04: Conditional Probability and Bayes

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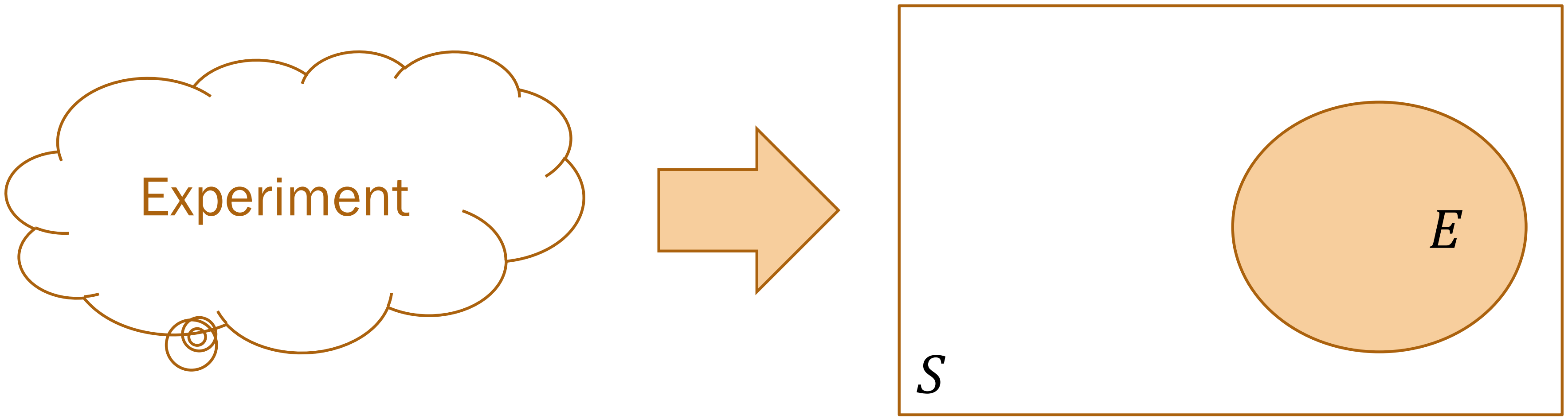
Adapted from slides by Lisa Yan

Key definitions

An experiment in probability:



Sample Space, S: The set of all possible outcomes of an experiment

Event, E: Some subset of S ($E \subseteq S$).

We have the power to redesign our experiment,
provided we can recreate the set of outcomes!

Card Flipping

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

In a 52 card deck, cards are flipped one at a time.
After the first ace (of any suit) appears, consider the next card.

Is P(next card = Ace Spades) < P(next card = 2 Clubs)?

Card Flipping

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

In a 52 card deck, cards are flipped one at a time.
After the first ace (of any suit) appears, consider the next card.

Is P(next card = Ace Spades) < P(next card = 2 Clubs)?

Sample space $S =$ 52 in-order cards (shuffle deck)

Event

E_{AS}, next card is Ace Spades

1. Take out Ace of Spades.
2. Shuffle leftover 51 cards.
3. Add Ace Spades after first ace.

$|E_{AS}| = 51! \cdot 1$

E_{2C}, next card is 2 Clubs

1. Take out 2 Clubs.
2. Shuffle leftover 51 cards.
3. Add 2 Clubs after first ace.

$|E_{2C}| = 51! \cdot 1$

$$P(E_{AS}) = P(E_{2C})$$

Today's plan

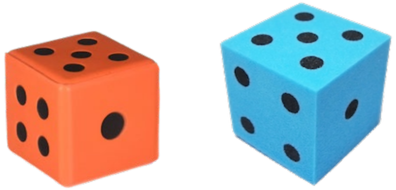
→ Conditional Probability and Chain Rule

Law of Total Probability

Bayes' Theorem

Dice, our misunderstood friends

Roll two 6-sided dice, yielding values D_1 and D_2.



Let E be event: $D_1 + D_2 = 4$.

What is $P(E)$?

$|S| = 36$

$E = \{(1,3), (2,2), (3,1)\}$

$P(E) = 3/36 = 1/12$

Let F be event: $D_1 = 2$.

What is $P(E$, given F already observed$)$?

Conditional Probability

The conditional probability of E given F is the probability that E occurs given that F has already occurred. This is known as conditioning on F.

Written as: $P(E|F)$

Means: "$P(E,$ given F already observed$)$"

Sample space → all possible outcomes consistent with F (i.e. $S \cap F$)

Event space → all outcomes in E consistent with F (i.e. $E \cap F$)

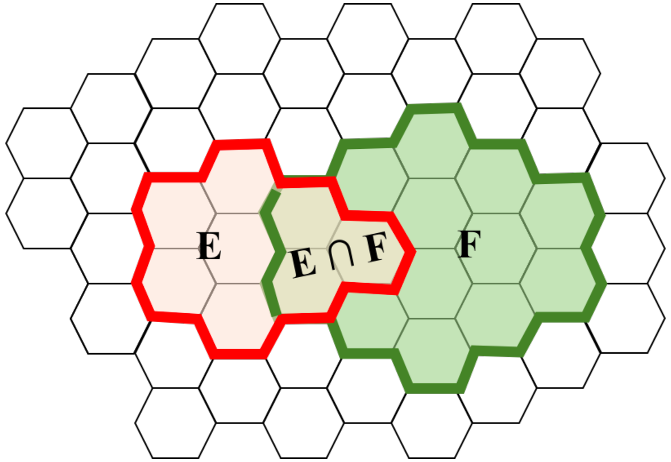
Conditional Probability, equally likely outcomes

The conditional probability of E given F is the probability that E occurs given that F has already occurred. This is known as conditioning on F.

With equally likely outcomes:

$$P(E|F) = \frac{\text{\# of outcomes in E consistent with F}}{\text{\# of outcomes in S consistent with F}} = \frac{|E \cap F|}{|S \cap F|}$$

$$= \frac{|EF|}{|F|}$$



$$P(E) = \frac{8}{50} \approx 0.16$$

$$P(E|F) = \frac{3}{14} \approx 0.21$$

Slicing up the spam

$P(E|F) = \frac{|EF|}{|F|}$ Equally likely outcomes

24 emails are sent, 6 each to 4 users.

