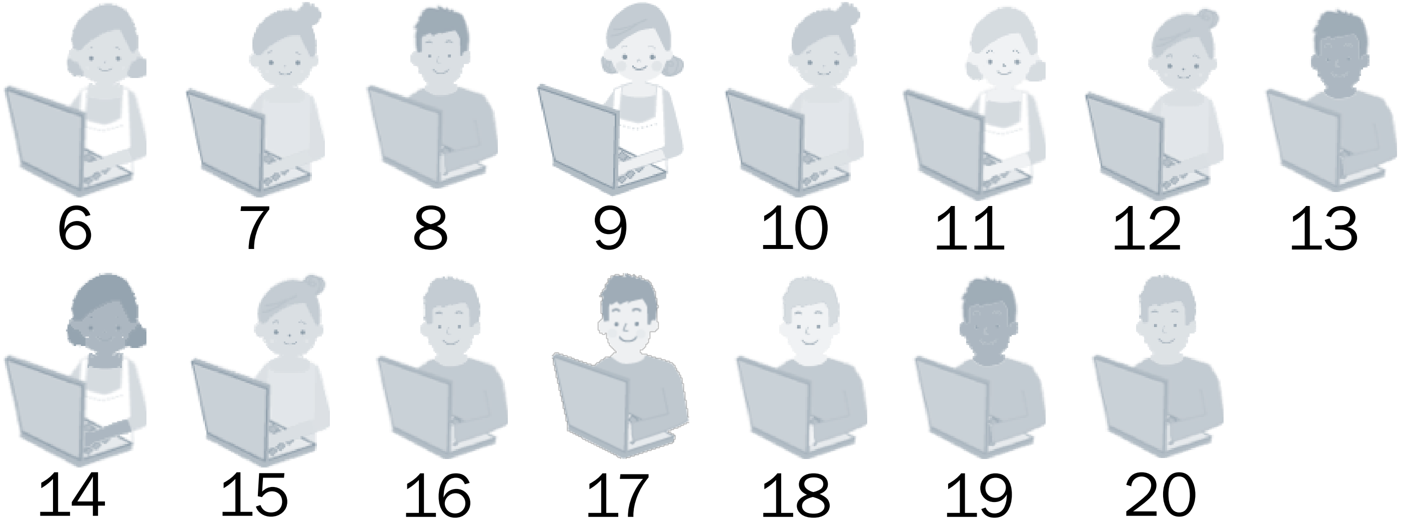

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

Combinations with cake

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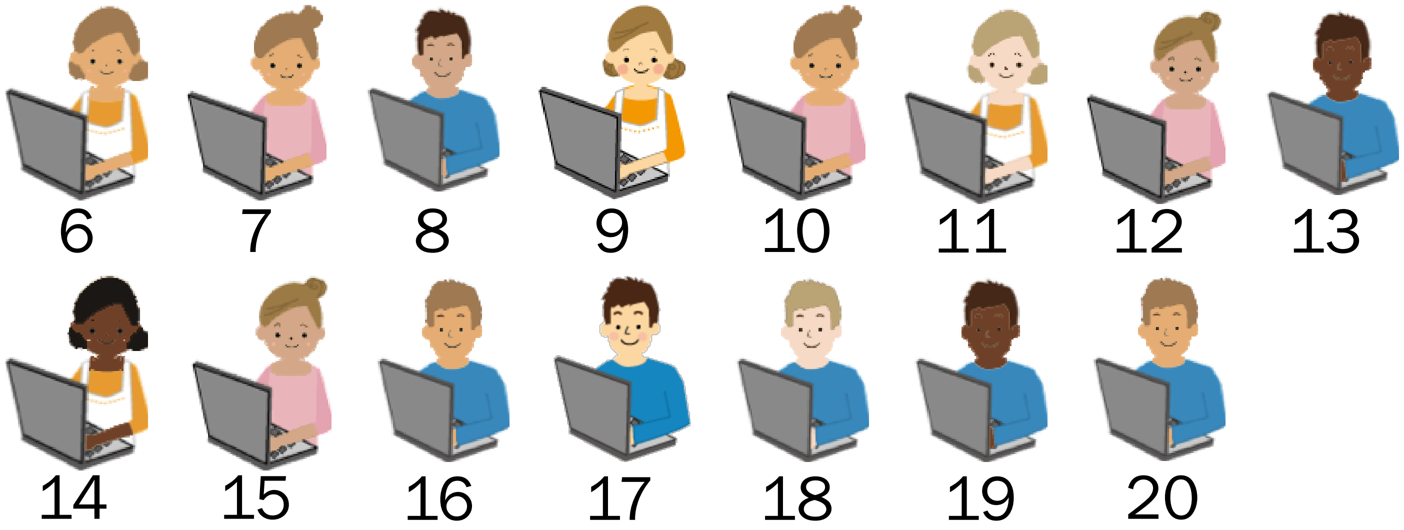

