05: Independence

Lisa Yan

April 15, 2020

Quick slide reference

Generalized Chain Rule

Chain Rule

Definition of conditional probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

The Chain Rule:

$$P(EF) = P(E|F)P(F)$$

Generalized Chain Rule

$$P(E_1 E_2 E_3 \dots E_n)$$
$$= P(E_1)P(E_2|E_1)P(E_3|E_1 E_2) \dots P(E_n|E_1 E_2 \dots E_{n-1})$$

Quick check

$$P(E_1 E_2 E_3 \dots E_n) = P(E_1)P(E_2|E_1) \dots P(E_n|E_1 E_2 \dots E_{n-1})$$ Chain Rule

You are going to a friend's Halloween party.

Let C = there is candy
M = there is music
W = you wear a costume
E = no one wears your costume

An awesome party means that all of these events must occur.

What is $P(\text{awesome party}) = P(CMWE)$?

A. $P(C)P(M|C)P(W|CM)P(E|CMW)$
B. $P(M)P(C|M)P(W|MC)P(E|MCW)$
C. $P(W)P(E|W)P(CM|EW)$
D. A, B, and C
E. None/other

Quick check

$P(E_1 E_2 E_3 \dots E_n) = P(E_1)P(E_2|E_1)\dots P(E_n|E_1 E_2 \dots E_{n-1})$ Chain Rule

You are going to a friend's Halloween party.

Let C = there is candy
M = there is music
E = no one wears your costume
W = you wear a costume

An awesome party means that all of these events must occur.

What is $P(\text{awesome party}) = P(CMEW)$?

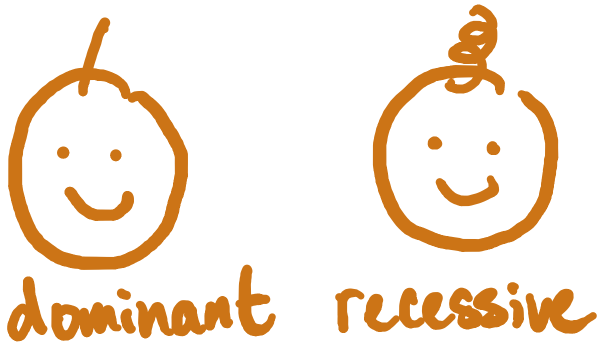
A. $P(C)P(M|C)P(E|CM)P(W|CME)$
B. $P(M)P(C|M)P(E|MC)P(W|MCE)$
C. $P(W)P(E|W)P(CM|EW)$
D. A, B, and C
E. None/other

Chain Rule is a way of introducing "order" and "procedure" into probability.

Think of the children

Two parents both have an (A, a) gene pair.

- Each parent will pass on one of their genes (each gene equally likely) to their child.
- The probability of a single child having curly hair (recessive trait) is 0.25.
- There are three children.



What is the probability that all three children have curly hair?

Let E_1, E_2, E_3 be the events that child 1, 2, and 3 have curly hair, respectively.

$$P(E_1 E_2 E_3) = P(E_1)P(E_2|E_1)P(E_3|E_1 E_2)$$

Independence I

Independence

Two events E and F are defined as independent if:

$$P(EF) = P(E)P(F)$$

Otherwise E and F are called dependent events.

If E and F are independent, then:

$$P(E|F) = P(E)$$

Intuition through proof

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$

Statement:

If E and F are independent, then $P(E|F) = P(E)$.

Proof:

$$P(E|F) = \frac{P(EF)}{P(F)}$$ Definition of conditional probability

$$= \frac{P(E)P(F)}{P(F)}$$ Independence of E and F

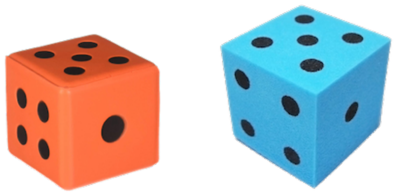
$$= P(E)$$ Taking the bus to cancellation city

Knowing that F happened does not change our belief that E happened.