- 10 of the 24 emails are spam.
- All possible outcomes are equally likely.

Let E = user 1 receives 3 spam emails.

What is $P(E)$?

Let F = user 2 receives 6 spam emails.

What is $P(E|F)$?

Let G = user 3 receives 5 spam emails.

What is $P(G|F)$?

Quick check

You have a flowering plant.

Let E = Flowers bloom
F = It gets watered
G = It gets sun

In English, how do you interpret $P(E|FG)$?

The probability that…

A. …flowers bloom given the probability that it gets water and it gets sun

B. …flowers bloom given it gets watered given it gets sun

C. …flowers bloom given (it gets watered and it gets sun)

D. All/none/other

Conditional probability in general

General definition of conditional probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

The Chain Rule (aka Product rule):

$$P(EF) = P(F)P(E|F)$$

These properties hold even when outcomes are not equally likely.

Netflix and Learn

$$P(E|F) = \frac{P(EF)}{P(F)}$$ Definition of Cond. Probability

Let E = a user watches Life is Beautiful.

What is $P(E)$?

$$P(E) \approx \frac{\text{\# people who have watched movie}}{\text{\# people on Netflix}}$$

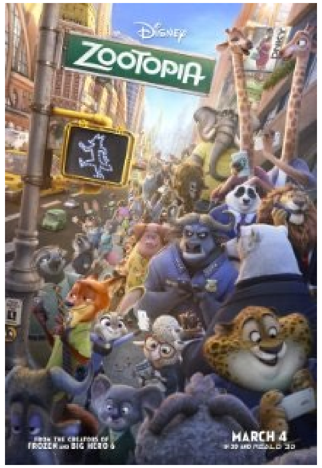
$$= 10{,}234{,}231 \,/\, 50{,}923{,}123 \approx 0.20$$

Netflix and Learn

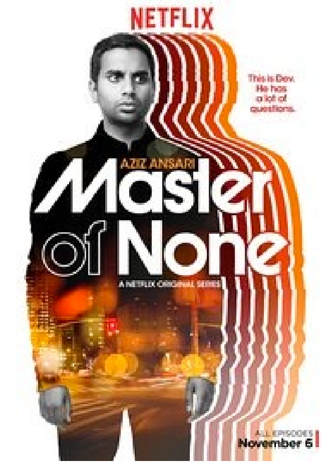
$$P(E|F) = \frac{P(EF)}{P(F)}$$ Definition of Cond. Probability

Let E be the event that a user watches the given movie.

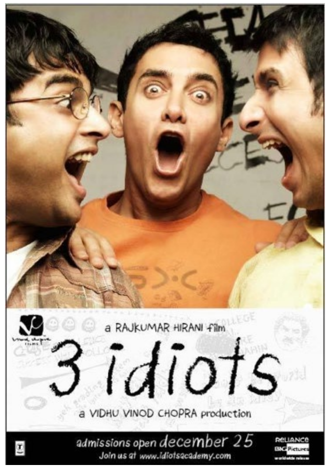
$P(E) = 0.19$



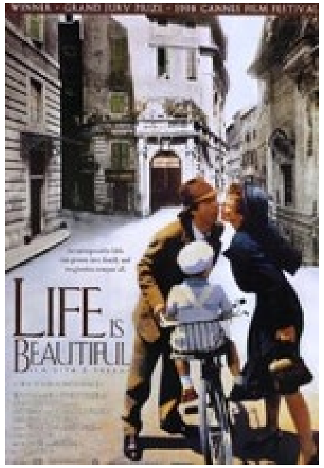
$P(E) = 0.32$



$P(E) = 0.20$



$P(E) = 0.09$



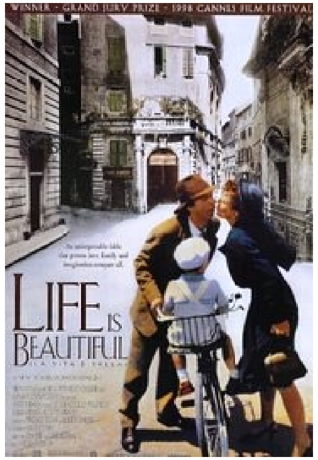
$P(E) = 0.20$

Netflix and Learn

$P(E|F) = \frac{P(EF)}{P(F)}$ Definition of Cond. Probability

Let E = a user watches Life is Beautiful.

Let F = a user watches Amelie.



What is the probability that a user watches Life is Beautiful, given they watched Amelie?

$$P(E|F)$$

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{\text{\# people who have watched both}}{\text{\# people on Netflix}}}{\frac{\text{\# people who have watched Amelie}}{\text{\# people on Netflix}}}$$

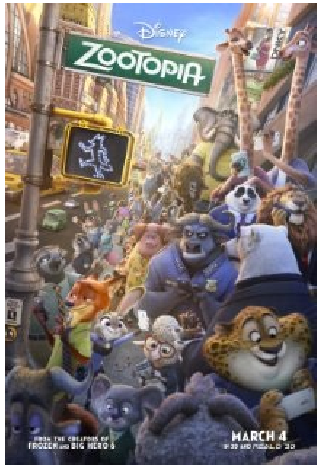
$$= \frac{\text{\# people who have watched both}}{\text{\# people who have watched Amelie}}$$