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

   $(n-k)!$ different permutations lead to the same mingle

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

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$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

2. What if we do not want to read both the 9th and 10th edition of Ross?

Probability textbooks (solution 2)

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

2. What if we do not want to read both the 9th and 10th edition of Ross?

Break

Announcements

PS#1

Out:	today
Due:	Friday 1/17, 1:00pm
Covers:	through Friday

Staff help

Piazza policy:	student discussion
Office hours:	start today

cs109.stanford.edu/staff.html

Python tutorial

When:	Friday 3:30-4:20pm
Location:	420-041
Recorded?	maybe
Notes:	to be posted online

Section sign-ups

Preference form:	today
Due:	Saturday 1/11
Results:	latest Monday

Handout: Calculation Reference

Week	Monday	Wednesday	Friday
1	JAN 6 1: Counting Slides Lecture Notes Administrivia Read: Ch 1.1-1.2	JAN 8 2: Permutations and Combinations Lecture Notes Calculation Ref Read: Ch 1.3-1.6 Out: PSet #1	JAN 10 3: Axioms of Probability Lecture Notes Python for Probability Serendipity Demo Read: Ch 2.1-2.5, 2.7
2	JAN 13 4: Conditional Probability and Bayes Lecture Notes Medical Bayes Demo	JAN 15 5: Independence Lecture Notes	JAN 17 6: Random Variables and Expectation Lecture Notes

Geometric series:

$$\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$$

$$\sum_{i=m}^{n} x^i = \frac{x^{n+1}-x^m}{x-1}$$

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \text{ if } |x| < 1$$

Integration by parts (everyone's favorite!):

Choose a suitable u and dv to decompose the integral of interest:

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

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    - r groups
- Put objects in r buckets

General approach to combinations

The number of ways to choose r groups of n distinct objects such that

For all $i = 1, \dots, r$, group i has size n_i, and

$\sum_{i=1}^{r} n_i = n$ (all objects are assigned), is

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \dots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	6
B	4
C	3

Datacenters (solution 2)

Choose k of n distinct objects into r groups of size $n_1, \dots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

Datacenter	# machines
A	6
B	4
C	3

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Steps:

1. Choose 6 computers for A

2. Choose 2 computers for B

3. Choose 3 computers for C

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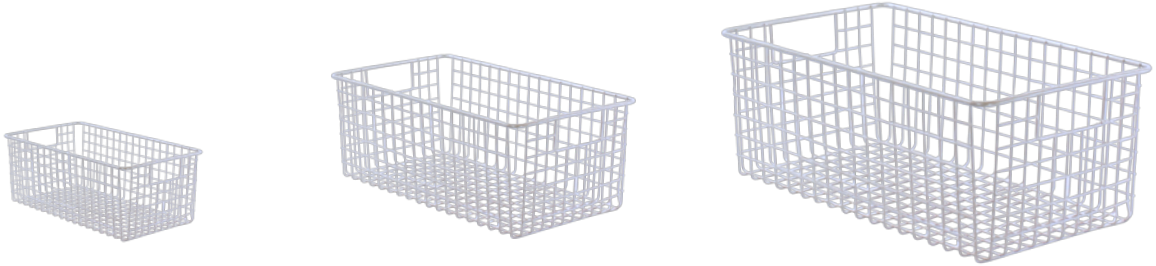
A trick question

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

How many ways are there to group 6 indistinct (indistinguishable) objects into 3 groups, where group A, B, and C have size 1, 2, and 3, respectively?



