03: Intro to Probability

David Varodayan

January 10, 2020

Adapted from slides by Lisa Yan

Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\, n_2! \cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct: r^n
  - Indistinct: $\frac{(n+r-1)!}{n!\,(r-1)!}$

Divider method

The number of ways to distribute n indistinct objects into r buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that

n are indistinct objects, and

$r - 1$ are indistinct dividers:

$$\text{Total} = (n + r - 1)! \times \frac{1}{n!} \times \frac{1}{(r-1)!}$$

$$= \binom{n + r - 1}{r - 1}$$

Integer solutions to equations

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

How many integer solutions are there to the following equation:

$$x_1 + x_2 + \cdots + x_r = n,$$

where for all i, x_i is an integer such that $0 \le x_i \le n$?

Positive integer equations can be solved with the divider method.

Venture capitalists

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

$$x_i \geq 0$$

Venture capitalists

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

⚠ $3 \leq x_1$

$x_i \geq 0$ for $i = 2, 3, 4$

Venture capitalists

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?
3. What if you don't invest all your money?

Set up

$$x_1 + x_2 + x_3 + x_4 \leq 10$$

x_i: amount invested in company i

$$x_i \geq 0$$

Today’s plan

Key definitions: sample spaces and events

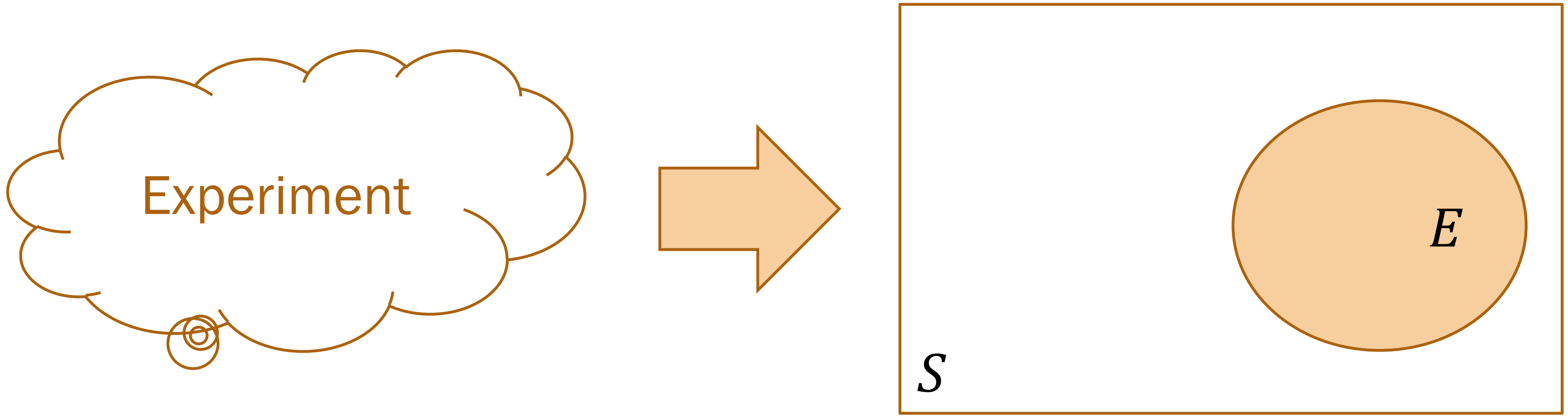
Axioms of Probability

Equally likely outcomes (counting)

Corollaries of Axioms of Probability

Key definitions

An experiment in probability:



Sample Space, S: The set of all possible outcomes of an experiment

Event, E: Some subset of S ($E \subseteq S$).

