25: Logistic Regression

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Adapted from slides by Lisa Yan

Announcements

Updated CS109 logistics and policies

We are implementing several changes to CS109 logistics and policies in response to growing concern about Covid-19. These changes affect lectures, section, office hours and the final exam.

Read the new policies in the announcements on the course webpage: http://web.stanford.edu/class/cs109/

We welcome your questions on piazza

Announcements

Problem Set 6

Due: Wednesday 3/11
Covers: Up to Lecture 25
Extra Python Office Hours: Saturday 3/7, 3-5PM

Regrades

Pset 1 to 5 and Midterm regrades to close on 3/11 at 1pm

Autograded Coding Problems

Run your code in the command line or install Pycharm following directions on Pset 6 webpage

Late Day Reminder

No late days permitted past last day of the quarter, 3/13

Today’s plan

Logistic Regression

- Chapter 0: Background
- Chapter 1: Big Picture
- Chapter 2: Details
- Chapter 3: Philosophy

Background: Weighted sum

If $\boldsymbol{X} = (X_1, X_2, \dots, X_m)$:

$$z = \theta^T \boldsymbol{X} = \sum_{j=1}^{m} \theta_j X_j$$

Weighted sum
(aka dot product)

$$= \theta_1 X_1 + \theta_2 X_2 + \cdots + \theta_m X_m$$

Weighted sum with an intercept term:

$$z = \theta_0 + \sum_{j=1}^{m} \theta_j X_j$$

$$= \theta_0 X_0 + \theta_1 X_1 + \theta_2 X_2 + \cdots + \theta_m X_m$$

Define $X_0 = 1$

$$= \theta^T \boldsymbol{X}$$

New $\boldsymbol{X} = (1, X_1, X_2, \dots, X_m)$

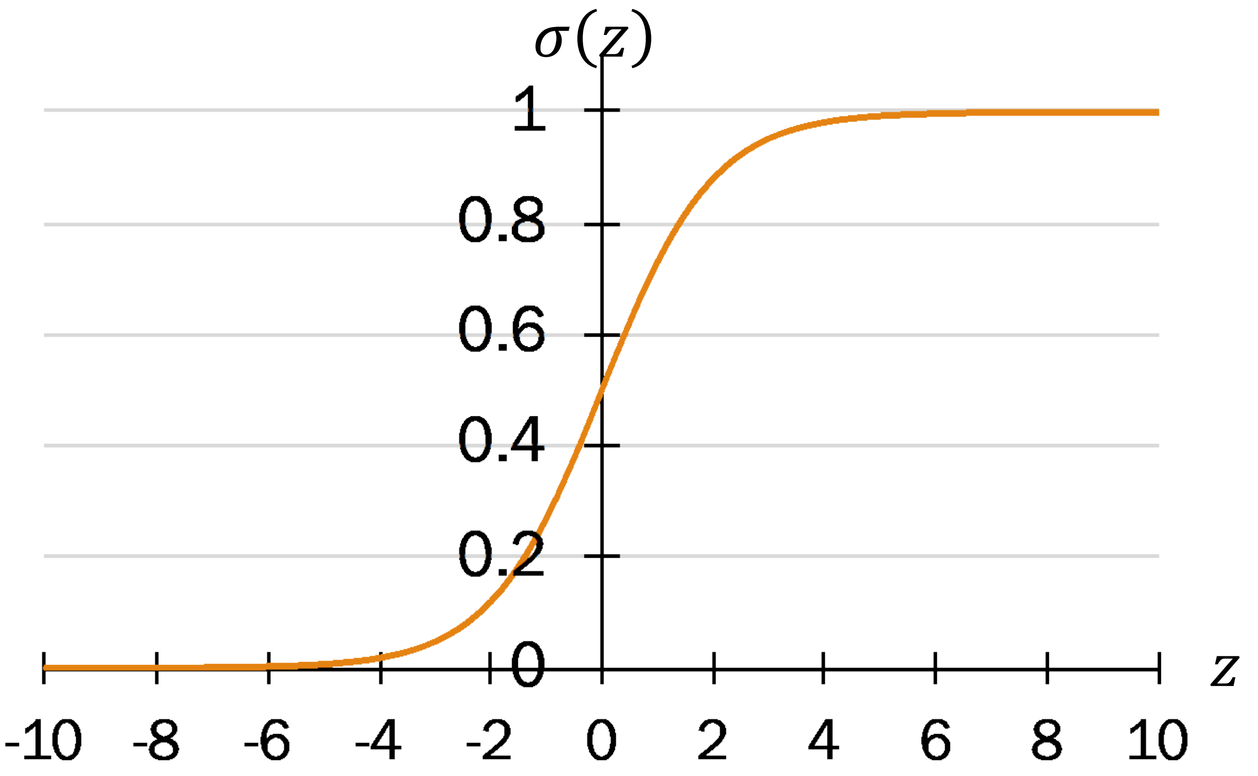
Background: Sigmoid function $\sigma(z)$

- The sigmoid function:

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

- Sigmoid squashes z to a number between 0 and 1.

- Recall definition of probability: A number between 0 and 1



$\sigma(z)$ can represent a probability.

Background: Chain Rule

$$f(x) = f\big(z(x)\big)$$

$$\frac{\partial f(x)}{\partial x} = \frac{\partial f(z)}{\partial z}\frac{\partial z}{\partial x}$$

Calculus
Chain Rule

Today's plan

Logistic Regression

- Chapter 0: Background
- Chapter 1: Big Picture
- Chapter 2: Details
- Chapter 3: Philosophy

From Naïve Bayes to Logistic Regression

Classification goal: Model $P(Y \mid \boldsymbol{X})$

$$\hat{Y} = \underset{y=\{0,1\}}{\arg\max}\, P(Y \mid \boldsymbol{X})$$

Predict the Y that is most likely given our observation $\boldsymbol{X}$

Naïve Bayes Classifier:

- Estimate $P(\boldsymbol{X} \mid Y)$ and $P(Y)$ because $\underset{y=\{0,1\}}{\arg\max}\, P(Y \mid \boldsymbol{X}) = \underset{y=\{0,1\}}{\arg\max}\, P(\boldsymbol{X}|Y)P(Y)$
- Actually modeling $P(\boldsymbol{X}, Y)$
- Assume $P(\boldsymbol{X}|Y) = P(X_1, X_2, \ldots, X_n|Y) = \prod_{i=1}^{m} P(X_i|Y)$

Can we model $P(Y \mid \boldsymbol{X})$ directly?

- Welcome our friend: Logistic Regression!

Logistic Regression

$$\hat{Y} = \underset{y=\{0,1\}}{\arg\max}\, P(Y \mid \boldsymbol{X})$$

Predict the Y that is most likely given our observation $\boldsymbol{X}$

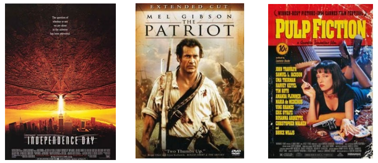
Logistic Regression Model

$$P(Y = 1 | \boldsymbol{X} = \boldsymbol{x}) = \sigma\left(\theta_0 + \sum_{j=1}^{m} \theta_j x_j\right)$$

models $P(Y \mid \boldsymbol{X})$ directly

Logistic Regression

$\boldsymbol{x} = [0,1,1]$



$\boldsymbol{X}$

input features

θ parameter

0.81

$P(Y = 1|\boldsymbol{X} = \boldsymbol{x}])$

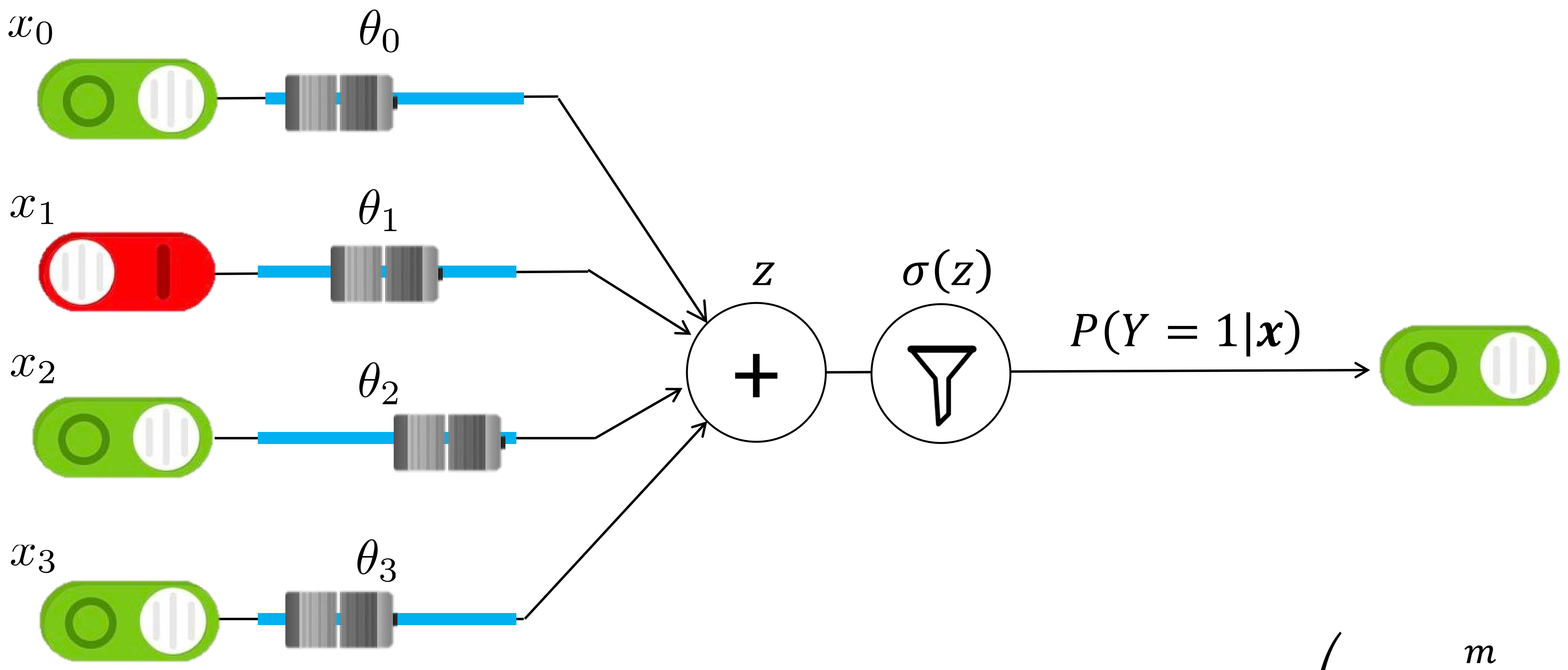
conditional likelihood

$$P(Y = 1|\boldsymbol{X} = x) = \sigma\left(\theta_0 + \sum_{j=1}^{m} \theta_j x_j\right)$$