A (fits 1) B (fits 2) C (fits 3)

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➡ Put objects into buckets

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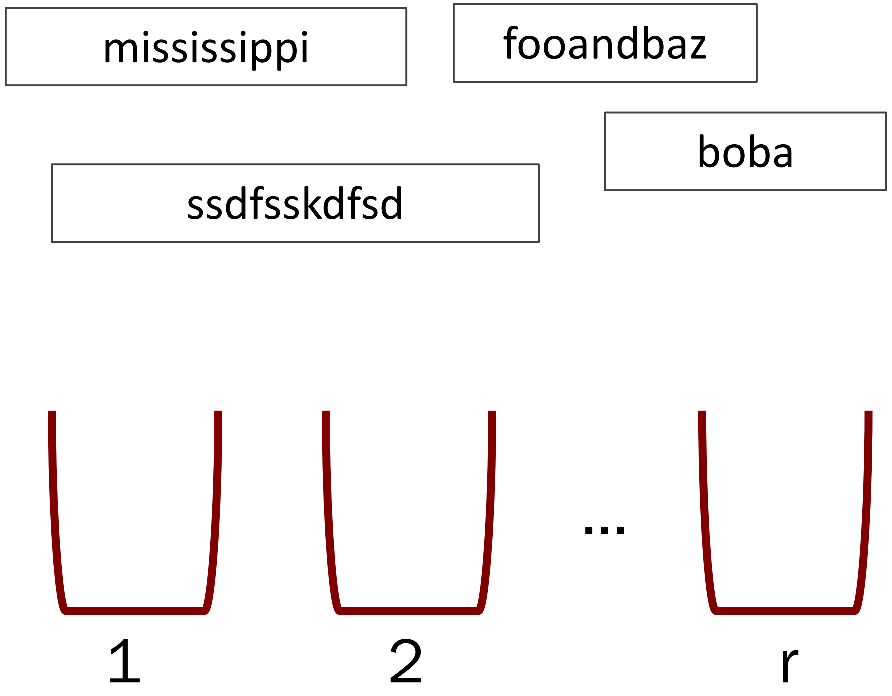
- **Put objects in r buckets**
  - Distinct
  - Indistinct

Hash tables and distinct strings

How many ways are there to hash n distinct strings to r buckets?



Steps:

1. Bucket 1st string
2. Bucket 2nd string

…

n. Bucket n^{th} string

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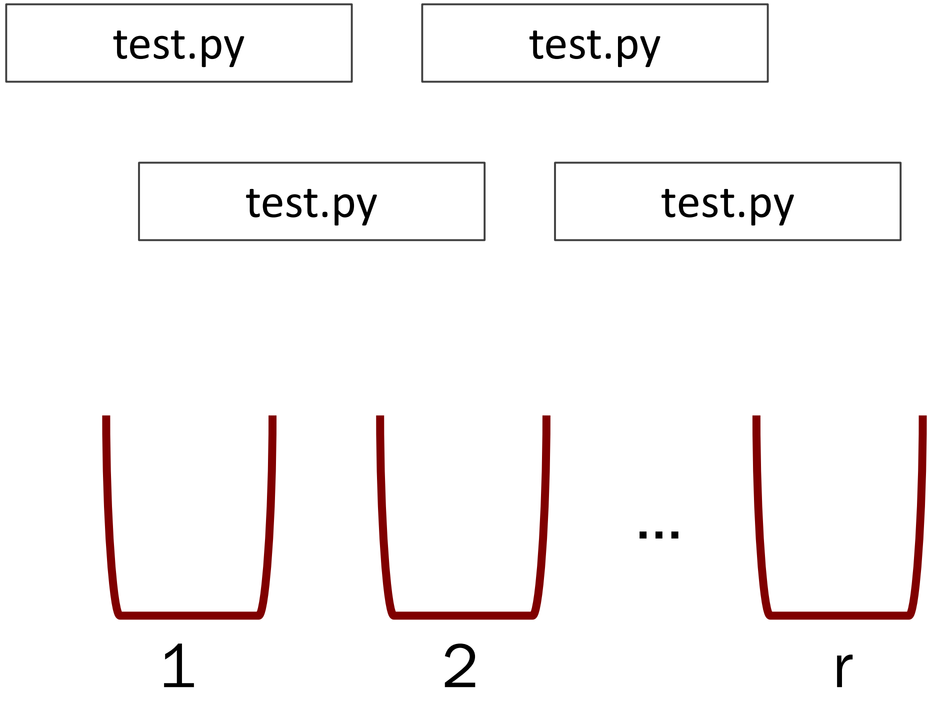
- **Put objects in r buckets**
  - Distinct: r^n
  - Indistinct

Hash tables and indistinct strings

How many ways are there to distribute n indistinct strings to r buckets?