Dice, our misunderstood friends

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$, $P(E|F) = P(E)$

- Roll two 6-sided dice, yielding values D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 5$



$G = \{(1,4), (2,3), (3,2), (4,1)\}$

1. Are E and F independent?

$$P(E) = 1/6$$
$$P(F) = 1/6$$
$$P(EF) = 1/36$$

<u>independent</u>

2. Are E and G independent?

$$P(E) = 1/6$$
$$P(G) = 4/36 = 1/9$$
$$P(EG) = 1/36 \neq P(E)P(G)$$

<u>dependent</u>

Generalizing independence

Three events E, F, and G are independent if:

$$\begin{cases} P(EFG) = P(E)P(F)P(G), \text{ and} \\ P(EF) = P(E)P(F), \text{ and} \\ P(EG) = P(E)P(G), \text{ and} \\ P(FG) = P(F)P(G) \end{cases}$$

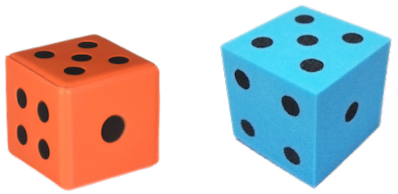
n events $E_1, E_2, \dots, E_n$ are independent if:

for $r = 1, \dots, n$:

  for every subset $E_1, E_2, \dots, E_r$:

    $P(E_1, E_2, \dots, E_r) = P(E_1)P(E_2)\cdots P(E_r)$

Dice, increasingly misunderstood (still our friends)



- Each roll of a 6-sided die is an independent trial.
- Two rolls: D_1 and D_2.
- Let event E: $D_1 = 1$

  event F: $D_2 = 6$

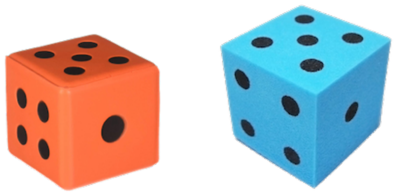
  event G: $D_1 + D_2 = 7$

$G = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

1. Are E and F independent?
2. Are E and G independent?
3. Are F and G independent?
4. Are E, F, G independent?

$P(EF) = 1/36$

Dice, increasingly misunderstood (still our friends)



- Each roll of a 6-sided die is an independent trial.
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  event G: $D_1 + D_2 = 7$

$G = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

1. Are E and F independent?
2. Are E and G ✅ independent?
3. Are F and G independent?
4. Are E, F, G independent?

$P(EF) = 1/36$

Pairwise independence is not sufficient to prove independence of >2 events!

Independence II

Independent trials

We often are interested in experiments consisting of n independent trials.

- n trials, each with the same set of possible outcomes
- n-way independence: an event in one subset of trials is independent of events in other subsets of trials

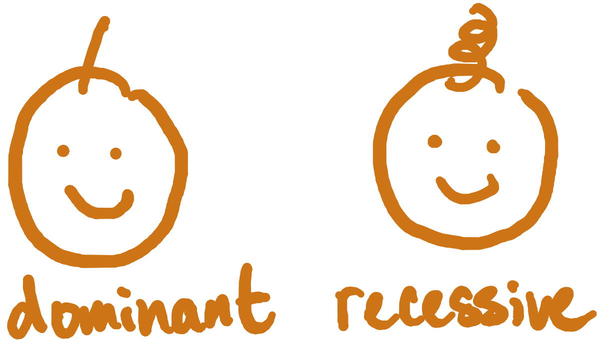
Examples:

- Flip a coin n times
- Roll a die n times
- Send a multiple choice survey to n people
- Send n web requests to k different servers

Think of the children as independent trials

Two parents both have an (A, a) gene pair.

- Each parent will pass on one of their genes (each gene equally likely) to their child.
- The probability of a single child having curly hair (recessive trait) is 0.25.
- There are three children. Each child is an independent trial.



What is the probability that all three children have curly hair?

Let E_1, E_2, E_3 be the events that child 1, 2, and 3 have curly hair, respectively.