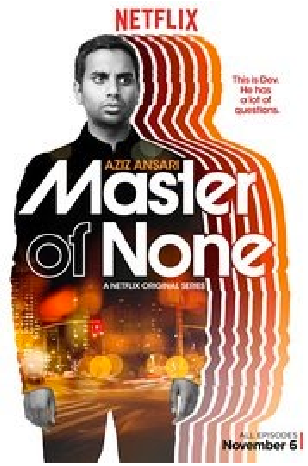
$$\approx 0.42$$

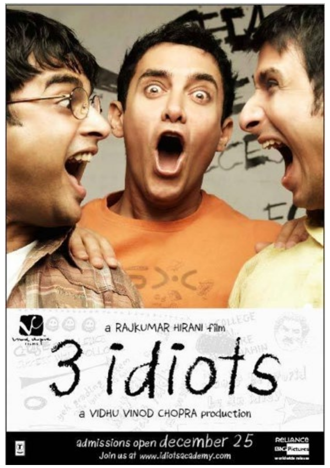
Netflix and Learn

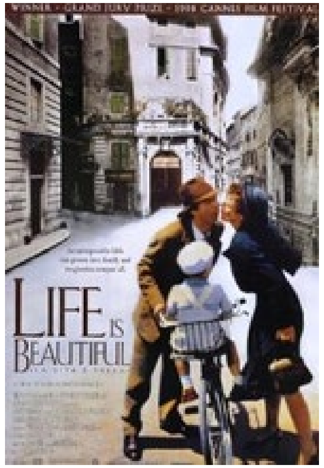
$$P(E|F) = \frac{P(EF)}{P(F)}$$ Definition of Cond. Probability

Let E be the event that a user watches the given movie.
Let F be the event that the same user watches Amelie.









$P(E) = 0.19$	$P(E) = 0.32$	$P(E) = 0.20$	$P(E) = 0.09$	$P(E) = 0.20$
$P(E\|F) = 0.14$	$P(E\|F) = 0.35$	$P(E\|F) = 0.20$	$P(E\|F) = 0.72$	$P(E\|F) = 0.42$

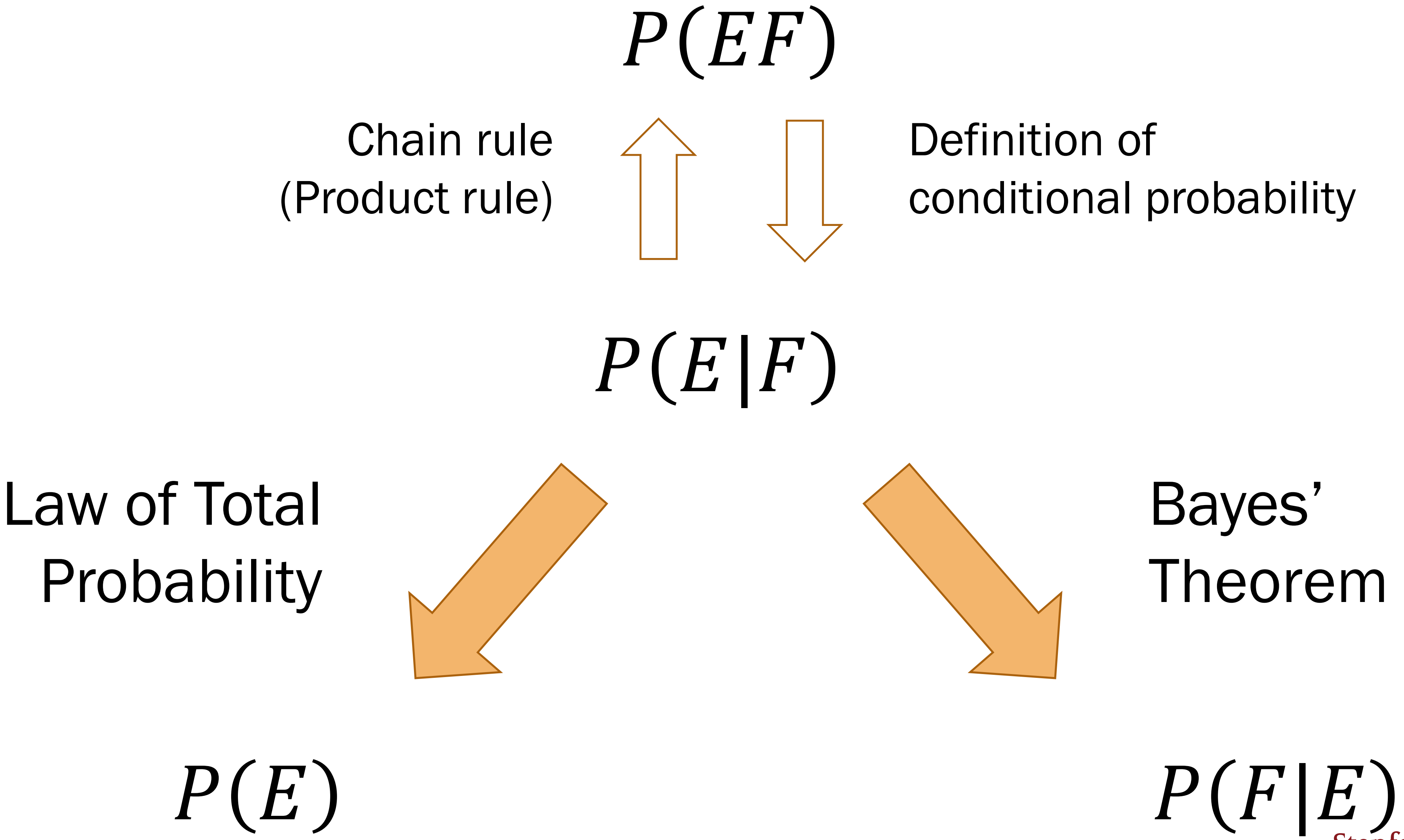
Today's plan

Conditional Probability and Chain Rule

Law of Total Probability

Bayes' Theorem

Today's plan in pictures



Law of Total Probability

$P(E|F) \Rightarrow P(E)$

<u>Thm</u> Let F be an event where $P(F) > 0$. For any event E,

$$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$$

<u>Proof</u>

1. F and F^C are disjoint s.t. $F \cup F^C = S$ — Def. of complement
2. $E = (EF) \cup (EF^C)$ — (see below)
3. $P(E) = P(EF) + P(EF^C)$ — Additivity axiom
4. $P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$ — Chain rule (product rule)

General Law of Total Probability

General Law of Total Probability

Thm For disjoint events $F_1, F_2, \ldots, F_n$ s.t. $F_1 \cup F_2 \cup \cdots \cup F_n = S$,

$$P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i)$$

Finding $P(E)$ from $P(E|F)$

$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$ Law of Total Probability

- Flip a fair coin.
- If heads: roll a fair 6-sided die.
- Else: roll a fair 3-sided die.