Key definitions

Sample Space, S

- Coin flip
  $S = \{\text{Heads}, \text{Tails}\}$
- Flipping two coins
  $S = \{(\text{H,H}), (\text{H,T}), (\text{T,H}), (\text{T,T})\}$
- Roll of 6-sided die
  $S = \{1, 2, 3, 4, 5, 6\}$
- # emails in a day
  $S = \{x \mid x \in \mathbb{Z}, \; x \geq 0\}$
- YouTube hours in a day
  $S = \{x \mid x \in \mathbb{R}, 0 \leq x \leq 24\}$

Event, E

- Flip lands heads
  $E = \{\text{Heads}\}$
- $\geq$ 1 head on 2 coin flips
  $E = \{(\text{H,H}), (\text{H,T}), (\text{T,H})\}$
- Roll is 3 or less:
  $E = \{1, 2, 3\}$
- Low email day ($\leq$ 20 emails)
  $E = \{x \mid x \in \mathbb{Z}, \; 0 \leq x \leq 20\}$
- Wasted day ($\geq$ 5 YT hours):
  $E = \{x \mid x \in \mathbb{R}, 5 \leq x \leq 24\}$

What is a probability?

A number between 0 and 1
to which we ascribe meaning.*

*our belief that an event E occurs.

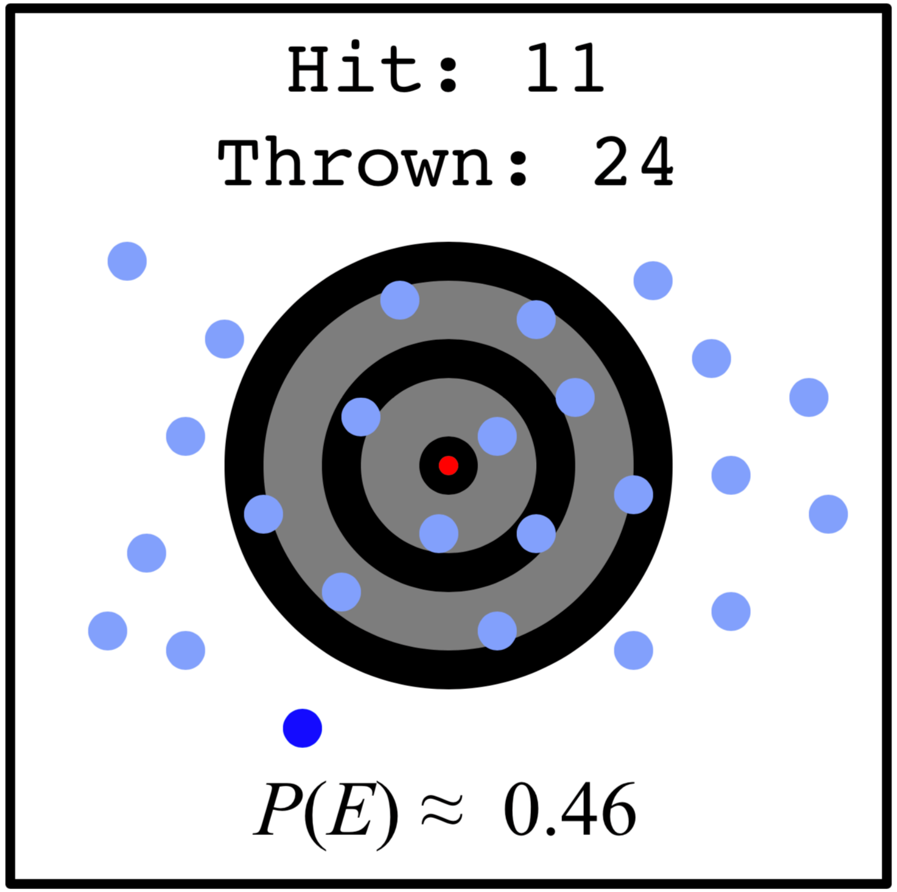
What is a probability?

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

n = # of total trials

$n(E)$ = # trials where E occurs

Let E = the set of outcomes where you hit the target.