Goal

Bucket 1 has x_1 strings,

Bucket 2 has x_2 strings,

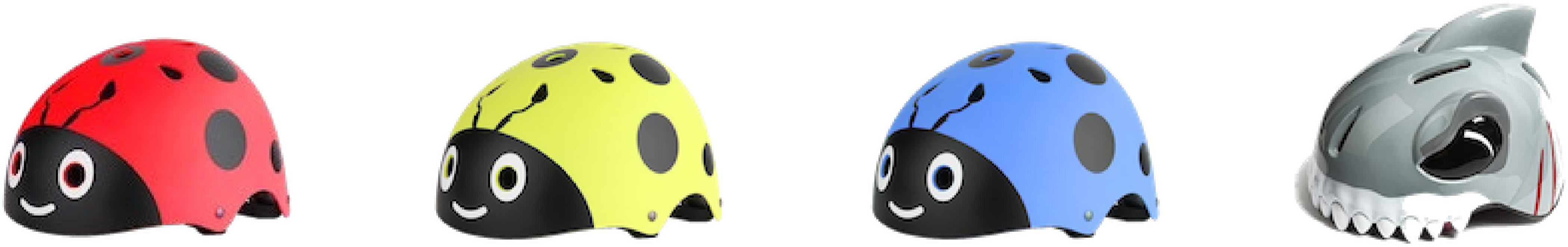
...

Bucket r has x_r strings (the rest)

Simple example: $n = 3$ strings and $r = 2$ buckets

Bicycle helmet sales

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?



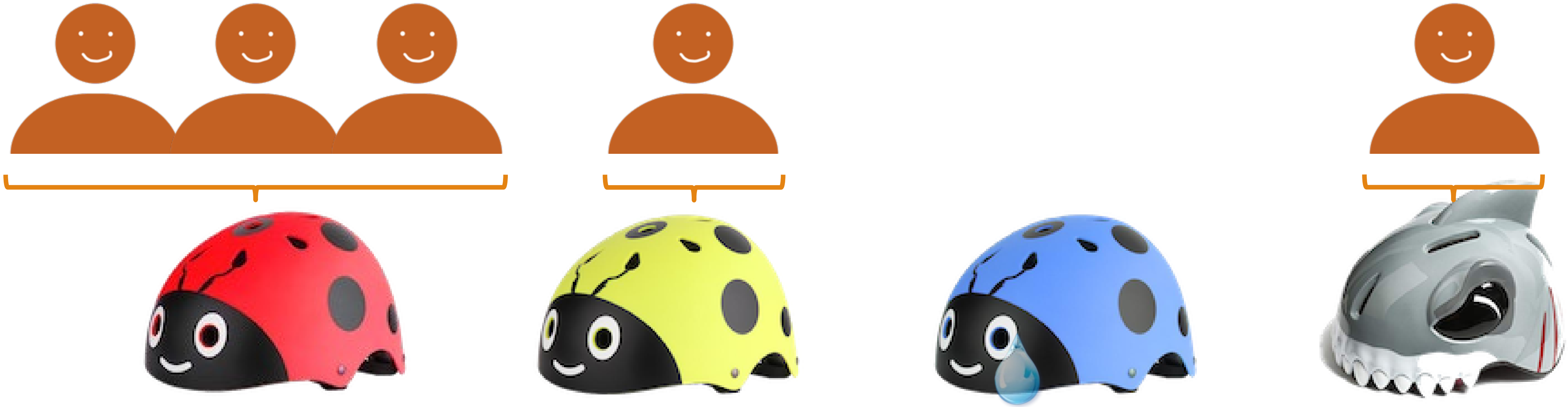
Consider the following generative process...