You win if you roll a 6. What is P(winning)?

Announcements

<u>Section sign-ups</u>

Results:	later today
Late signups/changes:	later today

<u>Problem set 1 autograder issues</u>

Read problem carefully:	see pinned Piazza post
Syntax issue:	`np.random.randint(1, 101)`

Today's plan

Conditional Probability and Chain Rule

Law of Total Probability

Bayes' Theorem

Today's tasks

$$P(EF)$$

Chain rule (Product rule)

Definition of conditional probability

$$P(E|F)$$

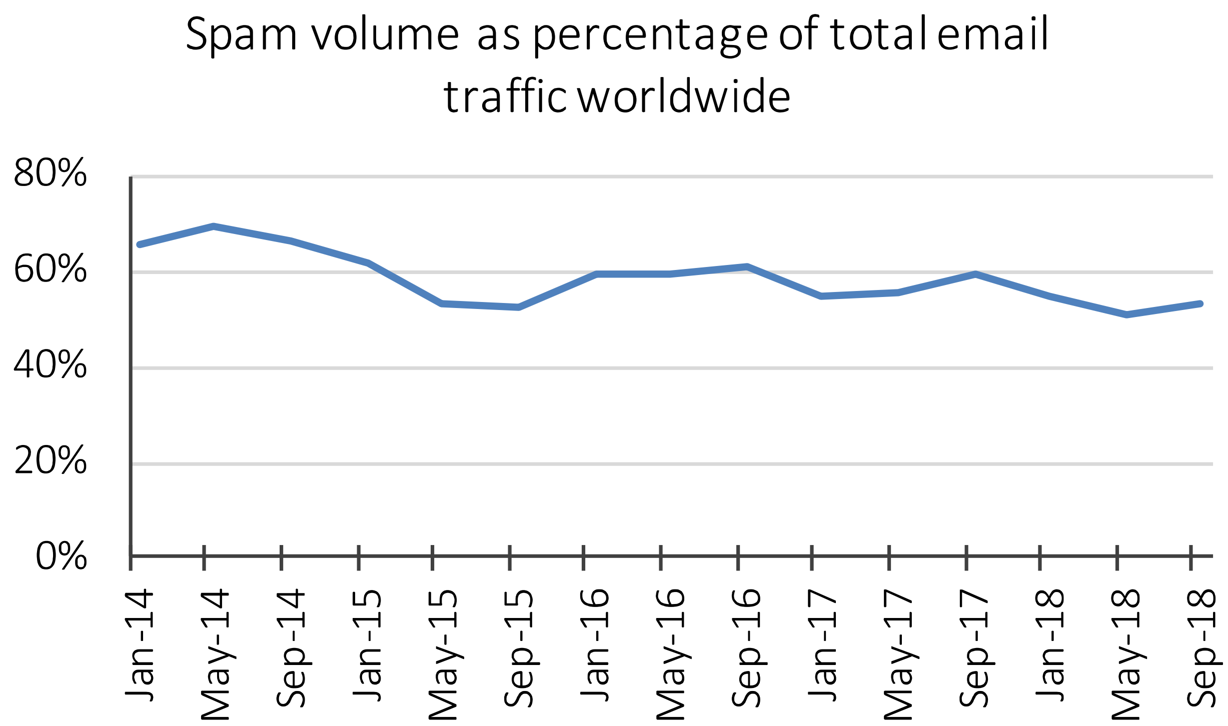
Law of Total Probability

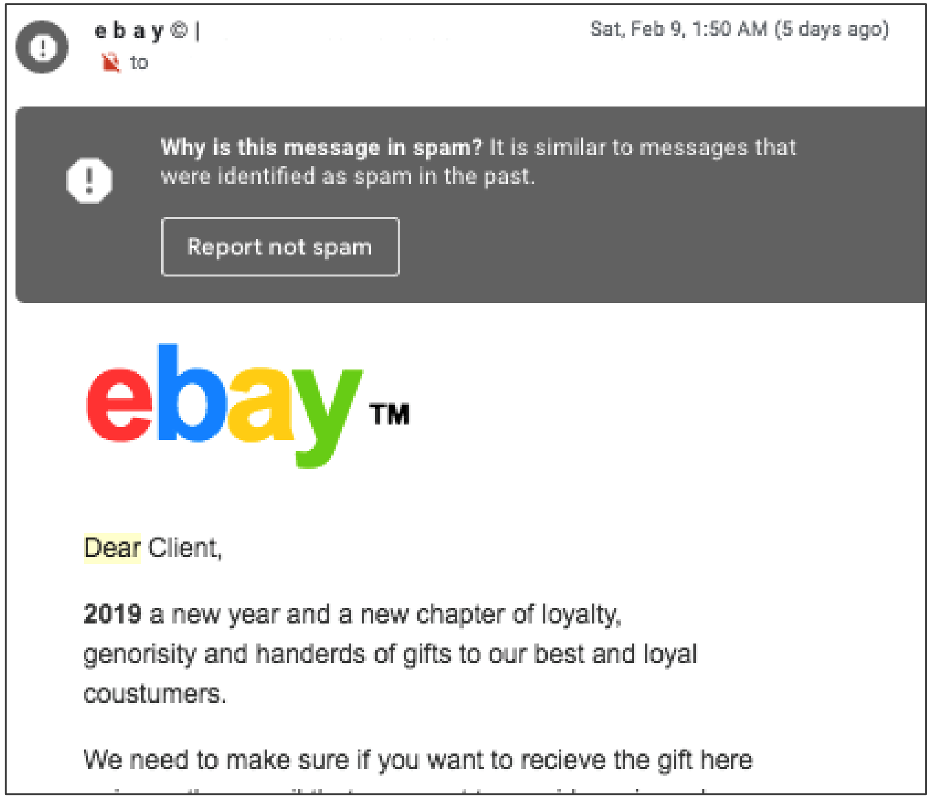
Bayes' Theorem

$$P(E)$$

$$P(F|E)$$

Detecting spam email





We can easily calculate how many spam emails contain "Dear":

$$P(E|F) = P\left(\text{“Dear”} \,\middle|\, \begin{matrix}\text{Spam}\\ \text{email}\end{matrix}\right)$$

