

## Calculation Reference

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Based on a handout by Mehran Sahami

### 1 Useful identities related to summations

Since it may have been a while since some folks have worked with summations, I just wanted to provide a reference on them that you may find useful in your future work. Here are some useful identities and rules related to working with summations. In the rules below,  $f$  and  $g$  are arbitrary real-valued functions.

Pulling a constant out of a summation:

$$\sum_{n=s}^t C \cdot f(n) = C \cdot \sum_{n=s}^t f(n), \text{ where } C \text{ is a constant.}$$

Eliminating the summation by summing over the elements:

$$\sum_{i=1}^n x = nx$$

$$\sum_{i=m}^n x = (n - m + 1)x$$

$$\sum_{i=s}^n f(C) = (n - s + 1)f(C), \text{ where } C \text{ is a constant.}$$

Combining related summations:

$$\sum_{n=s}^j f(n) + \sum_{n=j+1}^t f(n) = \sum_{n=s}^t f(n)$$

$$\sum_{n=s}^t f(n) + \sum_{n=s}^t g(n) = \sum_{n=s}^t [f(n) + g(n)]$$

Changing the bounds on the summation:

$$\sum_{n=s}^t f(n) = \sum_{n=s+p}^{t+p} f(n - p)$$

"Reversing" the order of the summation:

$$\sum_{n=a}^b f(n) = \sum_{n=b}^a f(n)$$

Arithmetic series:

$$\sum_{i=0}^n i = \sum_{i=1}^n i = \frac{n(n+1)}{2} \quad (\text{with a moment of silence for C. F. Gauss.})$$

$$\sum_{i=m}^n i = \frac{(n-m+1)(n+m)}{2}$$

Arithmetic series involving higher order polynomials:

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \left[\sum_{i=1}^n i\right]^2$$

Geometric series:

$$\sum_{i=0}^n x^i = \frac{1-x^{n+1}}{1-x}$$

$$\sum_{i=m}^n x^i = \frac{x^{n+1}-x^m}{x-1}$$

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \text{ if } |x| < 1$$

More exotic geometric series:

$$\sum_{i=0}^n i2^i = 2 + 2^{n+1}(n-1)$$

$$\sum_{i=0}^n \frac{i}{2^i} = \frac{2^{n+1}-n-2}{2^n}$$

Taylor expansion of exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Binomial coefficient:

$$\sum_{i=0}^n \binom{n}{i} = 2^n$$

Much more information on binomial coefficients is available in the Ross textbook.

## 2 Growth rates of summations

Besides solving a summation explicitly, it is also worthwhile to know some general growth rates on sums, so you can (tightly) bound a sum if you are trying to prove something in the big-Oh/Theta world. If you're not familiar with big-Theta ( $\Theta$ ) notation, you can think of it like big-Oh notation, but it actually provides a “tight” bound. Namely, big-Theta means that the function grows no more quickly and no more slowly than the function specified, up to constant factors, so it's actually more informative than big-Oh.

Here are some useful bounds:

$$\sum_{i=1}^n i^c = \Theta(n^{c+1}), \text{ for } c \geq 0.$$

$$\sum_{i=1}^n \frac{1}{i} = \Theta(\log n)$$

$$\sum_{i=1}^n c^i = \Theta(c^n), \text{ for } c \geq 2.$$

## 3 A few identities related to products

Recall that the mathematical symbol  $\Pi$  represents a product of terms (analogous to  $\Sigma$  representing a sum of terms). Below, we give some useful identities related to products.

Definition of factorial:

$$\prod_{i=1}^n i = n!$$

Note that  $0! = 1$  by definition.

Stirling's approximation for  $n!$  is given below. This approximation is useful when computing  $n!$  for large values of  $n$  (particularly when  $n > 30$ ).

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n, \text{ or equivalently } n! \approx \sqrt{2\pi n} n^{(n+\frac{1}{2})} e^{-n}$$

Eliminating the product by multiplying over the elements:

$$\prod_{i=1}^n C = C^n, \text{ where } C \text{ is a constant.}$$

Combining products:

$$\prod_{i=1}^n f(i) \prod_{i=1}^n g(i) = \prod_{i=1}^n f(i) \cdot g(i)$$

Turning products into summations (by taking logarithms, assuming  $f(i) > 0$  for all  $i$ ):

$$\log \left( \prod_{i=1}^n f(i) \right) = \sum_{i=1}^n \log f(i)$$

#### 4 Suggestions for computing permutations and combinations

For your problem set solutions it is fine for your answers to include factorials, exponentials, or combinations; you don't need to calculate those all out to get a single numeric answer. However, if you'd like to work with those in Python, R, or Microsoft Excel, here are a few functions you may find useful.

In Python:

<code>math.factorial(n)</code>	computes $n!$
<code>scipy.special.binom(n, m)</code>	computes $\binom{n}{m}$ (as a float)
<code>math.exp(n)</code>	computes $e^n$
<code>n ** m</code>	computes $n^m$

Names to the left of the dots (.) are modules that need to be imported before being used: `import math, scipy.special.`

In R:

`factorial(n)` computes  $n!$   
`choose(n, m)` computes  $\binom{n}{m}$   
`exp(n)` computes  $e^n$   
`n^m` computes  $n^m$

In Microsoft Excel:

`FACT(n)` computes  $n!$   
`COMBIN(n, m)`  $\binom{n}{m}$   
`EXP(n)` computes  $e^n$   
`POWER(n, m)` computes  $n^m$

To use functions in Excel, you need to set a cell to equal a function value. For example, to compute  $3! \cdot \binom{5}{2}$ , you would put the following in a cell:

`= FACT(3) * COMBIN(5, 2)`

Note the equals sign (=) at the beginning of the expression.

## 5 A little review of calculus

Since it may have been a while since you did calculus, here are a few rules that you might find useful.

Product Rule for derivatives:

$$d(u \cdot v) = du \cdot v + u \cdot dv$$

Derivative of exponential function:

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}$$

Integral of exponential function:

$$\int e^u du = e^u$$

Derivative of natural logarithm:

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

Integral of 1/x:

$$\int \frac{1}{x} dx = \ln(x)$$

Integration by parts (everyone's favorite!):

Choose a suitable  $u$  and  $dv$  to decompose the integral of interest:

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

Here's the underlying rule that integration by parts is derived from:

$$\int d(u \cdot v) = u \cdot v = \int du \cdot v + \int u \cdot dv$$

## 6 Bibliography

Additional information on sums and products can generally be found in a good calculus or discrete mathematics book. The discussion of summations above is based on Wikipedia (<http://en.wikipedia.org/wiki/Summation>).