Bicycle helmet sales: 1 possible assignment outcome

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

$n = 5$ indistinct objects

$r = 4$ distinct buckets



Bicycle helmet sales: 1 possible assignment outcome

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

$n = 5$ indistinct objects

$r = 4$ distinct buckets

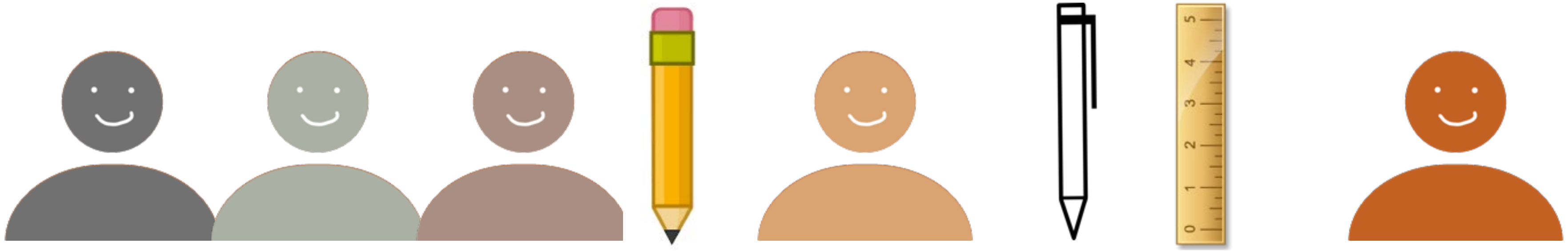
Goal Order n indistinct objects and $r - 1$ indistinct dividers.

Bicycle helmet sales: A generative proof

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

Goal Order n indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct



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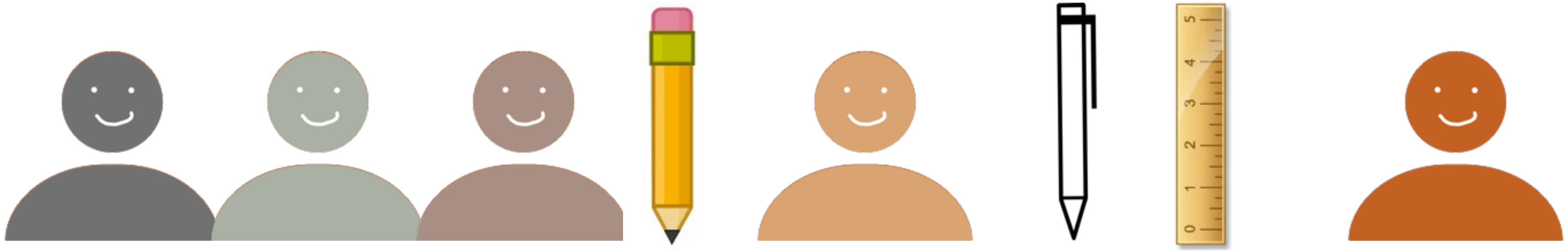

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

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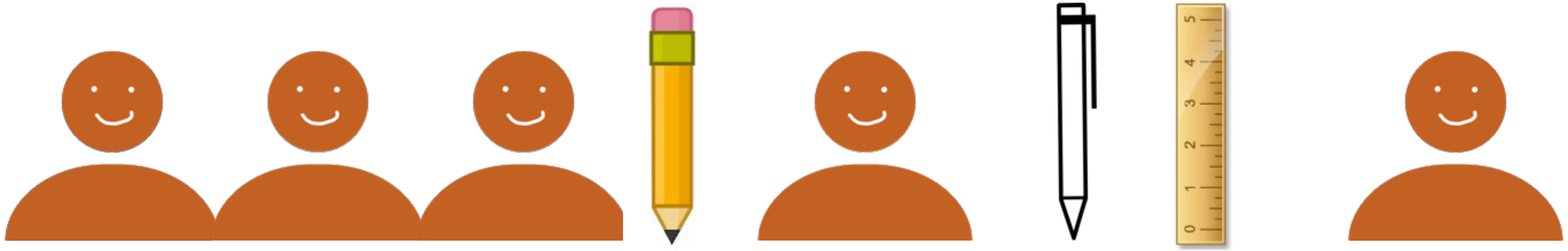