Logistic Regression Cartoon

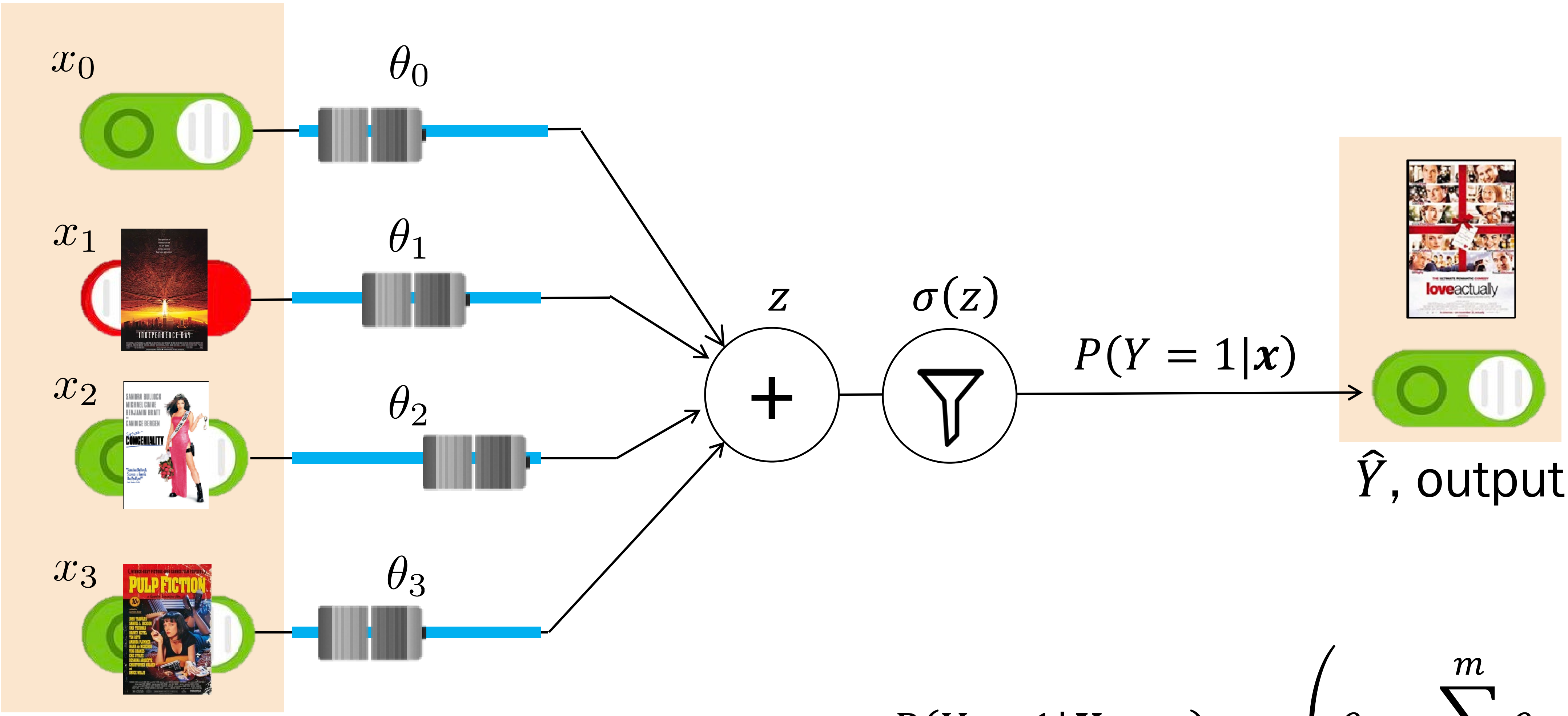
θ parameter

Logistic Regression cartoon



$$P(Y=1|\boldsymbol{X}=\boldsymbol{x})=\sigma\left(\theta_0+\sum_{j=1}^{m}\theta_j x_j\right)$$

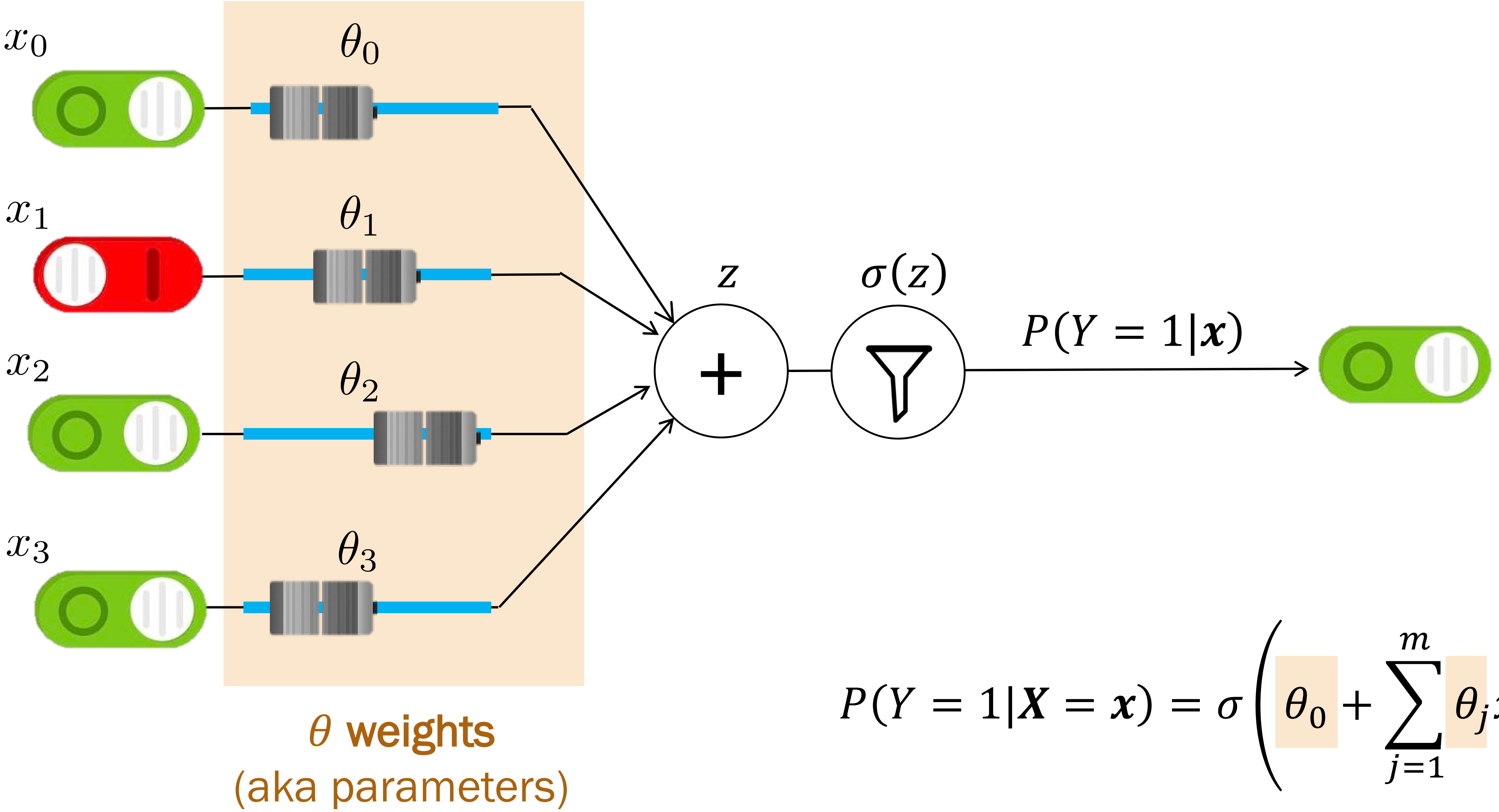
Logistic Regression input/output



$\boldsymbol{X}$, input features
[0,1,1]