But what is the probability that an email containing "Dear" is spam?

$$P(F|E) = P\left(\begin{matrix}\text{Spam}\\ \text{email}\end{matrix} \,\middle|\, \text{“Dear”}\right)$$

Bayes' Theorem

$P(E|F) \Rightarrow P(F|E)$

Thm For any events E and F where $P(E) > 0$ and $P(F) > 0$,

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

Proof

Bayes' Theorem (expanded form)

Thm   For any events E and F where $P(E) > 0$ and $P(F) > 0$,

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$

Proof

Detecting spam email

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$ Bayes' Theorem

- 60% of all email in 2019 is spam.
- 20% of spam has the word "Dear"
- 1% of non-spam (aka ham) has the word "Dear"

You get an email with the word "Dear" in it.

What is the probability that the email is spam?

Bayes' Theorem terminology

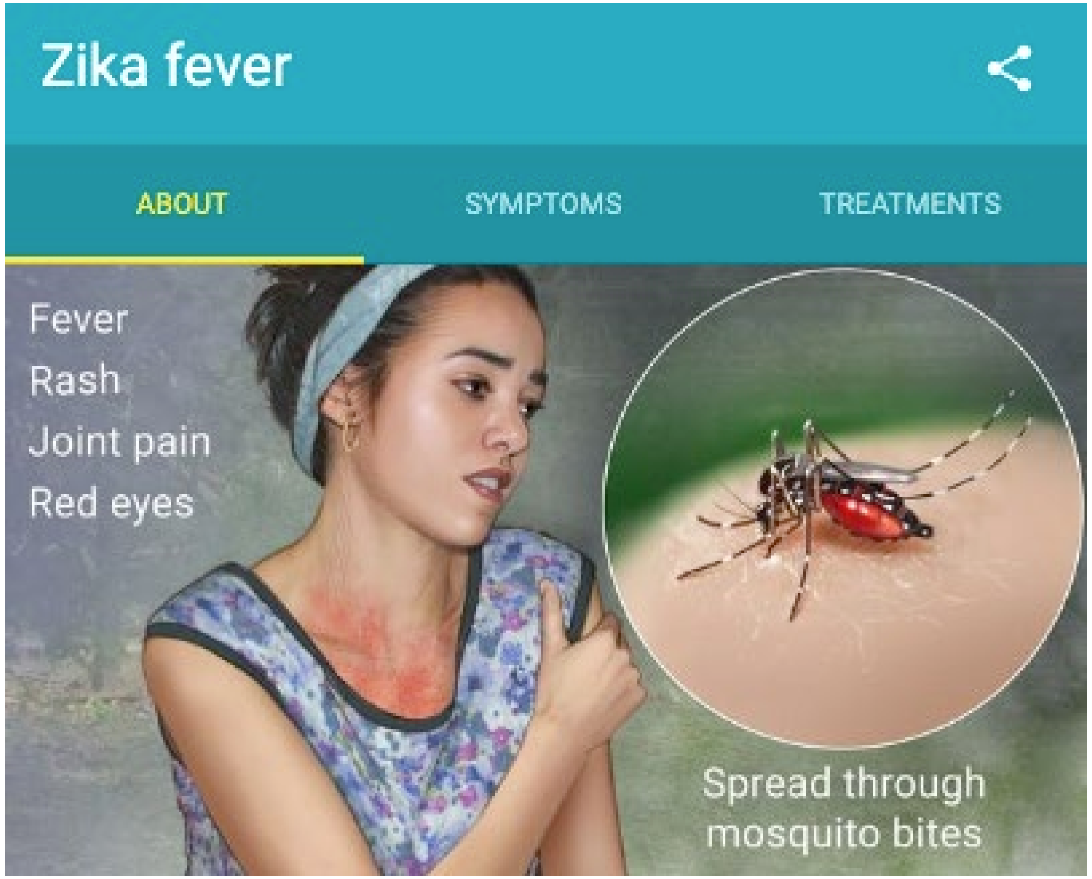
- 60% of all email in 2019 is spam. $P(F)$ prior
- 20% of spam has the word "Dear" $P(E|F)$ likelihood
- 1% of non-spam (aka ham) has the word "Dear" $P(E|F^C)$

You get an email with the word "Dear" in it.

What is the probability that the email is spam? Want: $P(F|E)$ posterior

$$\underbrace{P(F|E)}_{\text{posterior}} = \frac{\overbrace{P(E|F)}^{\text{likelihood}}\overbrace{P(F)}^{\text{prior}}}{\underbrace{P(E)}_{\text{normalization constant}}}$$

Zika, an autoimmune disease



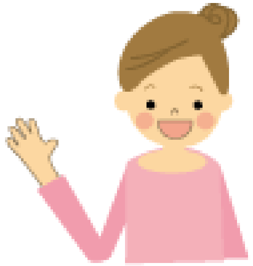
A disease spread through mosquito bites.
Usually no symptoms; worst case paralysis. During pregnancy: may cause birth defects

Ziika Forest, Uganda

Rhesus monkeys

If a test returns positive, what is the likelihood you have the disease?