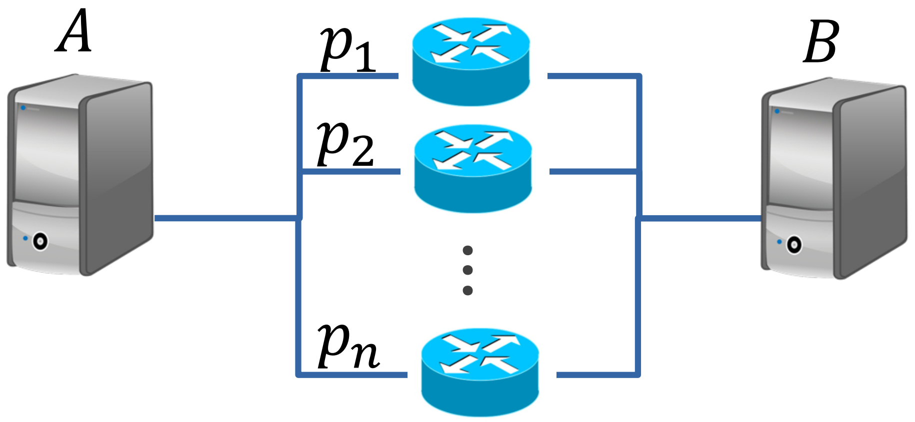
$$P(E_1E_2E_3) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)$$

Network reliability

Consider the following parallel network:

- n independent routers, each with probability p_i of functioning (where $1 \le i \le n$)
- E = functional path from A to B exists.

What is $P(E)$?



Network reliability

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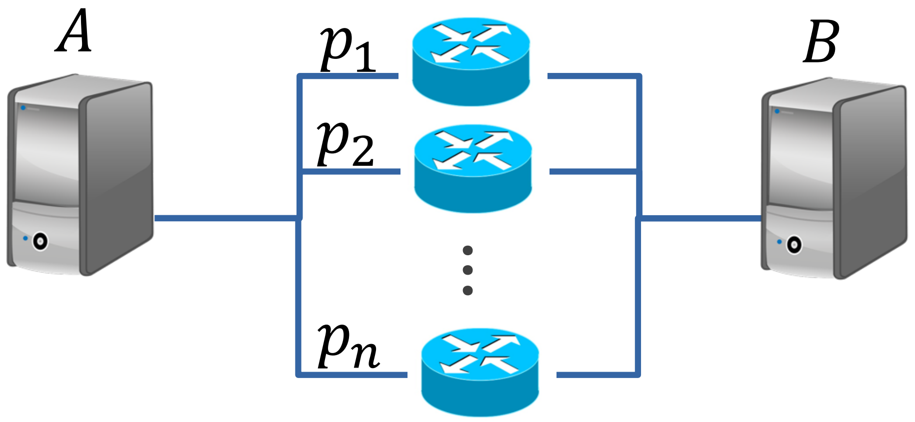


What is $P(E)$?

$$
\begin{aligned}
P(E) &= P(\geq 1 \text{ one router works}) \\
&= 1 - P(\text{all routers fail}) \\
&= 1 - (1 - p_1)(1 - p_2) \cdots (1 - p_n) \\
&= 1 - \prod_{i=1}^{n} (1 - p_i)
\end{aligned}
$$

≥ 1 with independent trials: take complement

05: Independence (live)

Lisa Yan

April 15, 2020

Independence

Two events E and F are defined as independent if:

$$P(EF) = P(E)P(F)$$

For independent events E and F,

- $P(E|F) = P(E)$

Think

Slide 24 has two questions to think over by yourself. We'll go over it together afterwards.