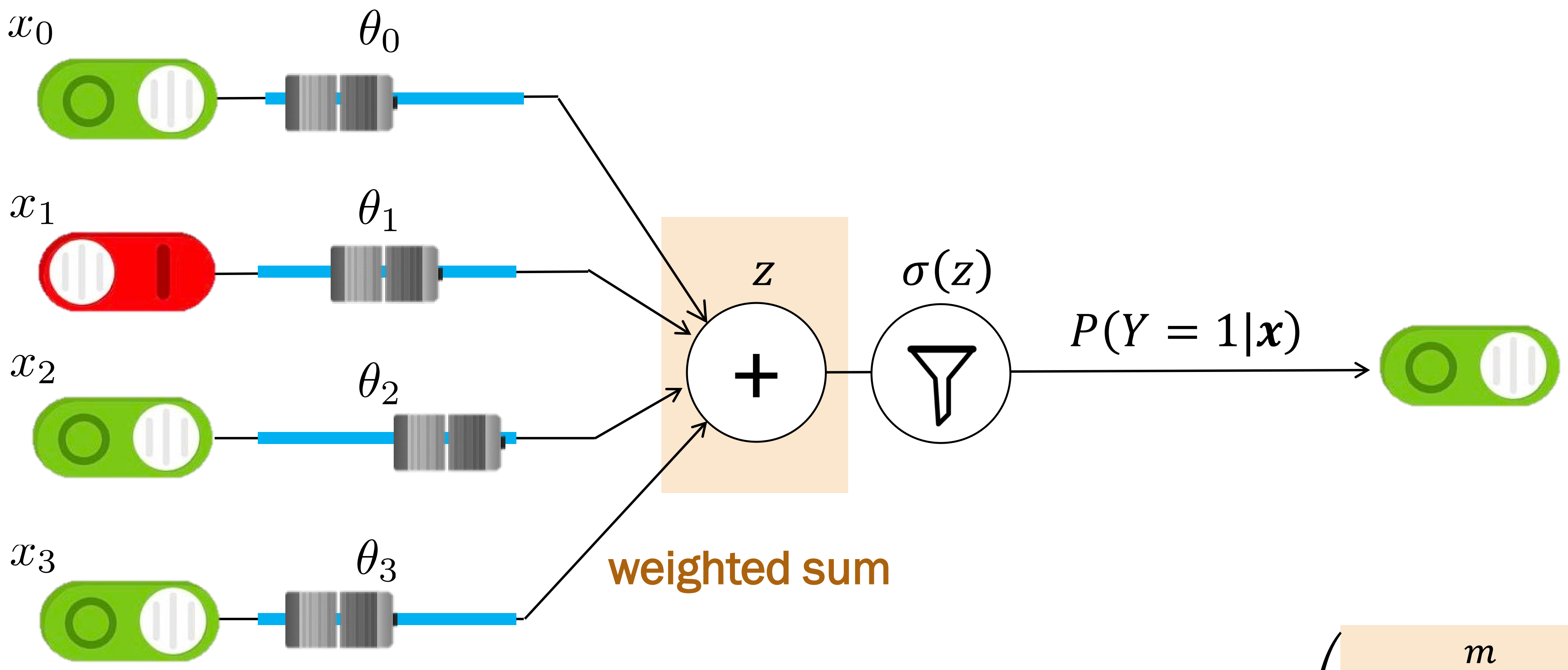
$$P(Y=1|\boldsymbol{X}=\boldsymbol{x}) = \sigma\left(\theta_0 + \sum_{j=1}^{m} \theta_j x_j\right)$$

Components of Logistic Regression



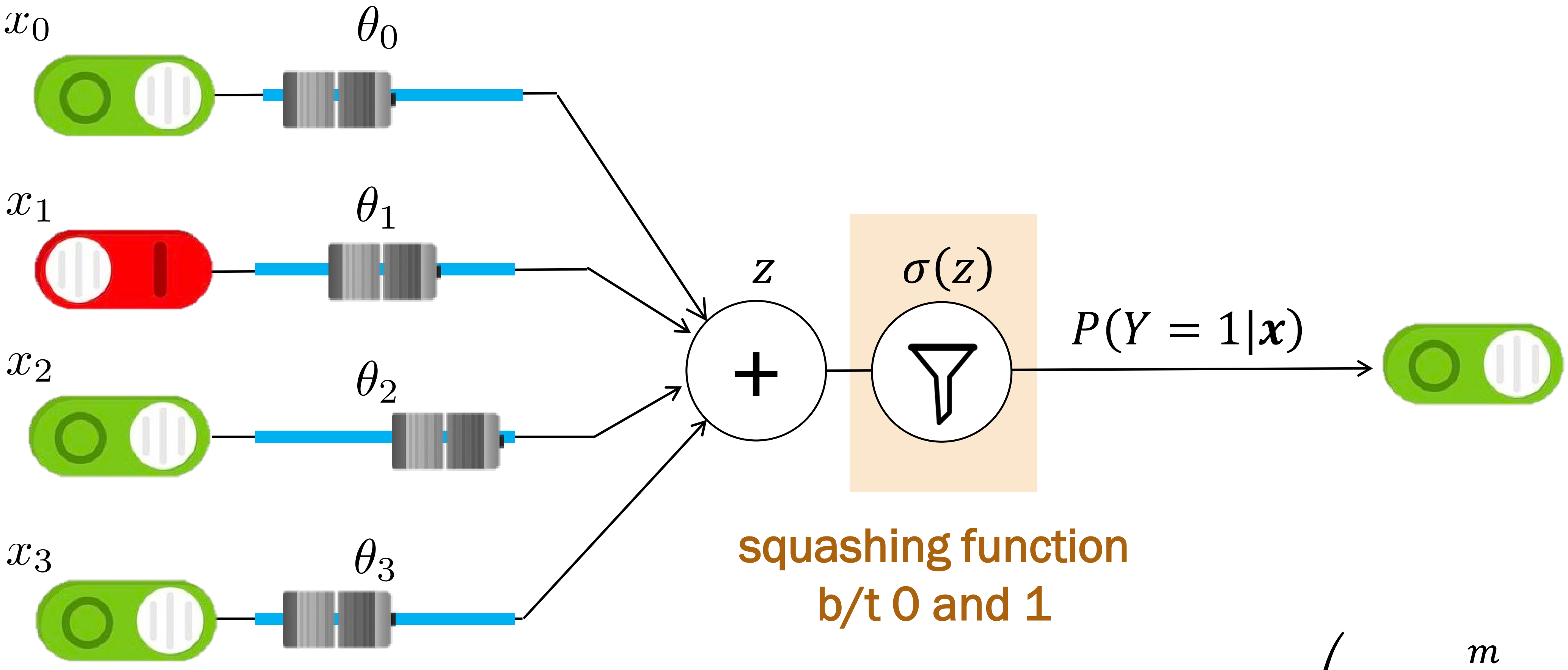
$$P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \sigma\left(\theta_0 + \sum_{j=1}^{m} \theta_j x_j\right)$$

Components of Logistic Regression



$$P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \sigma\left(\theta_0 + \sum_{j=1}^{m} \theta_j x_j\right)$$