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

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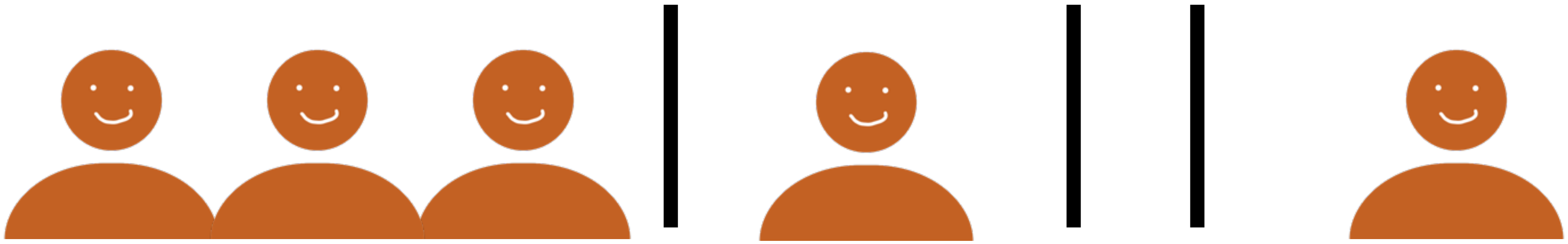

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

3. Make $r - 1$ dividers indistinct

$$\frac{1}{(r-1)!}$$

Divider method

The number of ways to distribute n indistinct objects into r buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that

n are indistinct objects, and

$r - 1$ are indistinct dividers:

Integer solutions to equations

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

How many integer solutions are there to the following equation:

$$x_1 + x_2 + \cdots + x_r = n,$$

where for all i, x_i is an integer such that $0 \le x_i \le n$?

Venture capitalists

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?

Venture capitalists

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have $10 million to invest in 4 companies (in $1 million increments).

1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1?

Venture capitalists

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have $10 million to invest in 4 companies (in $1 million increments).

1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1?
3. What if you don't invest all your money?

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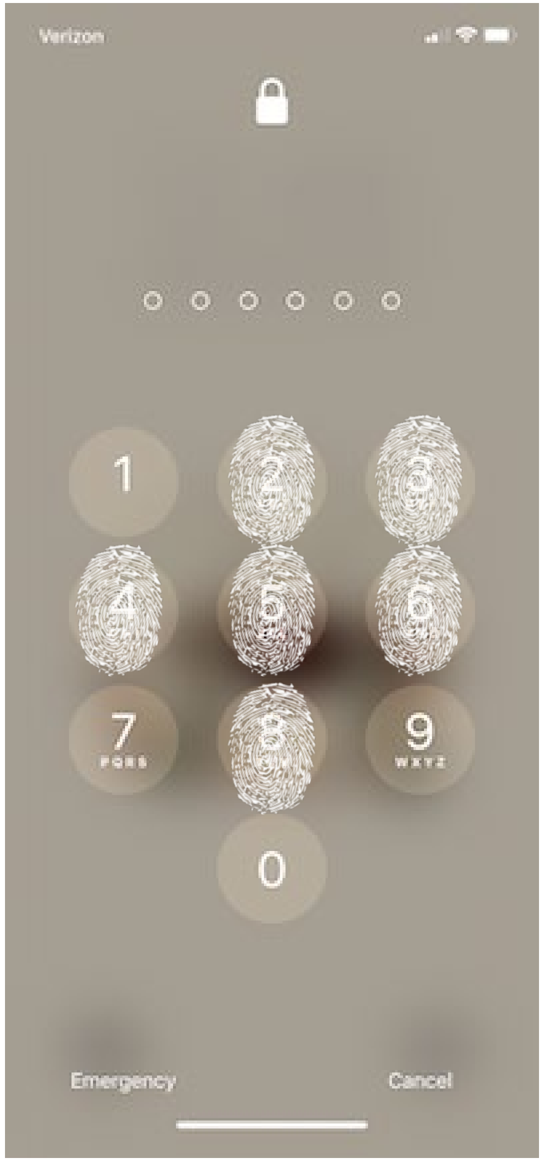
Unique 6-digit passcodes with six smudges