Taking tests: Confusion matrix



Take test

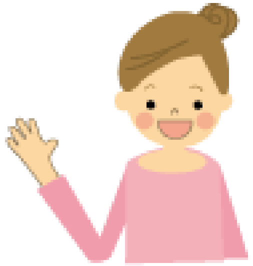
Fact, F — Has disease
or F^C — No disease

Evidence, E — Test positive
or E^C — Test negative

		Fact	
		F, disease +	F^C, disease –
Evidence	E, Test +	True positive $P(E \mid F)$	False positive $P(E \mid F^C)$
	E^C, Test –	False negative $P(E^C \mid F)$	True negative $P(E^C \mid F^C)$

If a test returns positive, what is the likelihood you have the disease?

Taking tests: Confusion matrix



Take test

Fact, F — Has disease
or F^C — No disease

Evidence, E — Test positive
or E^C — Test negative

		Fact	
		F, disease +	F^C, disease –
Evidence	E, Test +	True positive $P(E\|F)$	False positive $P(E\|F^C)$
	E^C, Test –	False negative $P(E^C\|F)$	True negative $P(E^C\|F^C)$

If a test returns positive, what is the likelihood you have the disease?

Zika Testing

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$ Bayes' Theorem

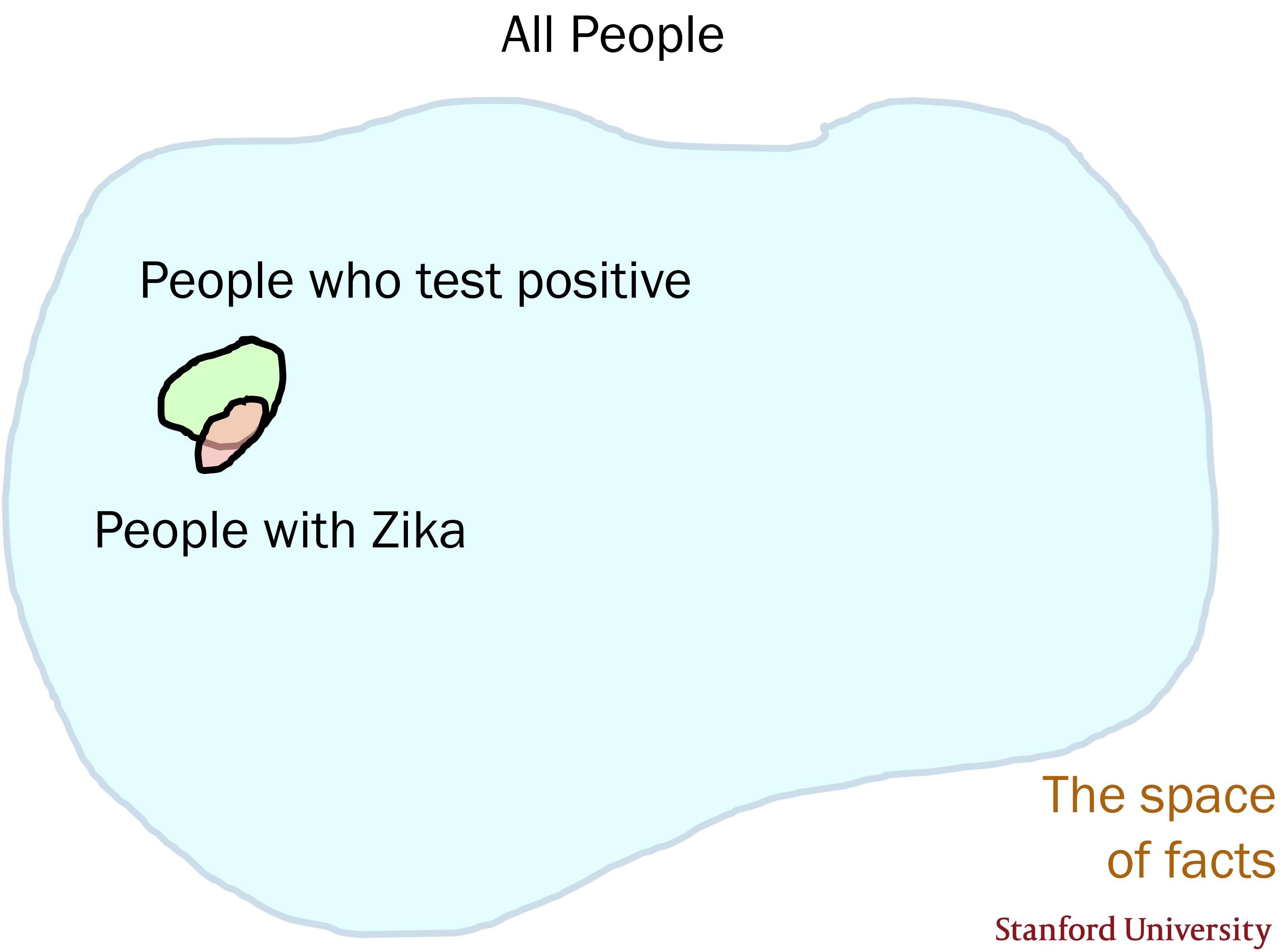
- A test is 98% effective at detecting Zika ("true positive").
- However, the test has a "false positive" rate of 1%.
- 0.5% of the US population has Zika.

What is the likelihood you have Zika if you test positive?

Bayes' Theorem intuition

<u>Original question</u>:

What is the likelihood you have Zika if you test positive for the disease?