Today’s plan

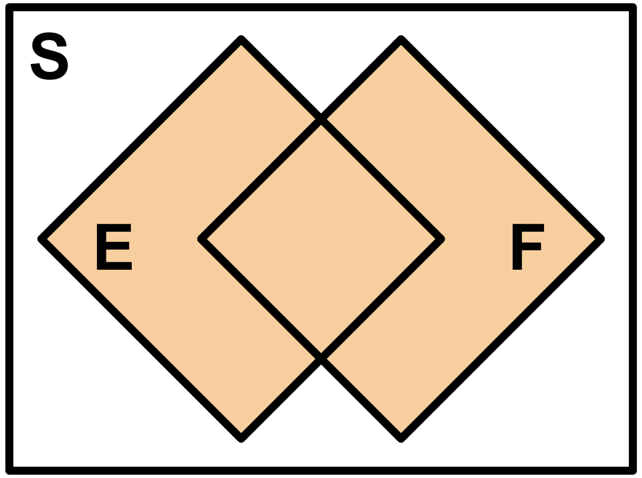
Key definitions: sample spaces and events

Axioms of Probability

Equally likely outcomes (counting)

Corollaries of Axioms of Probability

Quick review of sets



E and F are events in S.

Experiment:

Dice roll

$S = \{1, 2, 3, 4, 5, 6\}$

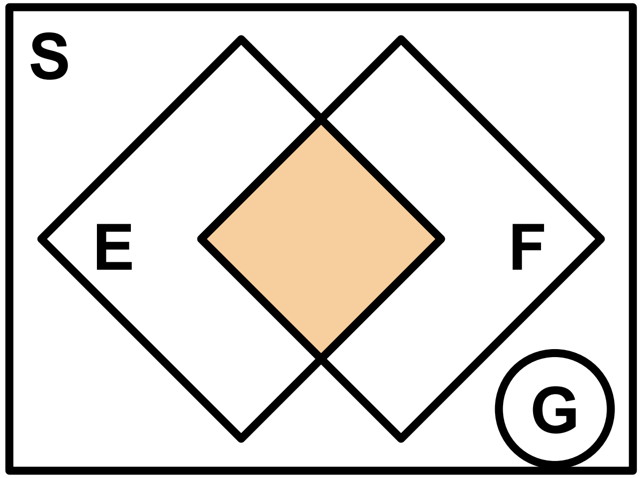
Let $E = \{1, 2\}$, and $F = \{2,3\}$

def Union of events, $E \cup F$

The event containing all outcomes in E *or* F.

$$E \cup F = \{1,2,3\}$$

Quick review of sets



E and F are events in S.

Experiment:

Dice roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

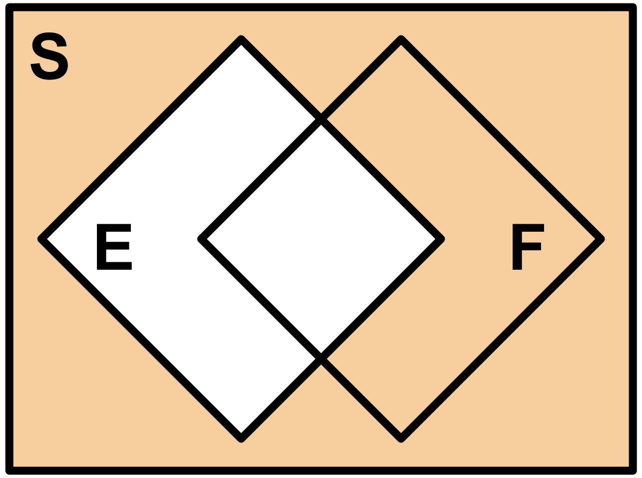
<u>def</u> Intersection of events, $E \cap F$

The event containing all outcomes in E ***and*** F.

$$E \cap F = EF = \{2\}$$

<u>def</u> Mutually exclusive events F and G means that $F \cap G = \emptyset$

Quick review of sets



E and F are events in S.

Experiment:

Dice roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

<u>def</u> Complement of event E, E^C

The event containing all outcomes in S that are ***<u>not</u>*** in E.

$$E^C = \{3, 4, 5, 6\}$$

3 Axioms of Probability

Definition of probability: $P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$

Axiom 1:

Axiom 2:

Axiom 3:

Axiom 3 is the (analytically) useful Axiom

Axiom 3: If E and F are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$

More generally, for any sequence of mutually exclusive events $E_1, E_2, \ldots$:

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

(like the Sum Rule of Counting, but for probabilities)

Today's plan

Key definitions: sample spaces and events

Axioms of Probability

Equally likely outcomes (counting)

Corollaries of Axioms of Probability

Equally Likely Outcomes

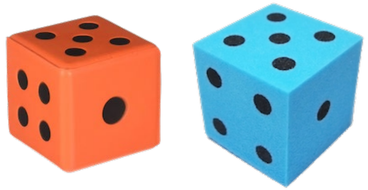
Some sample spaces have equally likely outcomes.