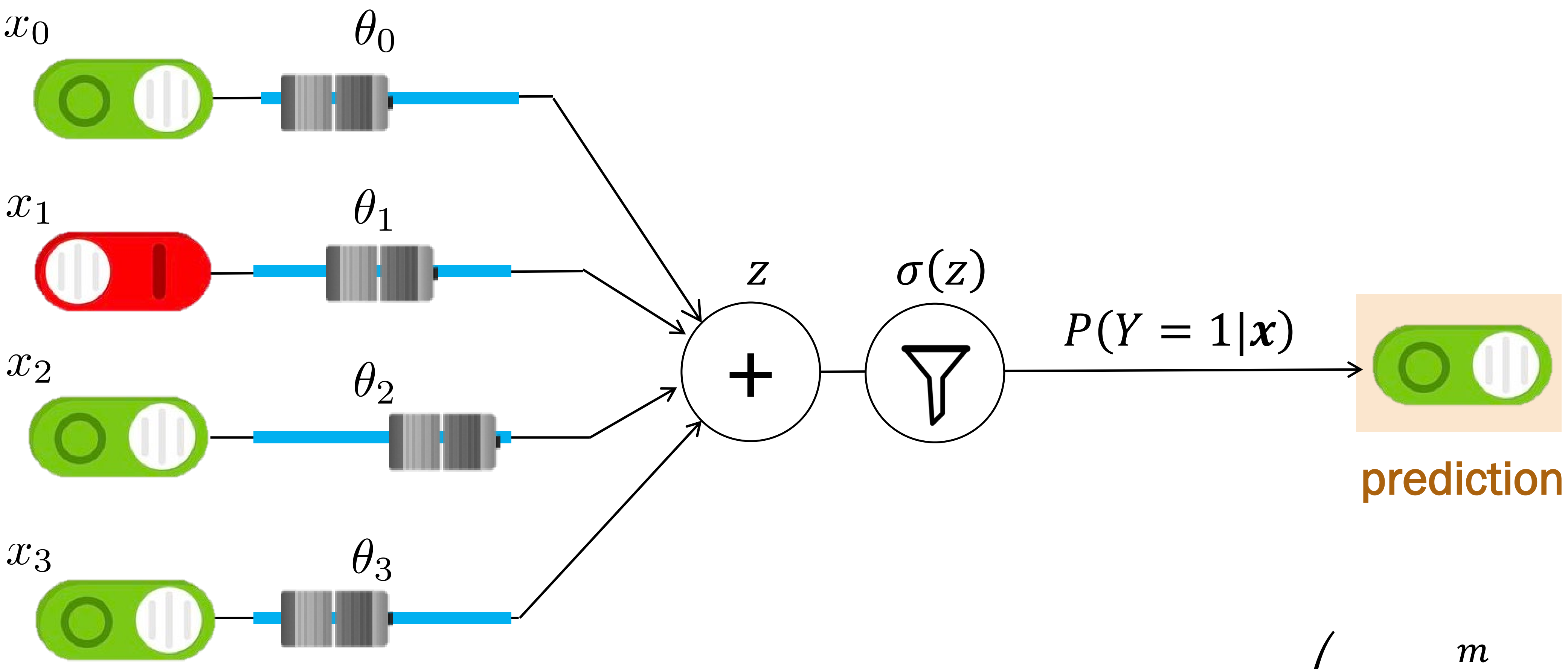
Components of Logistic Regression



squashing function
b/t 0 and 1

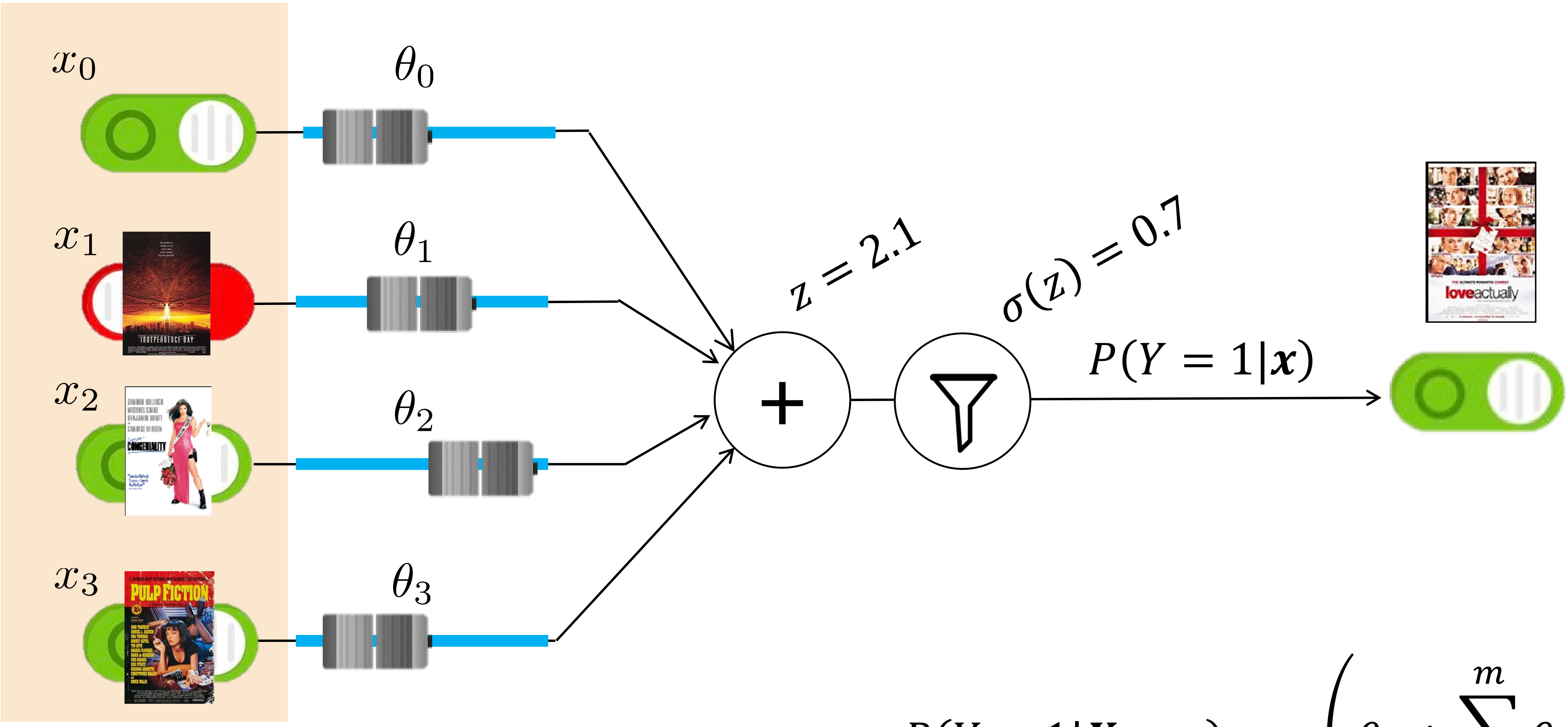
$$P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \sigma\left(\theta_0 + \sum_{j=1}^{m} \theta_j x_j\right)$$

Components of Logistic Regression



$$P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \sigma\left(\theta_0 + \sum_{j=1}^{m} \theta_j x_j\right)$$

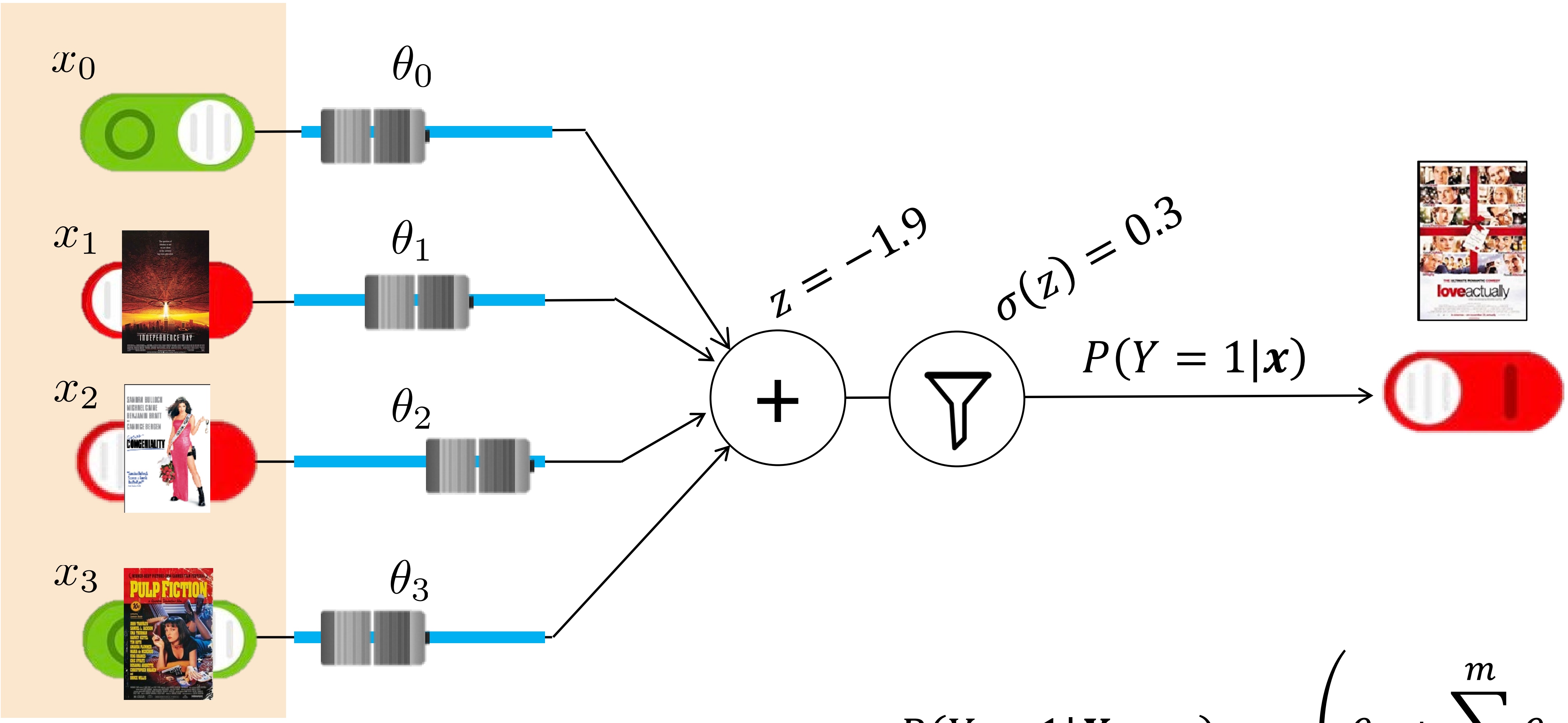
Different predictions for different inputs



$\boldsymbol{X}$, input features
[0,1,1]

$$P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \sigma\left(\theta_0 + \sum_{j=1}^{m} \theta_j x_j\right)$$

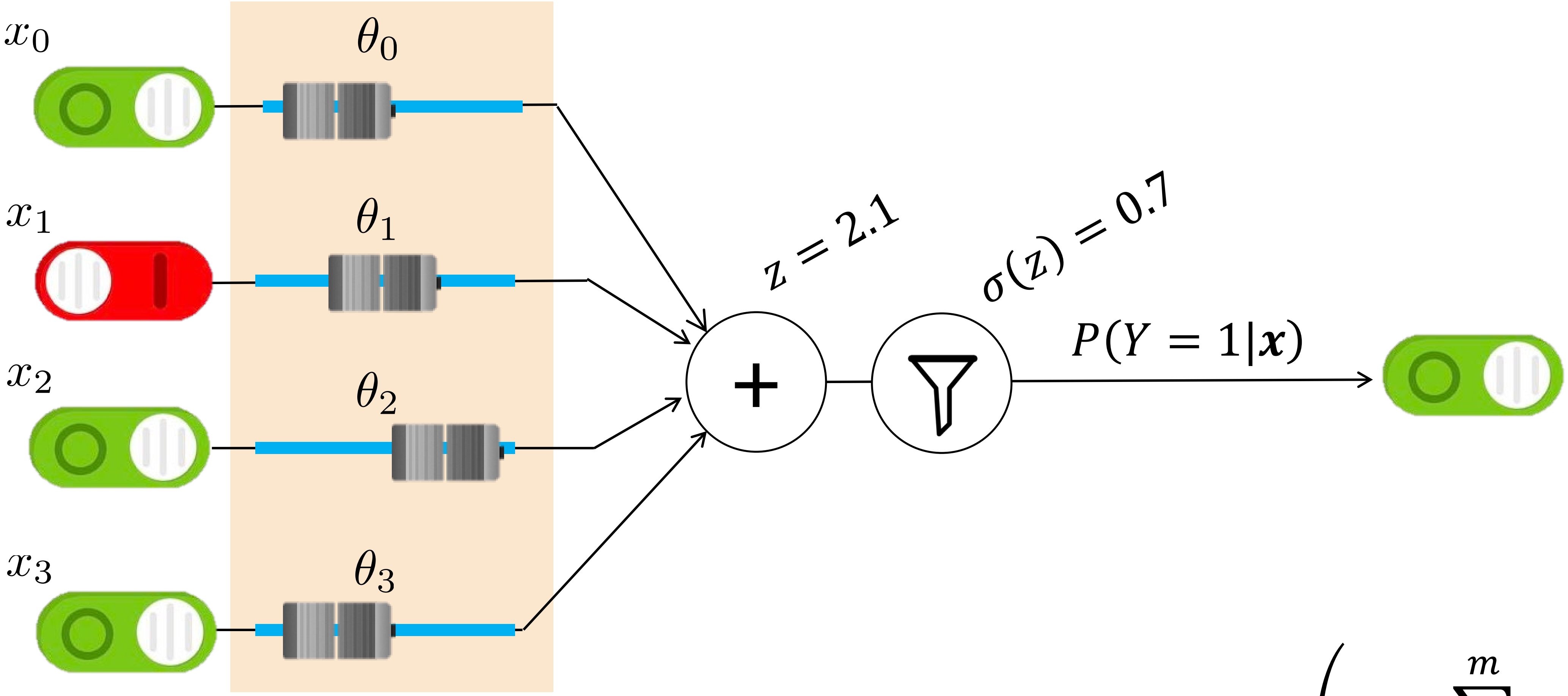
Different predictions for different inputs