Post any clarifications here!

https://us.edstem.org/courses/109/discussion/27279

Think by yourself: 2 min

(by yourself)

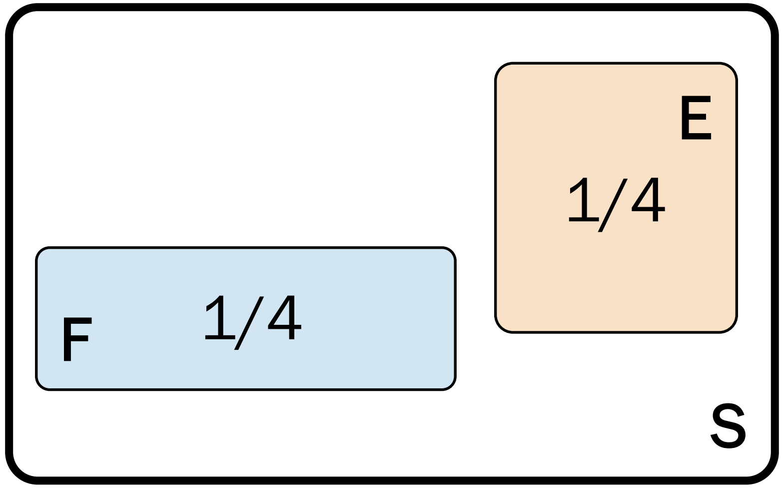
Independence?

Independent events E and F	$\Leftrightarrow$	$P(EF) = P(E)P(F)$ $P(E\|F) = P(E)$

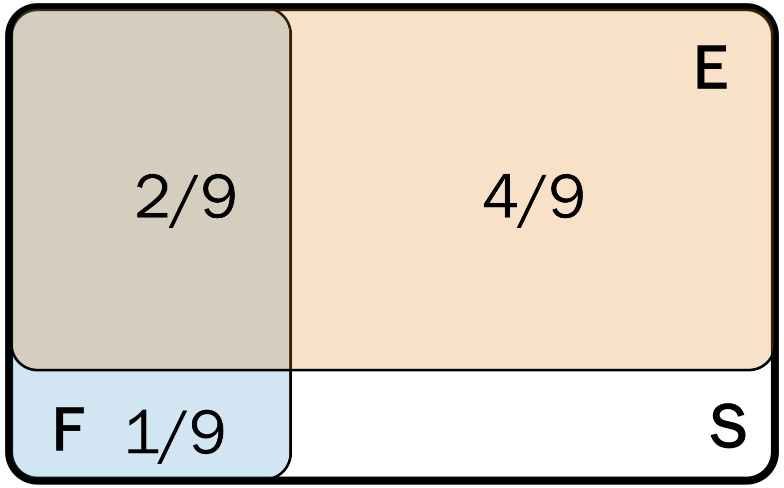
1. Two events E and F are independent if:
   A. Knowing that F happens means that E can't happen.
   B. Knowing that F happens doesn't change probability that E happened.

2. Are E and F independent in the following pictures?

A.



B.



(by yourself)

Independence?

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

1. Two events E and F are independent if:
   A. Knowing that F happens means that E can't happen.
   B. Knowing that F happens doesn't change probability that E happened.

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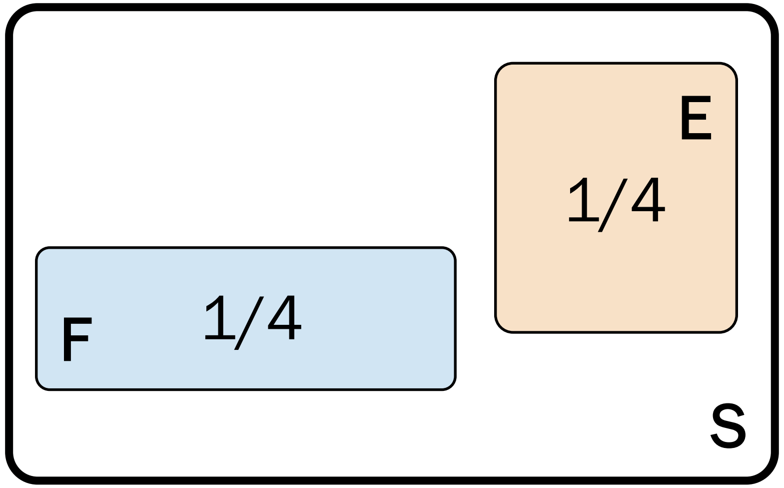
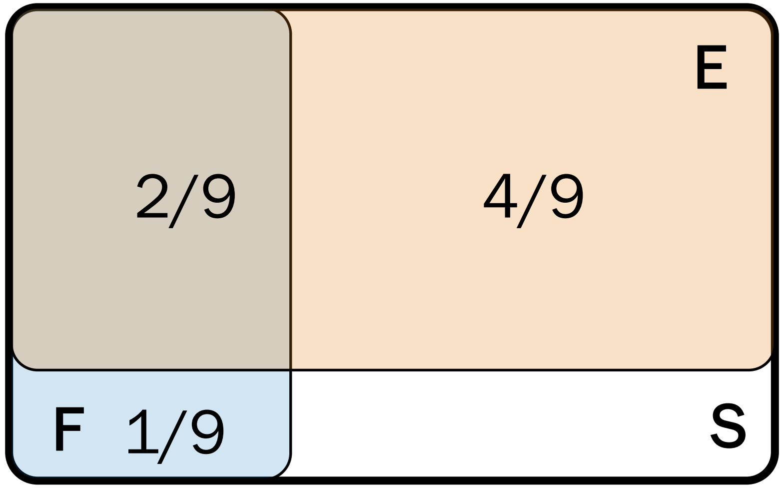
A.