- Coin flip: S = {Head, Tails}
- Flipping two coins: S = {(H, H), (H, T), (T, H), (T, T)}
- Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}

Roll two dice

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

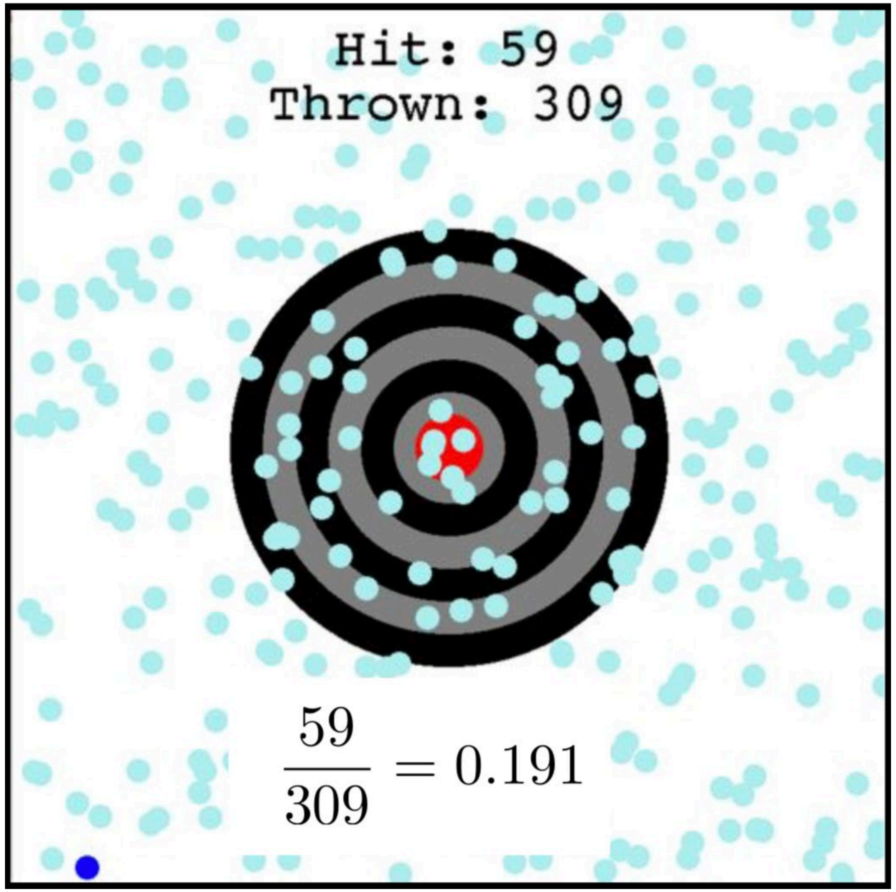
Roll two 6-sided dice. What is P(sum = 7)?



Target revisited

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Let E = the set of outcomes where you hit the target.



The dart is equally likely to land anywhere on the screen.

What is $P(E)$, the probability of hitting the target?