$\boldsymbol{X}$, input features
[0,0,1]

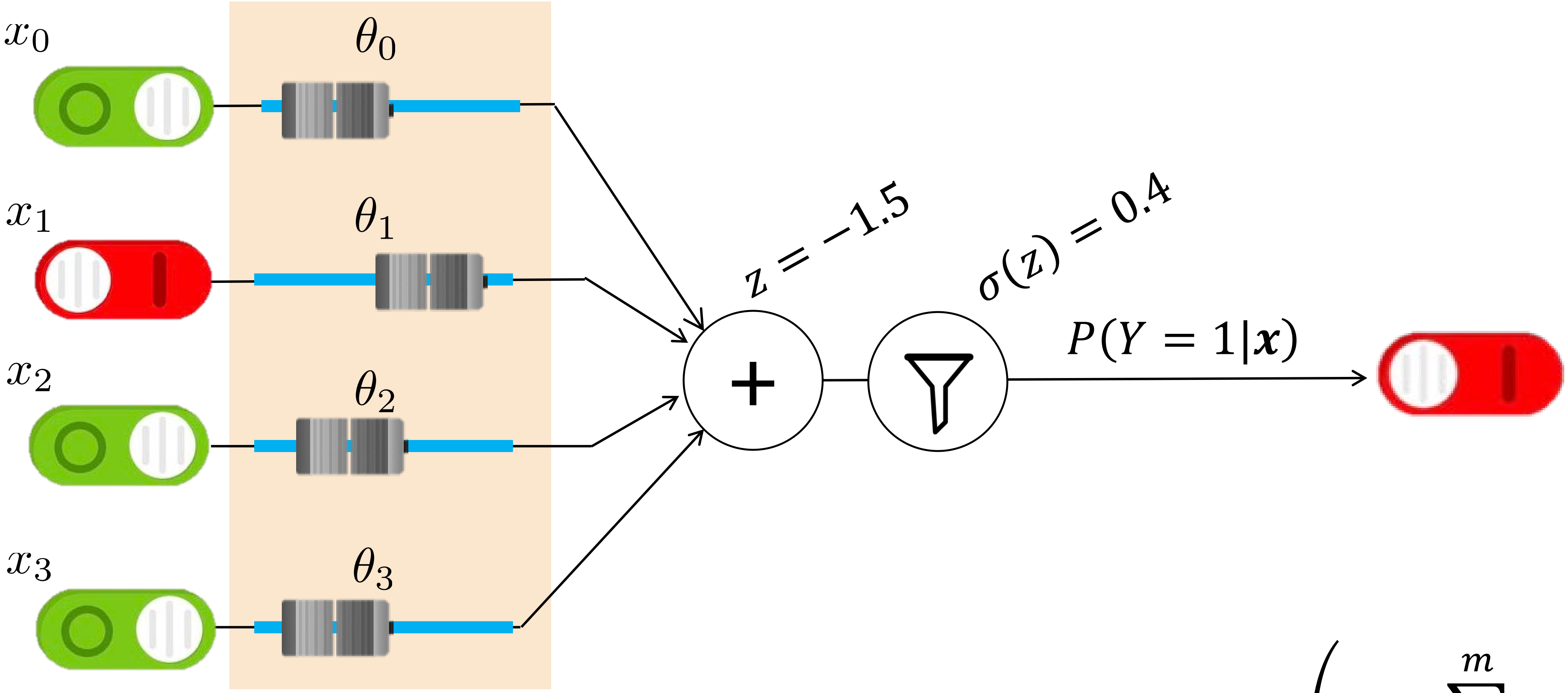
$$P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \sigma\left(\theta_0 + \sum_{j=1}^{m} \theta_j x_j\right)$$

Parameters affect prediction



$$P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \sigma\left(\theta_0 + \sum_{j=1}^{m} \theta_j x_j\right)$$

Parameters affect prediction



$$P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \sigma\left(\theta_0 + \sum_{j=1}^{m} \theta_j x_j\right)$$

Logistic Regression Model

$$\hat{Y} = \arg\max_{y=\{0,1\}} P(Y \mid \boldsymbol{X})$$

Predict the Y that is most likely given our observation $\boldsymbol{X}$

where

$$P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \sigma\left(\theta_0 + \sum_{j=1}^{m} \theta_j x_j\right)$$

models $P(Y \mid \boldsymbol{X})$ directly

- $\sigma(z) = \frac{1}{1+e^{-z}}$, the sigmoid function
- For simplicity, define $x_0 = 1$: $\boxed{P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \sigma(\theta^T \boldsymbol{x})}$
- Since $P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) + P(Y = 0|\boldsymbol{X} = \boldsymbol{x}) = 1$:

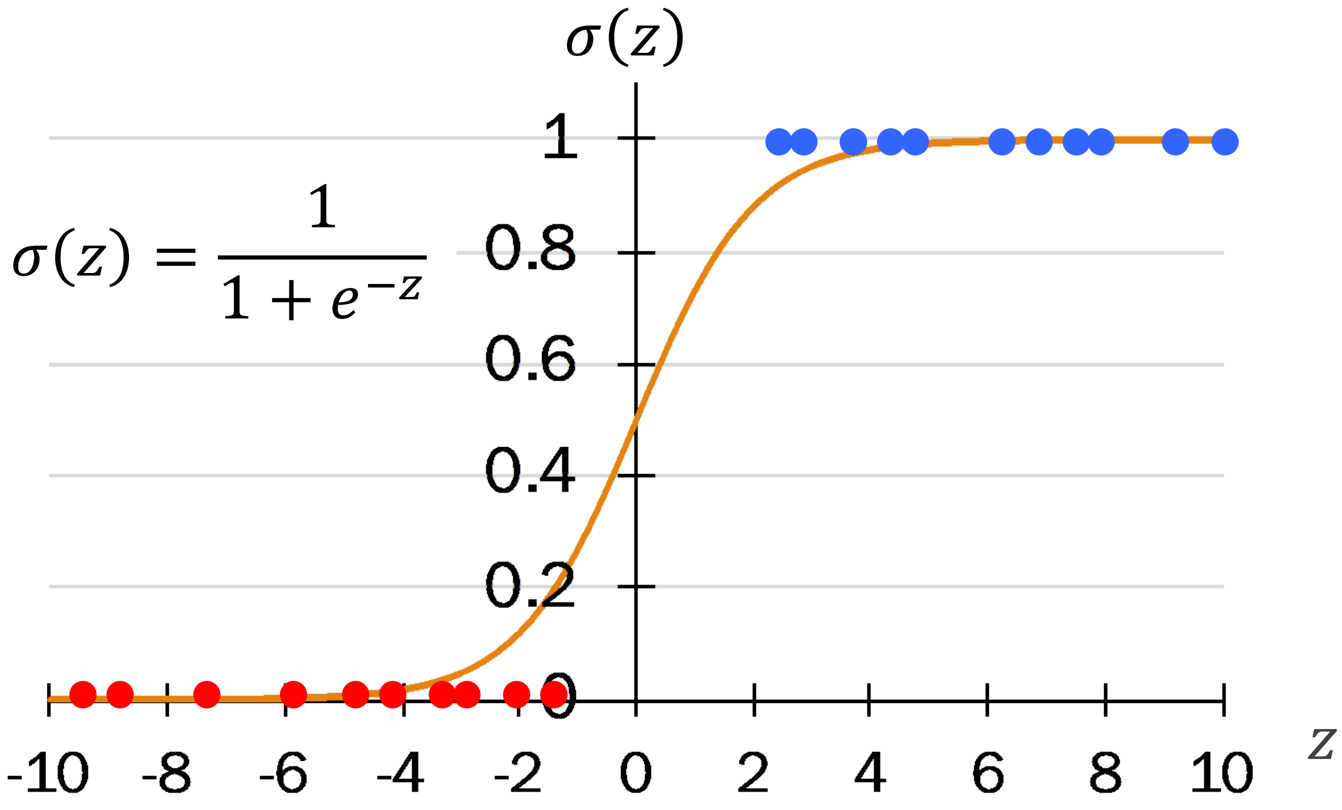
$$P(Y = 0|\boldsymbol{X} = \boldsymbol{x}) = 1 - \sigma(\theta^T \boldsymbol{x})$$

Classifying using the sigmoid function

Logistic Regression Model

$$\hat{Y} = \arg\max_{y=\{0,1\}} P(Y \mid \boldsymbol{X}) \quad \text{where} \quad P(Y=1|\boldsymbol{X}=\boldsymbol{x}) = \sigma\left(\theta_0 + \sum_{j=1}^{m} \theta_j x_j\right)$$