Bayes' Theorem intuition

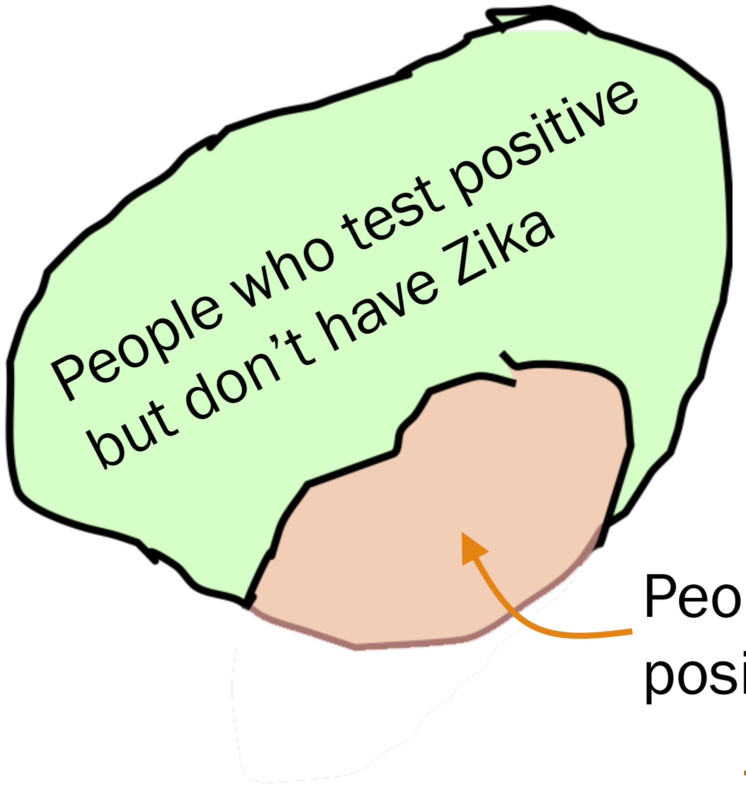
<u>Original question</u>:

What is the likelihood you have Zika if you test positive for the disease?

Interpret

<u>Interpretation</u>:
Of the people who test positive, how many actually have Zika?

People who test positive



People who test positive and have Zika

The space of facts, **<u>conditioned</u>** on a positive test result

Zika Testing

- A test is 98% effective at detecting Zika ("true positive").
- However, the test has a "false positive" rate of 1%.
- 0.5% of the US population has Zika.

What is the likelihood you have Zika if you test positive?

Say we have 1000 people:

5 have Zika
and test positive

985 **do not** have Zika
and test negative.

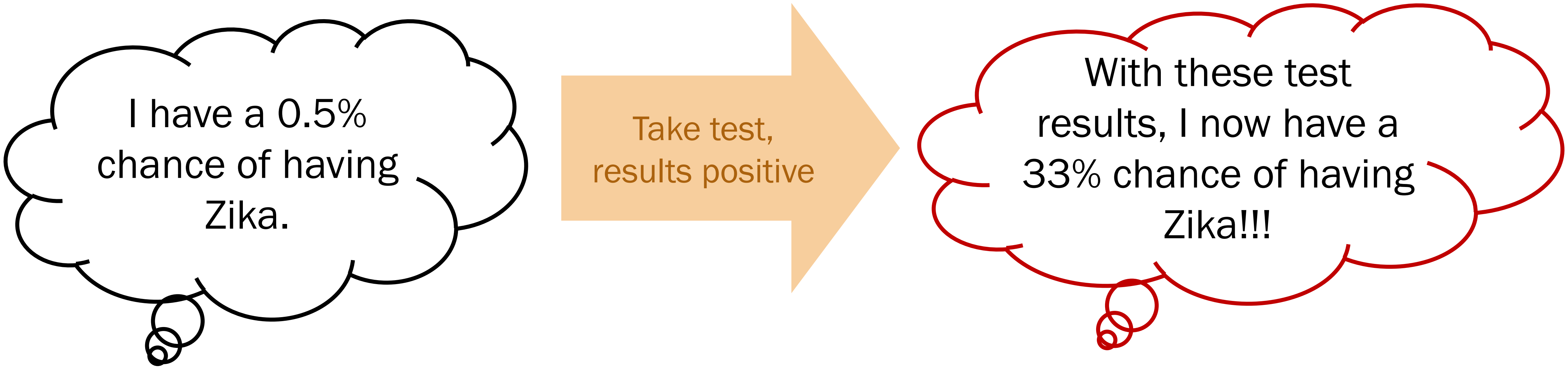
10 **do not** have Zika
and test positive.

≈ 0.333

Demo (class website)

Update your beliefs with Bayes' Theorem

E = you test positive for Zika
F = you actually have the disease



$P(F)$ $P(F|E)$

Why it's still good to get tested

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$ Bayes' Theorem

- A test is 98% effective at detecting Zika ("true positive"). $P(E|F)$
- However, the test has a "false positive" rate of 1%. $P(E|F^C)$
- 0.5% of the US population has Zika. $P(F)$

Let: E = you test positive
F = you actually have the disease
E^C = you test negative for Zika with this test.

	F, disease +	F^C, disease –
E, Test +	True positive $P(E\|F) = 0.98$	False positive $P(E\|F^C) = 0.01$

What is $P(F|E^C)$?

Why it's still good to get tested

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$ Bayes' Theorem

- A test is 98% effective at detecting Zika ("true positive"). $P(E|F)$
- However, the test has a "false positive" rate of 1%. $P(E|F^C)$
- 0.5% of the US population has Zika. $P(F)$

Let: E = you test positive
F = you actually have the disease
E^C = you test negative for Zika with this test.