B.



Be careful:
- Independence is NOT mutual exclusion.
- Independence is difficult to visualize graphically.

Independence

Two events E and F are defined as <u>independent</u> if:

$$P(EF) = P(E)P(F)$$

For independent events E and F,

- $P(E|F) = P(E)$
- E and F^C are independent.

new

Independence of complements

Statement:

If E and F are independent, then E and F^C are independent.

Proof:

$P(EF^C) = P(E) - P(EF)$	Intersection
$= P(E) - P(E)P(F)$	Independence of E and F
$= P(E)[1 - P(F)]$	Factoring
$= P(E)P(F^C)$	Complement
E and F^C are independent	Definition of independence

Knowing that F **did or didn't happen** does not change our belief that E happened.

Independence

Review

Two events E and F are defined as independent if:

$$P(EF) = P(E)P(F)$$

For independent events E and F,
- $P(E|F) = P(E)$
- E and F^C are independent

Independent trials are when we observe independent sub-experiments, each of which has the same set of possible outcomes.

Breakout Rooms

Check out the questions on the next slide (Slide 30). Post any clarifications here!

https://us.edstem.org/courses/109/discussion/27279

Breakout rooms: 5 min. Introduce yourself!

🤔

(biased) Coin Flips

Suppose we flip a coin n times. Each coin flip is an **independent trial** with probability p of coming up heads. Write an expression for the following:

1. $P(n$ heads on n coin flips$)$
2. $P(n$ tails on n coin flips$)$
3. $P($first k heads, then $n-k$ tails$)$
4. $P($exactly k heads on n coin flips$)$

(biased) Coin Flips

Suppose we flip a coin n times. Each coin flip is an **independent trial** with probability p of coming up heads. Write an expression for the following:

1. P(n heads on n coin flips)
2. P(n tails on n coin flips)
3. P(first k heads, then $n - k$ tails)
4. P(exactly k heads on n coin flips)

$$\binom{n}{k} p^k (1-p)^{n-k}$$

# of mutually exclusive outcomes

P(a particular outcome's k heads on n coin flips)

Make sure you understand #4! It will come up again.

Interlude for jokes/announcements

Announcements

Free Online CTL Tutoring

CTL offers appointment tutoring for CS 109, in addition to tutoring for a number of other courses. For more information and to schedule an appointment, visit our tutoring appointments and drop-in schedule page. We also have a variety of remote learning resources and academic coaching available to assist with all of your learning needs!

Sections start today!

Late signups/change form: end of day

Problem Set 1

due: **10am Friday** (not 10:30am)

Still confused about Monty Hall? Check out the code!
https://us.edstem.org/courses/109/discussion/27277?comment=93040

Probability of events

E or F
$P(E \cup F)$

Mutually exclusive?

Just add!

$P(E) + P(F)$

Inclusion-Exclusion Principle

$P(E) + P(F) - P(E \cap F)$

E and F
$P(EF)$

Independent?