$$\sigma(z) = \frac{1}{1+e^{-z}}$$

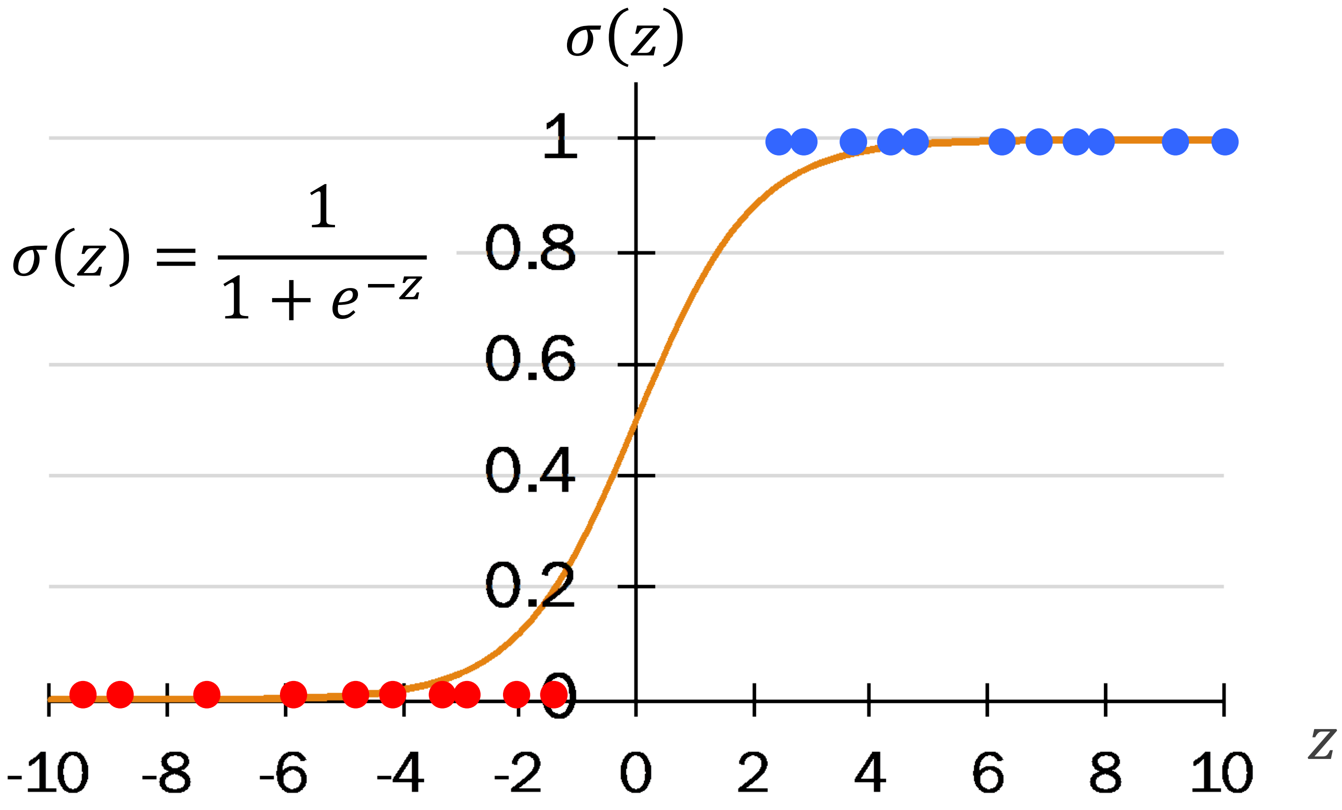


Logistic Regression uses the sigmoid function to try and distinguish $y = 1$ (blue) points from $y = 0$ (red) points.

Classifying using the sigmoid function

Logistic Regression Model $\hat{Y} = \underset{y=\{0,1\}}{\arg\max}\, P(Y \mid \boldsymbol{X})$ where $P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \sigma\left(\theta_0 + \sum_{j=1}^{m} \theta_j x_j\right)$

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$



When do we predict $\hat{Y} = 1$?

A. If $\sigma(\theta^T \boldsymbol{x}) > 1 - \sigma(\theta^T \boldsymbol{x})$

B. If $\sigma(\theta^T \boldsymbol{x}) > 0.5$

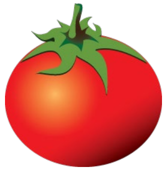
C. If $\theta^T \boldsymbol{x} > 0$

D. All are valid, but C is easiest

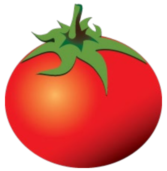
E. None/Other

Naming algorithms

Regression Algorithms

Linear Regression 

Classification Algorithms

Naïve Bayes 

Logistic Regression 

Awesome classifier,
terrible name

Training: Learning the parameters

Logistic regression gets its **intelligence** from its parameters $\theta = (\theta_0, \theta_1, \ldots, \theta_m)$.

- Logistic Regression Model:

$$P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \sigma(\theta^T \boldsymbol{x})$$

- Want to predict training data as correctly as possible:

$$\arg\max_{y=\{0,1\}} P\left(Y|\boldsymbol{X} = \boldsymbol{x}^{(i)}\right) = y^{(i)}$$ as often as possible

- Therefore, choose θ that maximizes the conditional likelihood of observing i.i.d. training data:

$$L(\theta) = \prod_{i=1}^{n} P\left(Y = y^{(i)}|\boldsymbol{X} = \boldsymbol{x}^{(i)}, \theta\right)$$

During training, find the θ that maximizes log-conditional likelihood of the training data. Use MLE!

Training: Learning the parameters via MLE

0. Add $x_0^{(i)} = 1$ to each $\boldsymbol{x}^{(i)}$
1. Logistic Regression model: $P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \sigma(\theta^T \boldsymbol{x})$
2. Compute log-likelihood of training data:

$$LL(\theta) = \sum_{i=1}^{n} y^{(i)} \log \sigma\left(\theta^T \boldsymbol{x}^{(i)}\right) + \left(1 - y^{(i)}\right) \log\left(1 - \sigma\left(\theta^T \boldsymbol{x}^{(i)}\right)\right)$$

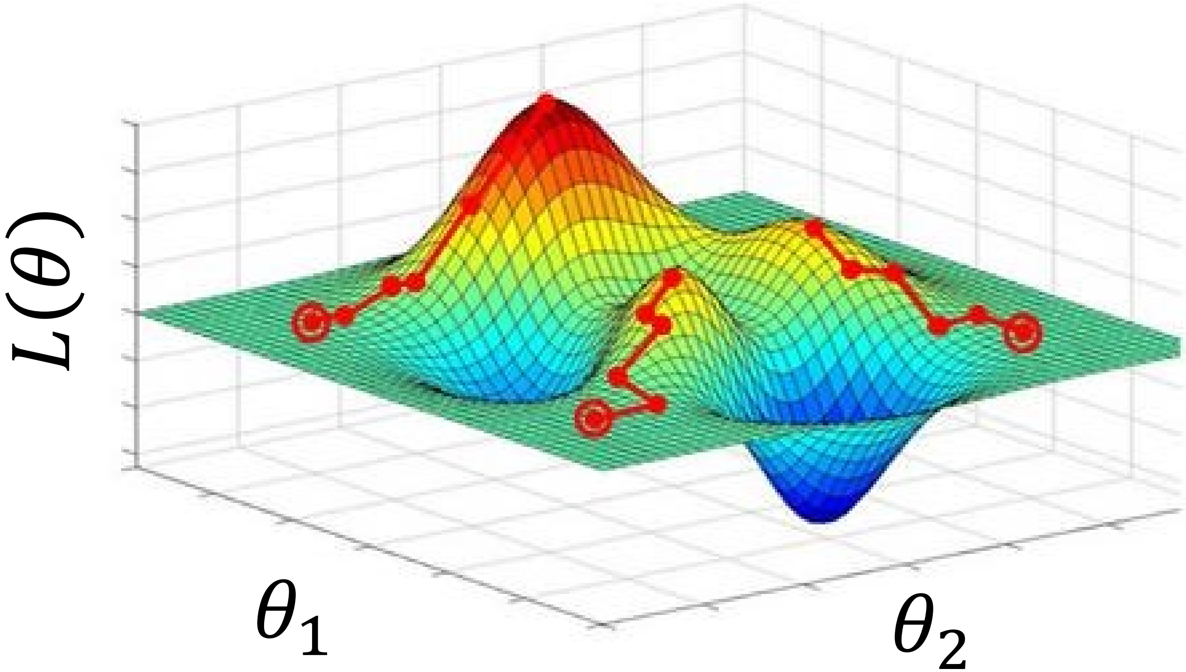
3. Compute derivative of log-likelihood with respect to each $\theta_j, j = 0, 1, \ldots, m$:

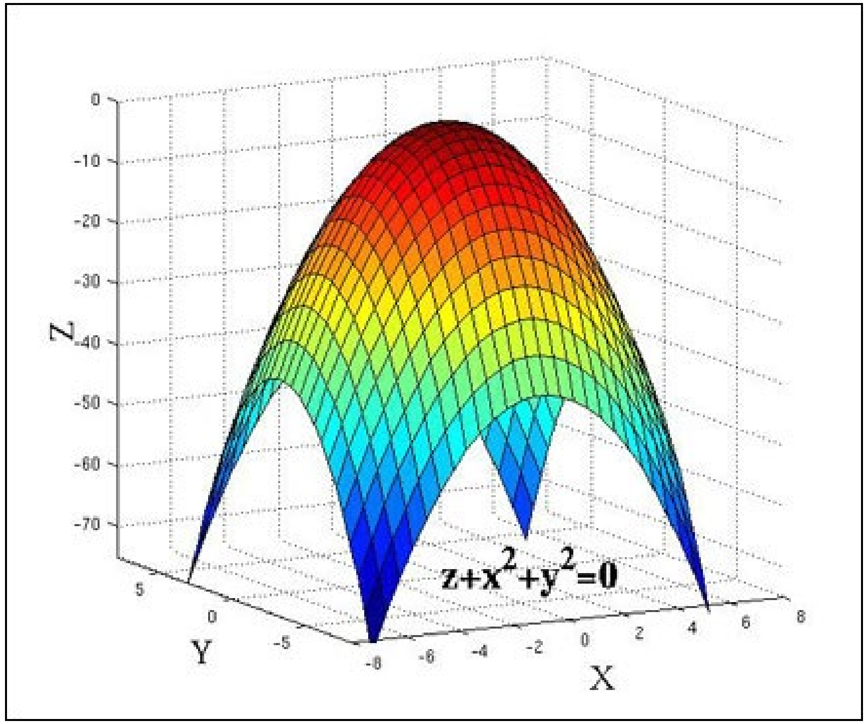
$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=1}^{n} \left[y^{(i)} - \sigma\left(\theta^T \boldsymbol{x}^{(i)}\right)\right] x_j^{(i)}$$

Gradient Ascent

Review

Walk uphill and you will find a local maxima
(if your step is small enough).





Logistic regression $LL(\theta)$
is convex

Training: Gradient ascent step

4. Optimize.

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=1}^{n} \left[y^{(i)} - \sigma\left(\theta^T \boldsymbol{x}^{(\boldsymbol{i})}\right) \right] x_j^{(i)}$$

Repeat many times:

For all thetas:

$$\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \frac{\partial LL\left(\theta^{\text{old}}\right)}{\partial \theta_j^{\text{old}}}$$

$$= \theta_j^{\text{old}} + \eta \cdot \sum_{i=1}^{n} \left[y^{(i)} - \sigma\left({\theta^{\text{old}}}^T \boldsymbol{x}^{(\boldsymbol{i})} \right) \right] x_j^{(i)}$$

What does this look like in code?