Screen size = 800×800 $|S| = 800^2$

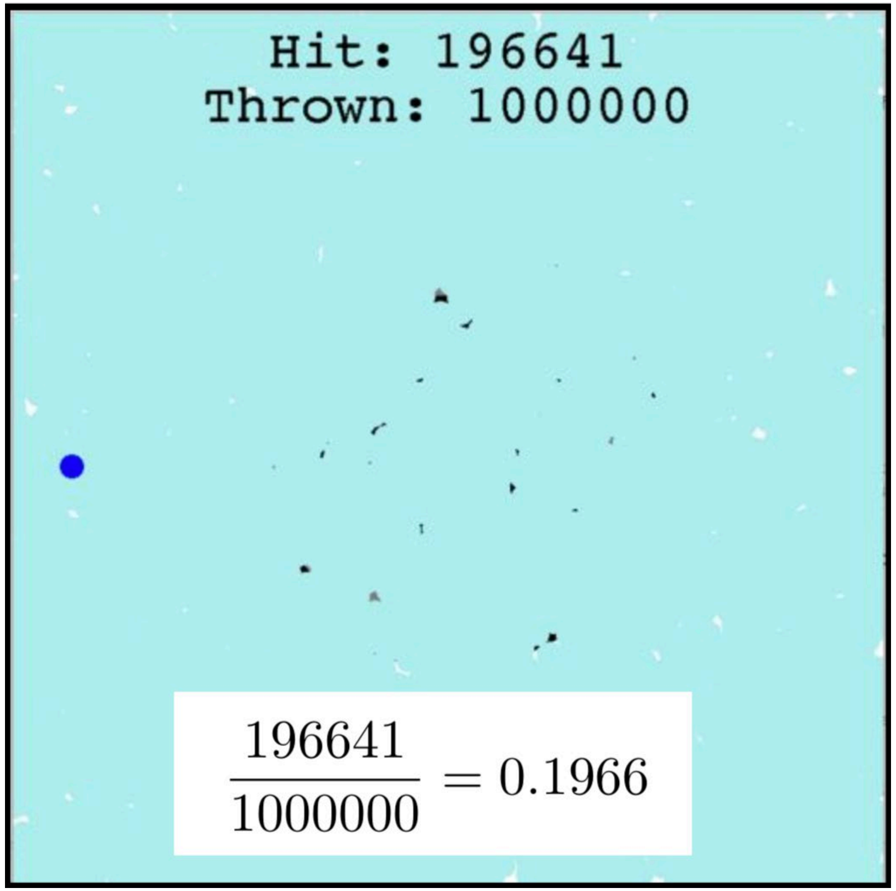
Radius of target: 200 $|E| = \pi \cdot 200^2$

$$P(E) = \frac{|E|}{|S|} = \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$

Target revisited

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Let E = the set of outcomes where you hit the target.



The dart is equally likely to land anywhere on the screen.

What is $P(E)$, the probability of hitting the target?

Screen size = 800×800 $|S| = 800^2$

Radius of target: 200 $|E| = \pi \cdot 200^2$

$$P(E) = \frac{|E|}{|S|} = \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$

Not equally likely outcomes

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Play the lottery.

What is $P(\text{win})$?

$S = \{\text{Lose}, \text{Win}\}$

$E = \{\text{Win}\}$

$$P(E) = \frac{|E|}{|S|} = \frac{1}{2} = 50\%?$$

The hard part: defining equally likely outcomes ***consistently*** across sample space and events

Cats and carrots

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 carrots in a bag. 3 drawn.

What is P(1 cat and 2 carrots drawn)?

Note: Do indistinct objects give you an equally likely sample space?

Cats and carrots (ordered solution)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 carrots in a bag. 3 drawn.

What is P(1 cat and 2 carrots drawn)?

Define

- $S =$ Pick 3 distinct items
- $E =$ 1 distinct cat,
  2 distinct carrots

Cats and carrots (unordered solution)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 carrots in a bag. 3 drawn.

What is P(1 cat and 2 carrots drawn)?

Define

- S = Pick 3 distinct items
- E = 1 distinct cat,
  2 distinct carrots

Announcements

Section sign-ups

Preference form:	out
Due:	Saturday 1/11
Results:	latest Monday

Python tutorial

When:	Friday 3:30-4:20pm
Location:	420-040
Notes:	to be posted online
Installation:	On Piazza

Any Poker Straight

Consider 5-card poker hands.

- "straight" is 5 consecutive rank cards of any suit

What is P(Poker straight)?

- What is an example of an outcome?
- Is each outcome equally likely?
- Should objects be ordered or unordered?

Any Poker Straight

Consider 5-card poker hands.

- "straight" is 5 consecutive rank cards of any suit

What is P(Poker straight)?

Define

- S (unordered)

- E (unordered,
  consistent with S)

"Official" Poker Straight

Consider 5-card poker hands.

- "straight" is 5 consecutive rank cards of any suit
- "straight flush" is 5 consecutive rank cards of same suit

What is P(Poker straight, but not straight flush)?