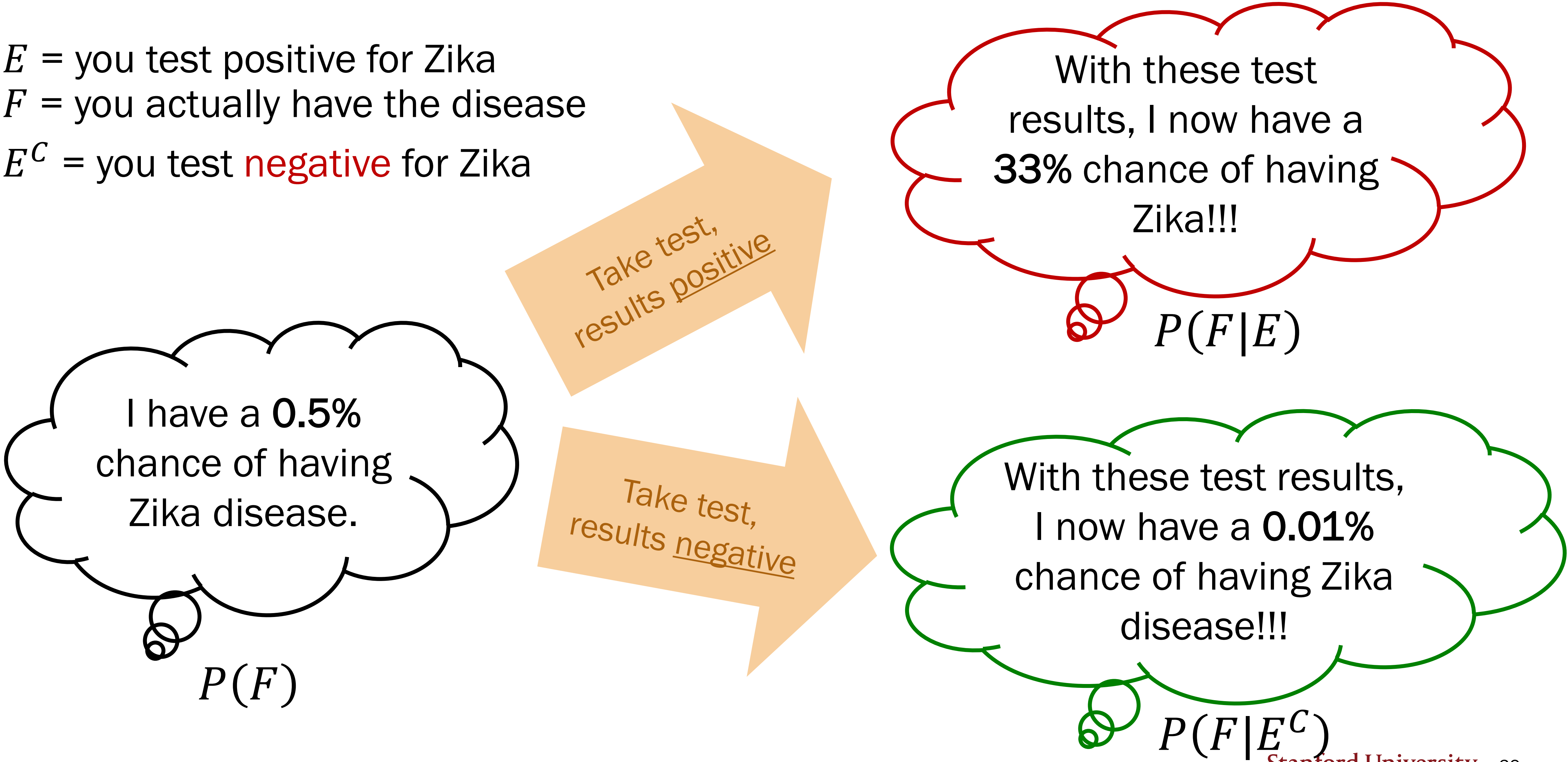
	F, disease +	F^C, disease –
E, Test +	True positive $P(E\|F) = 0.98$	False positive $P(E\|F^C) = 0.01$
E^C, Test –	False negative $P(E^C\|F) = 0.02$	True negative $P(E^C\|F^C) = 0.99$

What is $P(F|E^C)$?

$$P(F|E^C) = \frac{P(E^C|F)P(F)}{P(E^C|F)P(F) + P(E^C|F^C)P(F^C)}$$

Why it's still good to get tested

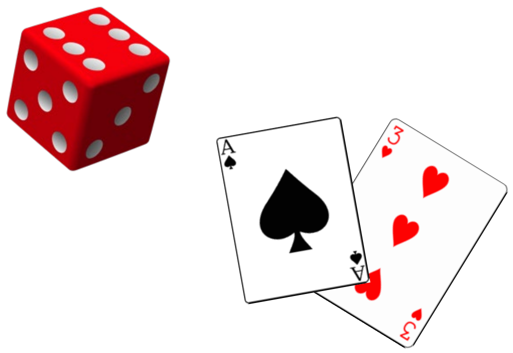
E = you test positive for Zika
F = you actually have the disease
E^C = you test negative for Zika



This class going forward

<u>Last week</u>

Equally likely events



$P(E \cap F)$ $P(E \cup F)$

(counting, combinatorics)

<u>Today and for most of this course</u>

Not equally likely events

$P(E = \text{Evidence} \mid F = \text{Fact})$

(collected from data)

$P(F = \text{Fact} \mid E = \text{Evidence})$

(categorize a new datapoint)

Another conditional probability example

Monty Hall Problem from Let's Make a Deal

Behind one door is a car (equally likely to be any door).

Behind the other two doors are goats

1. You choose a door
2. Host opens 1 of other 2 doors, revealing a goat
3. You are given an option to change to the other door.

Should you switch?

Doors A, B, C

What happens if you switch

You picked:

1/3 1/3 1/3

A = prize
- Host opens
- You switch to
- Result

P(win | A prize,
you picked ___,
switch) =

B = prize
- Host opens
- You switch to
- Result

P(win | B prize,
you picked ___,
switch) =

C = prize
- Host opens
- You switch to
- Result

P(win | C prize,
you picked ___,
switch) =

P(win | you picked ___, switched) =

Monty Hall, 1000 envelope version

Start with 1000 envelopes (of which 1 is the prize).

1. You choose 1 envelope.
   - $\frac{1}{1000}$ = P(envelope is prize)
   - $\frac{999}{1000}$ = P(other 999 envelopes have prize)

2. I open 998 of remaining 999 (showing they are empty).
   - $\frac{999}{1000}$ = P(998 empty envelopes had prize) + P(last other envelope has prize) = P(last other envelope has prize)

3. Should you switch?
   - P(you win without switching) = $\frac{1}{\text{original \# envelopes}}$
   - P(you win with switching) = $\frac{\text{original \# envelopes} - 1}{\text{original \# envelopes}}$