Training: Gradient Ascent

Gradient Ascent Step $\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \sum_{i=1}^{n} \left[y^{(i)} - \sigma\left({\theta^{\text{old}}}^T \boldsymbol{x}^{(i)} \right) \right] x_j^{(i)}$

initialize θ_j = 0 for 0 ≤ j ≤ m
repeat many times:

gradient[j] = 0 for 0 ≤ j ≤ m

// compute all gradient[j]'s
// based on n training examples

θ_j += η * gradient[j] for all 0 ≤ j ≤ m

Training: Gradient Ascent

Gradient Ascent Step $\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \sum_{i=1}^{n} \left[y^{(i)} - \sigma\left({\theta^{\text{old}}}^T \boldsymbol{x}^{(i)} \right) \right] x_j^{(i)}$

initialize θ_j = 0 for 0 ≤ j ≤ m

repeat many times:

```
    gradient[j] = 0 for 0 ≤ j ≤ m
    for each training example (x, y):
        for each 0 ≤ j ≤ m:
            // update gradient[j] for
            // current (x,y) example
    θ_j += η  * gradient[j] for all 0 ≤ j ≤ m
```

Training: Gradient Ascent

Gradient Ascent Step $\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \sum_{i=1}^{n} \left[y^{(i)} - \sigma\left({\theta^{\text{old}}}^T \boldsymbol{x}^{(i)} \right) \right] x_j^{(i)}$

initialize θ_j = 0 for 0 ≤ j ≤ m

repeat many times:

gradient[j] = 0 for 0 ≤ j ≤ m

for each training example (x, y):

for each 0 ≤ j ≤ m:

gradient[j] += $\left[y - \frac{1}{1 + e^{-\theta^T \boldsymbol{x}}} \right] x_j$

θ_j += η * gradient[j] for all 0 ≤ j ≤ m

What are important implementation details?

Training: Gradient Ascent

Gradient Ascent Step $\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \sum_{i=1}^{n} \left[y^{(i)} - \sigma\left({\theta^{\text{old}}}^T \boldsymbol{x}^{(i)} \right) \right] x_j^{(i)}$

initialize θ_j = 0 for 0 ≤ j ≤ m
repeat many times:

gradient[j] = 0 for 0 ≤ j ≤ m

for each training example (x, y):

for each 0 ≤ j ≤ m:

gradient[j] += $\left[y - \frac{1}{1 + e^{-\theta^T \boldsymbol{x}}} \right] \boxed{x_j}$

θ_j += η * gradient[j] for all 0 ≤ j ≤ m

- x_j is j-th feature of input var $x = (x_1, \ldots, x_m)$

Training: Gradient Ascent

Gradient Ascent Step $\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \sum_{i=1}^{n} \left[y^{(i)} - \sigma\left({\theta^{\text{old}}}^T \boldsymbol{x}^{(i)} \right) \right] x_j^{(i)}$

initialize θ_j = 0 for 0 ≤ j ≤ m

repeat many times:

    gradient[j] = 0 for 0 ≤ j ≤ m

    for each training example (x, y):

        for each 0 ≤ j ≤ m:

            gradient[j] += $\left[y - \frac{1}{1 + \boxed{e^{-\theta^T \boldsymbol{x}}}} \right] x_j$

    θ_j += η * gradient[j] for all 0 ≤ j ≤ m

- x_j is j-th feature of input var $x = (x_1, \dots, x_m)$
- Insert $x_0 = 1$ before training

Training: Gradient Ascent

Gradient Ascent Step $\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \sum_{i=1}^{n} \left[y^{(i)} - \sigma\left(\theta^{\text{old}^T} \boldsymbol{x}^{(i)} \right) \right] x_j^{(i)}$

initialize θ_j = 0 for 0 ≤ j ≤ m

repeat many times:

gradient[j] = 0 for 0 ≤ j ≤ m

for each training example (x, y):

for each 0 ≤ j ≤ m:

gradient[j] += $\left[y - \frac{1}{1 + e^{-\theta^T \boldsymbol{x}}} \right] x_j$

θ_j += η * gradient[j] for all 0 ≤ j ≤ m

- x_j is j-th feature of input var $x = (x_1, \ldots, x_m)$
- Insert $x_0 = 1$ before training
- Finish computing gradient before updating any part of θ

Training: Gradient Ascent

Gradient Ascent Step $\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \sum_{i=1}^{n} \left[y^{(i)} - \sigma\left(\theta^{\text{old}^T} \boldsymbol{x}^{(i)} \right) \right] x_j^{(i)}$

initialize θ_j = 0 for 0 ≤ j ≤ m
repeat many times:

gradient[j] = 0 for 0 ≤ j ≤ m

for each training example (x, y):

for each 0 ≤ j ≤ m:

gradient[j] += $\left[y - \frac{1}{1 + e^{-\theta^T \boldsymbol{x}}} \right] x_j$

θ_j += $\boxed{\eta}$ * gradient[j] for all 0 ≤ j ≤ m

- x_j is j-th feature of input var $x = (x_1, \ldots, x_m)$
- Insert $x_0 = 1$ before training
- Finish computing gradient before updating any part of θ
- Learning rate η is a constant you set before training

Training: Gradient Ascent

Gradient Ascent Step $\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \sum_{i=1}^{n} \left[y^{(i)} - \sigma\left({\theta^{\text{old}}}^T \boldsymbol{x}^{(i)} \right) \right] x_j^{(i)}$

initialize θ_j = 0 for 0 ≤ j ≤ m
repeat many times:

    gradient[j] = 0 for 0 ≤ j ≤ m

    for each training example (x, y):

        for each 0 ≤ j ≤ m:

            gradient[j] += $\left[y - \frac{1}{1 + e^{-\theta^T \boldsymbol{x}}} \right] x_j$

    θ_j += η * gradient[j] for all 0 ≤ j ≤ m

- x_j is j-th feature of input var $x = (x_1, \ldots, x_m)$
- Insert $x_0 = 1$ before training
- Finish computing gradient before updating any part of θ
- Learning rate η is a constant you set before training

Testing: Classification with Logistic Regression

Training

Learn parameters $\theta = (\theta_0, \theta_1, \ldots, \theta_m)$

via gradient ascent:

$$\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \sum_{i=1}^{n} \left[y^{(i)} - \sigma\left(\theta^{\text{old}^T} \boldsymbol{x}^{(\boldsymbol{i})} \right) \right] x_j^{(i)}$$

Testing

- Compute $\hat{y} = P(Y = 1 | \boldsymbol{X} = \boldsymbol{x}) = \sigma(\theta^T \boldsymbol{x}) = \dfrac{1}{1 + e^{-\theta^T \boldsymbol{x}}}$
- Classify instance as:

$$\begin{cases} 1 & \hat{y} > 0.5, \text{ equivalently } \theta^T \boldsymbol{x} > 0 \\ 0 & \text{otherwise} \end{cases}$$

Parameters θ_j are <u>not</u> updated during testing phase

Today's plan

Logistic Regression

- Chapter 0: Background
- Chapter 1: Big Picture
- Chapter 2: Details
- Chapter 3: Philosophy

Introducing notation $\hat{y}$

Logistic Regression model:

$$\hat{y} = P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \sigma(\theta^T \boldsymbol{x})$$

$$P(Y = y|\boldsymbol{X} = \boldsymbol{x}) = \begin{cases} \hat{y} & \text{if } y = 1 \\ 1 - \hat{y} & \text{if } y = 0 \end{cases}$$

Prediction:

$$\hat{Y} = \underset{y=\{0,1\}}{\arg\max}\, P(Y|\boldsymbol{X} = \boldsymbol{x}) \quad = \begin{cases} 1 & \text{if } \hat{y} > 0.5 \\ 0 & \text{otherwise} \end{cases}$$

Training: Learning the parameters via MLE

0. Add $x_0^{(i)} = 1$ to each $\boldsymbol{x}^{(i)}$
1. Logistic Regression model:

$$P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \hat{y}$$
$$\hat{y} = \sigma(\theta^T \boldsymbol{x})$$

2. Compute log-likelihood of training data:

$$LL(\theta) = \sum_{i=1}^{n} y^{(i)} \log \sigma\left(\theta^T \boldsymbol{x}^{(i)}\right) + \left(1 - y^{(i)}\right) \log\left(1 - \sigma\left(\theta^T \boldsymbol{x}^{(i)}\right)\right)$$

How did we get this log-likelihood function?