Define

- S (unordered)
- E (unordered,
  consistent with S)

Chip defect detection

n chips are manufactured, 1 of which is defective.
k chips are randomly selected from n for testing.

What is P(defective chip is in k selected chips?)

Define

- S (unordered)
- E (unordered, consistent with S)

Chip defect detection (solution 2)

n chips are manufactured, 1 of which is defective.
k chips are randomly selected from n for testing.

What is P(defective chip is in k selected chips?)

Redefine experiment

1. Choose k indistinct chips (1 way)
2. Wave a wand and make one defective

Define

- S (unordered)
- E (unordered, consistent with S)

Today's plan

Key definitions: sample spaces and events

Axioms of Probability

Equally likely outcomes (counting)

➡ Corollaries of Axioms of Probability

Axioms of Probability

Definition of probability: $P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$

Axiom 1: $0 \le P(E) \le 1$

Axiom 2: $P(S) = 1$

Axiom 3: If E and F are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$

3 Corollaries of Axioms of Probability

Corollary 1: $P(E^C) = 1 - P(E)$

Proof of Corollary 1

Corollary 1: $P(E^C) = 1 - P(E)$

Proof:

E, E^C are mutually exclusive	Definition of E^C
$P(E \cup E^C) = P(E) + P(E^C)$	Axiom 3
$S = E \cup E^C$	Everything must either be in E or E^C, by definition
$1 = P(S) = P(E) + P(E^C)$	Axiom 2
$P(E^C) = 1 - P(E)$	Rearrange

3 Corollaries of Axioms of Probability

Corollary 1: $P(E^C) = 1 - P(E)$

Corollary 2: If $E \subseteq F$, then $P(E) \leq P(F)$

Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$
(Inclusion-Exclusion Principle for Probability)

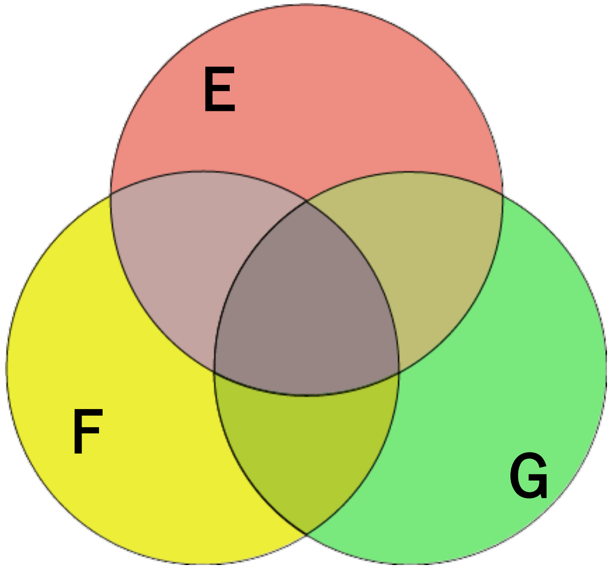
Inclusion-Exclusion Principle (Corollary 3)

Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$

(Inclusion-Exclusion Principle for Probability)

General form:

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{(r+1)} \sum_{i_1 < \cdots < i_r} P\left(\bigcap_{j=1}^{r} E_{i_j}\right)$$