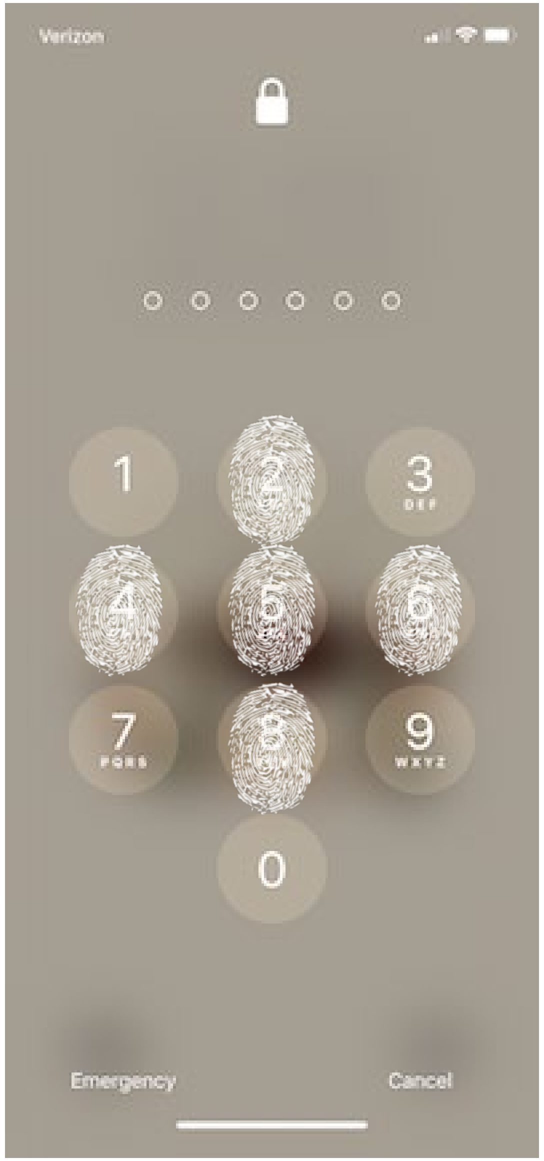
Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$



How many unique 6-digit passcodes are possible if a phone password uses each of six distinct numbers?

Unique 6-digit passcodes with five smudges

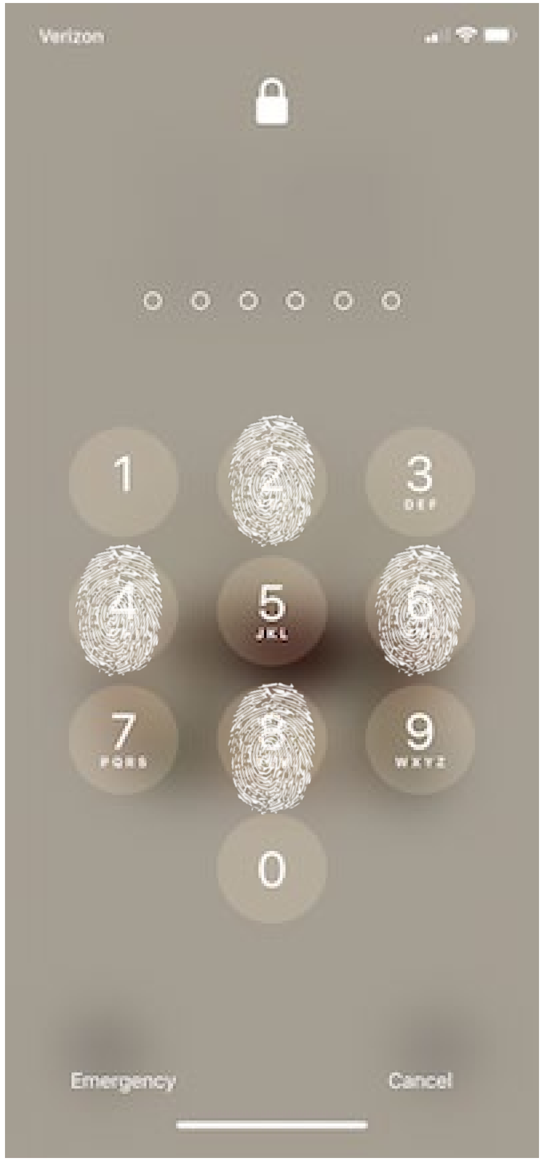
Order n semi-distinct objects	$\frac{n!}{n_1!\,n_2!\cdots n_r!}$



How many unique 6-digit passcodes are possible if a phone password uses each of five distinct numbers?

