## Problem One

**Prove that the harmonic series diverges, i.e. has an infinite sum.**

We know that the harmonic series is defined as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \tag{1}$$

We also know that

$$\lim_{k \to \infty} 1 + \frac{k}{2} = \infty. \tag{2}$$

We will see that by comparing (2) to (1), we can conclude that the harmonic series is divergent. We will do so by substituting and grouping terms in (1) together.

$$\begin{aligned} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots &> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots \\ &> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \dots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots \end{aligned}$$

The above shows that

$$\sum_{n=1}^{2^k} \frac{1}{n} \geq 1 + \frac{k}{2}$$

so the harmonic series must be convergent.

## Problem Two

**Use the Pigeonhole Principle to solve the following problems.** Remember that Pigeonhole Principle states that if you put $n$ pigeons into $k$ holes, then one of the holes contains $\lceil n/k \rceil$ pigeons.

1. **Prove that, in a group of $n \in \mathbb{N}$ people, there are two people with the same number of friends.** (This problem assumes that if person A is friends with person B, then person B is friends with person A.)

   A person can have between 0 and $n-1$ friends. However, if a person has $n-1$ friends, then everyone has at least one friend. There are $n$ people, but each person can have between 1 to $n-1$ friends, so by the Pigeonhole Principle, two people have the same number of friends.

   A similar situation occurs if a person has 0 friends. If a person has 0 friends, then no one can have $n-1$ friends, so by the Pigeonhole Principle, two people have the same number of friends.

   If neither case applies, then everyone has between 1 and $n-2$ friends, and by applying the Pigeonhole Principle one more time, we find that two people must have the same number of friends. So two people must have the same number of friends.

2. **Prove that, given a set of a hundred whole numbers, one number is either divisible by 100 or there are several numbers whose sum is divisible by 100.**

   Say there is no number that is divisible by 100. We will show that there are several numbers who sum is divisible by 100. First, we label the numbers $x_1, x_2, x_3$, and so on. Second, we consider the subsets $\{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}$, and so on. Compute the sums; if no sum is divisible by 100, then there are 100 subsets but only 99 possible residues (i.e. numbers modulo 100). Thus we have some $i$ and $j$, where $i < j$, such that

   $$x_1 + \cdots + x_i \equiv x_1 + \cdots + x_j \mod 100$$

   which, if we subtract $x_1 + \ldots x_{i-1}$ from each side, we find that

   $$x_i + \cdots + x_j \equiv 0 \mod 100.$$

   In other words, we have found some numbers whose sum is divisible by 100!