3. Compute derivative of log-likelihood with respect to each $\theta_j, j = 0, 1, \ldots, m$:

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=1}^{n} \left[y^{(i)} - \sigma\left(\theta^T \boldsymbol{x}^{(i)}\right)\right] x_j^{(i)}$$

Log-likelihood of data

Logistic Regression model:

$$P(Y = y|\boldsymbol{X} = \boldsymbol{x}) = \begin{cases} \hat{y} & \text{if } y = 1 \\ 1 - \hat{y} & \text{if } y = 0 \end{cases}$$

where $\hat{y} = \sigma(\theta^T \boldsymbol{x})$

$$= (\hat{y})^y (1 - \hat{y})^{1-y}$$

(see Bernoulli MLE PMF)

Likelihood of training data:

$$L(\theta) = \prod_{i=1}^{n} P\left(Y = y^{(i)} | \boldsymbol{X} = \boldsymbol{x}^{(i)}, \theta\right)$$

Notes:

- Actually **conditional likelihood**
- Still correctly gets correct θ_{MLE} since $\boldsymbol{X}, \theta$ independent
- See lecture notes

Log-likelihood of data

Logistic Regression model:

$$P(Y = y|\boldsymbol{X} = \boldsymbol{x}) = \begin{cases} \hat{y} & \text{if } y = 1 \\ 1 - \hat{y} & \text{if } y = 0 \end{cases}$$

where $\hat{y} = \sigma(\theta^T \boldsymbol{x})$

$$= (\hat{y})^y (1 - \hat{y})^{1-y}$$

(see Bernoulli MLE PMF)

Likelihood of training data:

$$L(\theta) = \prod_{i=1}^{n} P\left(Y = y^{(i)}|\boldsymbol{X} = \boldsymbol{x}^{(i)}, \theta\right) = \prod_{i=1}^{n} \left(\hat{y}^{(i)}\right)^{y^{(i)}} \left(1 - \hat{y}^{(i)}\right)^{1-y^{(i)}}$$

Log-likelihood:

$$LL(\theta) = \sum_{i=1}^{n} y^{(i)} \log \hat{y}^{(i)} + \left(1 - y^{(i)}\right) \log\left(1 - \hat{y}^{(i)}\right)$$

$$= \sum_{i=1}^{n} y^{(i)} \log \sigma\left(\theta^T \boldsymbol{x}^{(i)}\right) + \left(1 - y^{(i)}\right) \log\left(1 - \sigma\left(\theta^T \boldsymbol{x}^{(i)}\right)\right)$$

Training: Learning the parameters via MLE

0. Add $x_0^{(i)} = 1$ to each $\boldsymbol{x}^{(i)}$
1. Logistic Regression model: $P(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \sigma(\theta^T \boldsymbol{x})$
2. Compute log-likelihood of training data: $LL(\theta) = \sum_{i=1}^{n} y^{(i)} \log \sigma\left(\theta^T \boldsymbol{x}^{(i)}\right) + \left(1 - y^{(i)}\right) \log\left(1 - \sigma\left(\theta^T \boldsymbol{x}^{(i)}\right)\right)$
3. Compute derivative of log-likelihood with respect to each $\theta_j, j = 0, 1, \ldots, m$:

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=1}^{n} \left[y^{(i)} - \sigma\left(\theta^T \boldsymbol{x}^{(i)}\right)\right] x_j^{(i)}$$

How did we get this gradient?

Aside: Sigmoid has a beautiful derivative

Sigmoid function:

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

Derivative:

$$\frac{d}{dz}\sigma(z) = \sigma(z)[1 - \sigma(z)]$$

What is $\frac{\partial}{\partial \theta_j}\sigma(\theta^T \boldsymbol{x})$?

A. $\sigma(x_j)[1 - \sigma(x_j)]x_j$

B. $\sigma(\theta^T \boldsymbol{x})[1 - \sigma(\theta^T \boldsymbol{x})]\boldsymbol{x}$

C. $\sigma(\theta^T \boldsymbol{x})[1 - \sigma(\theta^T \boldsymbol{x})]x_j$

D. $\sigma(\theta^T \boldsymbol{x})x_j[1 - \sigma(\theta^T \boldsymbol{x})x_j]$

E. None/other

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D. $\sigma(\theta^T \boldsymbol{x})x_j[1 - \sigma(\theta^T \boldsymbol{x})x_j]$

E. None/other

Let $z = \theta^T \boldsymbol{x} = \sum_{k=0}^{m} \theta_k x_k$.

$$\frac{\partial}{\partial \theta_j}\sigma(\theta^T \boldsymbol{x}) = \frac{\partial}{\partial z}\sigma(z) \cdot \frac{\partial z}{\partial \theta_j}$$

(Chain Rule)

$$= \sigma(\theta^T \boldsymbol{x})[1 - \sigma(\theta^T \boldsymbol{x})]x_j$$

Compute gradient of log-conditional likelihood

Find: $\dfrac{\partial LL(\theta)}{\partial \theta_j}$

where

Log-conditional Likelihood: $LL(\theta) = \sum_{i=1}^{n} y^{(i)} \log \sigma\left(\theta^T \boldsymbol{x}^{(\boldsymbol{i})}\right) + \left(1 - y^{(i)}\right) \log\left(1 - \sigma\left(\theta^T \boldsymbol{x}^{(i)}\right)\right)$

Compute gradient of log-likelihood

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta_j} \left[y^{(i)} \log\left(\hat{y}^{(i)}\right) + \left(1 - y^{(i)}\right) \log\left(1 - \hat{y}^{(i)}\right) \right]$$

Let $\hat{y}^{(i)} = \sigma\left(\theta^T \boldsymbol{x}^{(\boldsymbol{i})}\right)$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial \hat{y}^{(i)}} \left[y^{(i)} \log\left(\hat{y}^{(i)}\right) + \left(1 - y^{(i)}\right) \log\left(1 - \hat{y}^{(i)}\right) \right] \cdot \frac{\partial \hat{y}^{(i)}}{\partial \theta_j}$$

(Chain Rule)

$$= \sum_{i=1}^{n} \left[y^{(i)} \frac{1}{\hat{y}^{(i)}} - \left(1 - y^{(i)}\right) \frac{1}{1 - \hat{y}^{(i)}} \right] \cdot \hat{y}^{(i)} \left(1 - \hat{y}^{(i)}\right) x_j^{(i)}$$

(calculus)

$$= \sum_{i=1}^{n} \left[y^{(i)} - \hat{y}^{(i)} \right] x_j^{(i)} \qquad = \sum_{i=1}^{n} \left[y^{(i)} - \sigma\left(\theta^T \boldsymbol{x}^{(i)}\right) \right] x_j^{(i)}$$

(simplify)

Today's plan

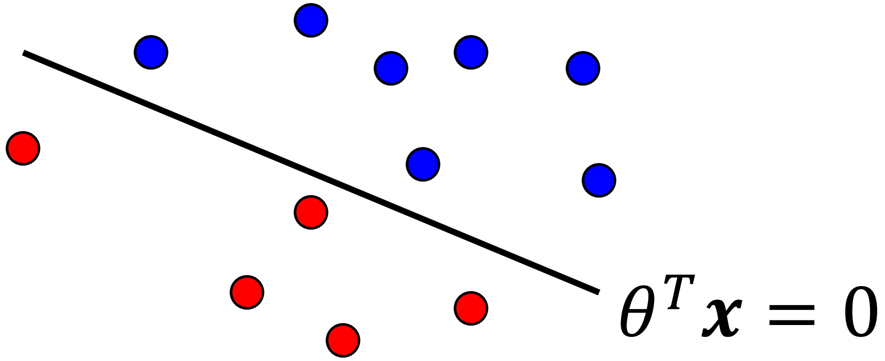
Logistic Regression

- Chapter 0: Background
- Chapter 1: Big Picture
- Chapter 2: Details
- Chapter 3: Philosophy

Intuition about Logistic Regression

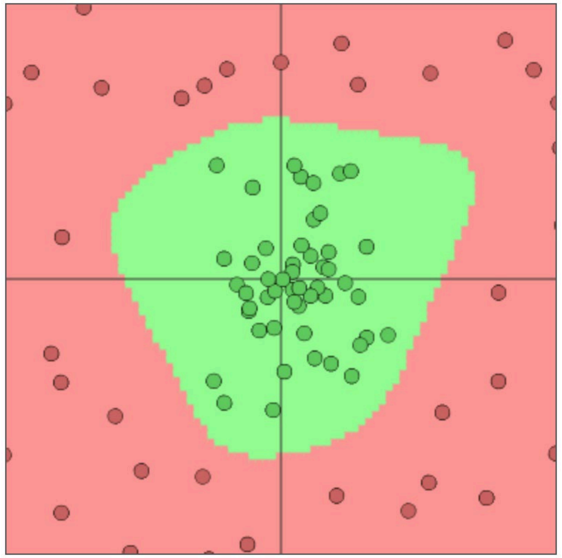
Logistic Regression Model $P(Y=1|\boldsymbol{X}=\boldsymbol{x})=\sigma(\theta^T\boldsymbol{x})$ where $\theta^T\boldsymbol{x}=\sum_{j=0}^{m}\theta_j x_j$

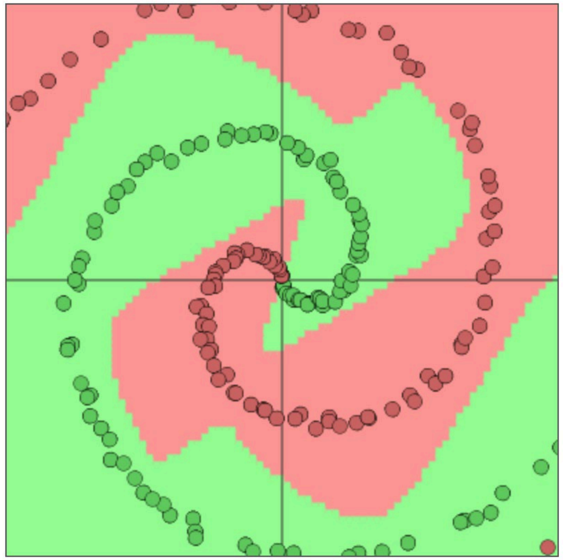
Logistic Regression is trying to fit a **line** that separates data instances where $y=1$ from those where $y=0$:



- We call such data (or functions generating the data **linearly separable**.
- Naïve Bayes is linear too, because there is no interaction between different features.

Data is often not linearly separable





- Not possible to draw a line that successfully separates all the $y = 1$ points (green) from the $y = 0$ points (red)
- Despite this fact, Logistic Regression and Naive Bayes still often work well in practice

Many tradeoffs in choosing an algorithm

	Naïve Bayes	Logistic Regression
Modeling goal	$P(\boldsymbol{X}, Y)$	$P(Y \mid \boldsymbol{X})$
Generative or **discriminative**?	**Generative:** could use joint distribution to generate new points (but you might not need this extra effort)	**Discriminative:** just tries to discriminate $y = 0$ vs $y = 1$ (Cannot generate new points b/c no $P(\boldsymbol{X}, Y)$)
Continuous input features?	Needs parametric form (e.g., Gaussian) or discretized buckets (for multinomial features)	Yes, easily
Discrete input features?	Yes, multi-value discrete data = multinomial $P(X_i \mid Y)$	Multi-valued discrete data hard (e.g., if $X_i \in \{A, B, C\}$, not necessarily good to encode as $\{1, 2, 3\}$