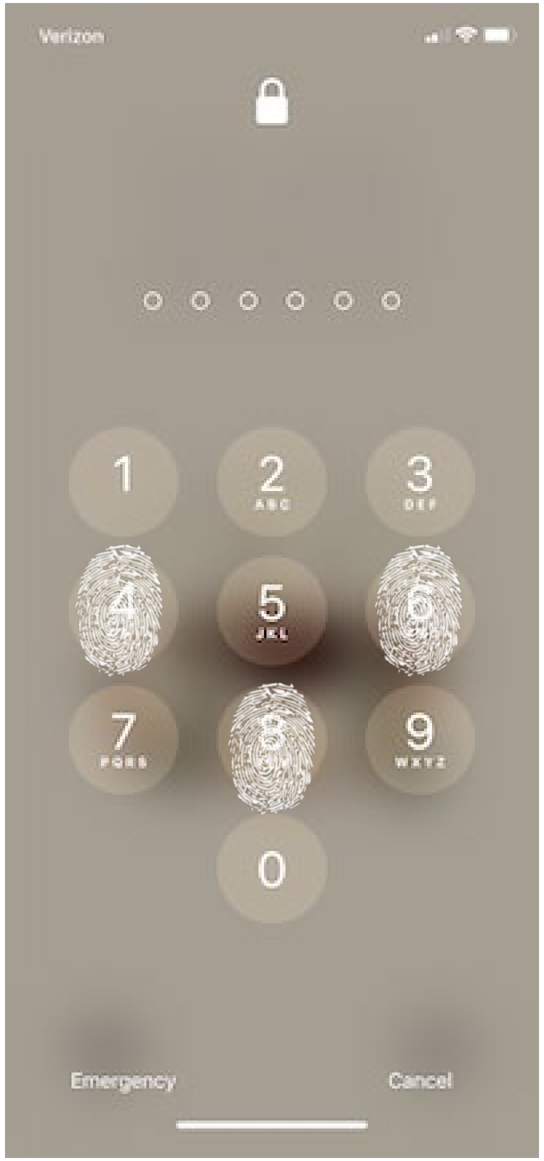
Unique 6-digit passcodes with four smudges

Order n semi-distinct objects	$\frac{n!}{n_1!\,n_2!\cdots n_r!}$



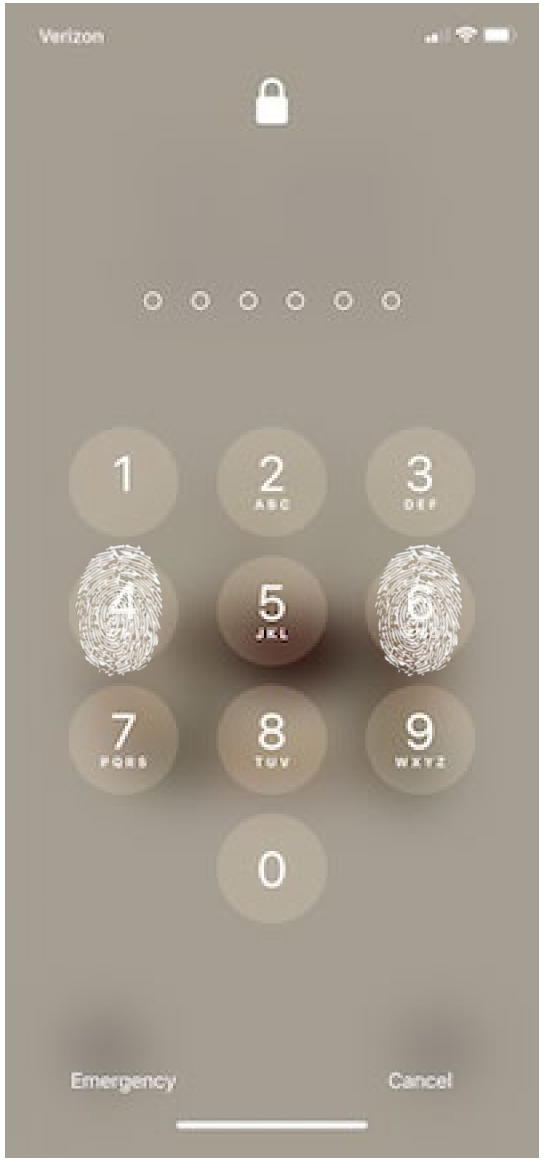
How many unique 6-digit passcodes are possible if a phone password uses each of four distinct numbers?

Unique 6-digit passcodes with three smudges



How many unique 6-digit passcodes are possible if a phone password uses each of **three** distinct numbers?

Unique 6-digit passcodes with two smudges



How many unique 6-digit passcodes are possible if a phone password uses each of two distinct numbers?

Unique 6-digit passcodes with one smudge

How many unique 6-digit passcodes are possible if a phone password uses one number?