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CS 109

Section #6
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Section 6 Solution

Adapted for Winter 2020 by Alex Tsun

1. **Random Number of Random Variables**: *law of total expectation*

   Let N be a non-negative integer-valued random variable; that is, takes values in $\{0, 1, 2, \dots\}$. Let $X_1, X_2, X_3, \dots$ be an infinite sequence of iid random variables (independent of N), each with mean μ, and $X = \sum_{i=1}^{N} X_i$ be the sum of the first N of them. Before doing any work, what do you think $E[X]$ will turn out to be? Show it mathematically.

$$E[X] = E\left[\sum_{i=1}^{N} X_i\right] = \sum_n E\left[\sum_{i=1}^{N} X_i \mid N = n\right] p_N(n) = \sum_n E\left[\sum_{i=1}^{n} X_i \mid N = n\right] p_N(n)$$

$$= \sum_n E\left[\sum_{i=1}^{n} X_i\right] p_N(n) = \sum_n n\mu p_N(n) = \mu \sum_n n p_N(n) = \mu E[N]$$

Alternatively,

$$E[X] = E[E[X|N]] = E[N\mu] = \mu E[N]$$

2. **Beta Sum**: *beta distribution and sum of RVs*

   What is the distribution of the sum of 100 IID Betas? Let X be the sum

$$X = \sum_{i=1}^{100} X_i \qquad \text{Where each } X_i \sim \text{Beta}(a = 3, b = 4)$$

   Note the variance of a Beta:

$$\text{Var}(X_i) = \frac{ab}{(a+b)^2(a+b+1)} \qquad \text{Where } X_i \sim \text{Beta}(a, b)$$

By the Central Limit Theorem, the sum of equally weighted IID random variables will be Normally distributed. We calculate the expectation and variance of X_i using the beta formulas:

$$
\begin{aligned}
E(X_i) &= \frac{a}{a+b} && \text{Expectation of a Beta} \\
&= \frac{3}{7} \approx 0.43 \\
\text{Var}(X_i) &= \frac{ab}{(a+b)^2(a+b+1)} && \text{Variance of a Beta} \\
&= \frac{3 \cdot 4}{(3+4)^2(3+4+1)} \\
&= \frac{12}{49 \cdot 8} \approx 0.03
\end{aligned}
$$

$$
\begin{aligned}
X &\sim N(\mu = n \cdot E[X_i], \sigma^2 = n \cdot \text{Var}(X_i)) \\
&\sim N(\mu = 43, \sigma^2 = 3)
\end{aligned}
$$

3. **Medicine Doses**:

   Megha has a health condition that requires unpredictable amounts of medication. Every day, there is a 20% chance that she feels perfectly fine and requires no medicine. Otherwise, she needs to take a dose of medication. The necessary dose is equally likely to be any value in the continuous range 1 to 5 ounces. How much medicine she needs on any given day is independent of all other days.

   Megha's insurance will fully cover 90 ounces of medicine for each 30-day period. What is the probability that 90 ounces will be enough for the next 30 days? Make your life easier by using Central Limit Theorem.

Let M be the amount of medicine Megha will need in the next thirty days. Let M_i be the amount of medicine Megha needs on the ith day. M is a sum of M_1 through M_{30} and can be modeled with the CLT.

To use the CLT, we need to first know the mean and variance of M_i. To do this, let D_i be the event that she needs to take a dose on the ith day. Note that $M_i|D_i \sim Uni(1,5)$ and $M_i|D_i^C = 0$. Using the law of total expectation, we have:

$$E[M_i] = E[M_i|D_i]P(D_i) + E[M_i|D_i^C]P(D_i^C) = 3 * 0.8 + 0 * 0.2 = 2.4$$

To find the variance of M_i, we need to know $E[M_i^2]$. We can use a similar approach as the previous problem along with the law of the unconscious statistician:

$$
\begin{aligned}
E[M_i^2] &= E[M_i^2|D_i]P(D_i) + E[M_i^2|D_i^C]P(D_i^C) \\
&= \frac{4}{5}\int_{m=1}^{5} m^2 f_M(m)dm + 0 * .2 \\
&= \frac{4}{5}\int_{m=1}^{5} m^2 \frac{1}{4}dt \approx 8.267
\end{aligned}
$$

We then have $Var(M_i) = E[M_i^2] - E[M_i]^2 = 8.267 - 2.4^2 = 2.507$. According to the CLT:

$$\sum_{i=1}^{30} M_i \approx N(30*2.4, 30*2.507) \implies M \sim N(72, 75.21)P\left(M < 90\right) \approx \Phi\left(\frac{90-72}{\sqrt{75.21}}\right) \approx 0.98$$