$P(E \cup F \cup G) =$

r = 1: $P(E) + P(F) + P(G)$

r = 2: $- P(E \cap F) - P(E \cap G) - P(F \cap G)$

r = 3: $+ P(E \cap F \cap G)$

Selecting Programmers

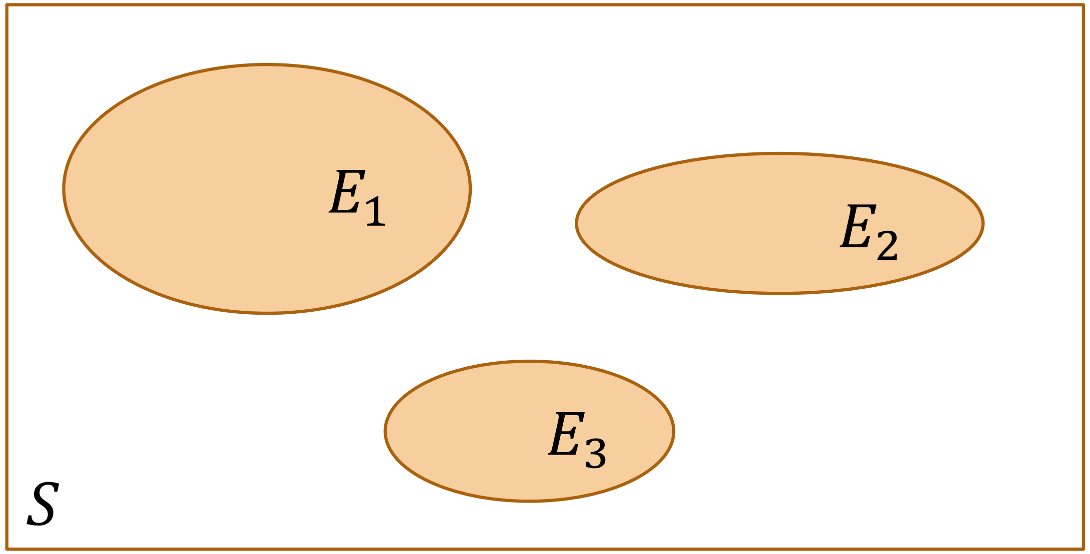
$$P(E \cup F) = P(E) + P(F) - P(EF)$$ Inclusion-Exclusion

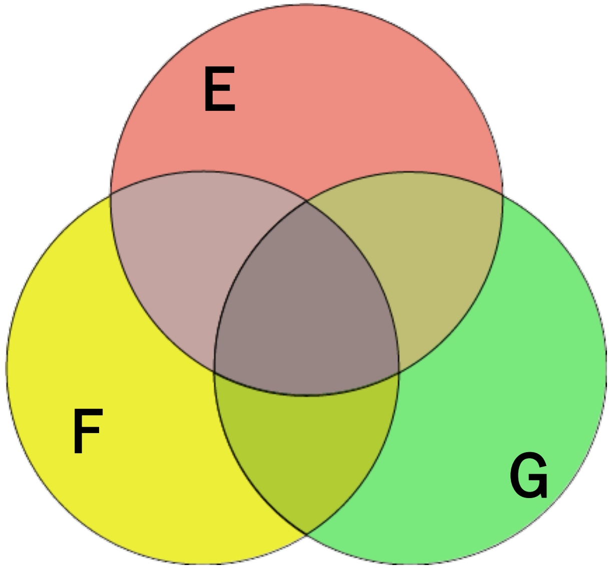
- P(student programs in Java) = 0.28
- P(student programs in Python) = 0.07
- P(student programs in Java and Python) = 0.05.

What is P(student does not program in (Java or Python))?

1. Define events & state goal
2. Identify known probabilities
3. Solve

Takeaway: Mutually exclusive events





Axiom 3,
Mutually exclusive events

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Inclusion-Exclusion Principle

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{(r+1)} \sum_{i_1 < \cdots < i_r} P\left(\bigcap_{j=1}^{r} E_{i_j}\right)$$

Design your experiment to compute easier probabilities.

Serendipity

- The population of Stanford is $n = 17{,}000$ people.
- You are friends with $r = 100$ people.
- Walk into a room, see $k = 268$ random people.
- Assume you are equally likely to see each person at Stanford.

What is the probability that you see someone you know?

Serendipity

- The population of Stanford is $n = 17{,}000$ people.
- You are friends with $r = 100$ people.
- Walk into a room, see $k = 268$ random people.
- Assume you are equally likely to see each person at Stanford.

What is the probability that you see someone you know?

Define

- S (unordered)
- E: see ≥ 1 friend in the room

$$|S| = \binom{n}{k} = \binom{17000}{268}$$

How should we compute $P(E)$?

A. $P(\text{exactly } 1) + P(\text{exactly } 2)$
$P(\text{exactly } 3) + \cdots$

B. $1 - P(\text{see no friends})$

It is often much easier to compute $P(E^c)$.

The Birthday ~~Paradox~~ Problem

What is the probability that in a set of *n* people, at least one pair of them will share the same birthday?

For you to think about (and discuss in section!)