Just multiply!

Chain Rule

Probability of events

E or F
$P(E \cup F)$

Mutually exclusive?

Just add!

$P(E) + P(F)$

Inclusion-Exclusion Principle

$P(E) + P(F) - P(E \cap F)$

E and F
$P(EF)$

Independent?

Just multiply!

$P(E)P(F)$

Chain Rule
$P(E)P(F|E)$
or
$P(F)P(E|F)$

Probability of events

E or F
$P(E \cup F)$

De Morgan's

E and F
$P(EF)$

Mutually exclusive?

Just add!

Inclusion-Exclusion Principle

Independent?

Just multiply!

Chain Rule

Augustus De Morgan

Augustus De Morgan (1806–1871):

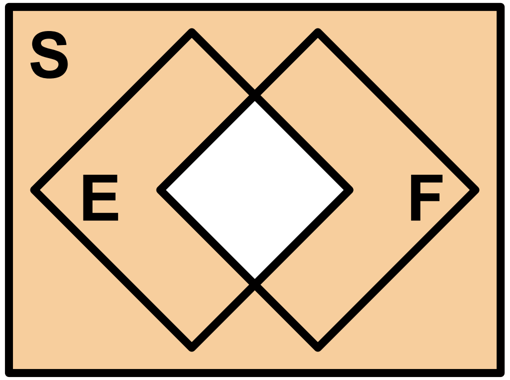
British mathematician who wrote the book *Formal Logic* (1847).

He looked remarkably similar to Jason Alexander (George from Seinfeld)

(but that's not important right now)

De Morgan's Laws

DeMorgan's lets you switch from AND to OR.



$$(E \cap F)^C = E^C \cup F^C$$

$$\left(\bigcap_{i=1}^{n} E_i\right)^C = \bigcup_{i=1}^{n} E_i^C$$

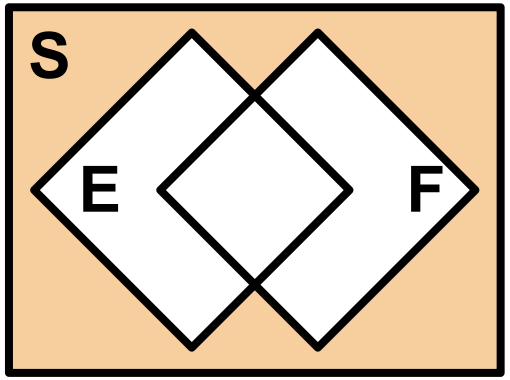
In probability:

$$P(E_1 E_2 \cdots E_n)$$

$$= 1 - P\Big((E_1 E_2 \cdots E_n)^C\Big)$$

$$= 1 - P\big(E_1^C \cup E_2^c \cup \cdots \cup E_n^c\big)$$

Great if E_i^C mutually exclusive!



$$(E \cup F)^C = E^C \cap F^C$$

$$\left(\bigcup_{i=1}^{n} E_i\right)^C = \bigcap_{i=1}^{n} E_i^C$$

In probability:

$$P(E_1 \cup E_2 \cup \cdots \cup E_n)$$

$$= 1 - P\Big((E_1 \cup E_2 \cup \cdots \cup E_n)^C\Big)$$

$$= 1 - P\big(E_1^c E_2^c \cdots E_n^c\big)$$

Great if E_i independent!

Think, then Breakout Rooms

Check out the questions on the next slide (Slide 40). **These are challenging problems.**

Post any clarifications here!

https://us.edstem.org/courses/109/discussion/27279

Think by yourself: 2 min

Breakout rooms: 5 min

🤔

Hash table fun

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?

2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?

Hash table fun

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?

Define S_i = string i is hashed into bucket 1
S_i^C = string i is not hashed into bucket 1

$$P(S_i) = p_1$$
$$P\left(S_i^C\right) = 1 - p_1$$

Hash table fun

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?

Define S_i = string i is hashed into bucket 1
S_i^C = string i is not hashed into bucket 1

$P(S_i) = p_1$
$P(S_i^C) = 1 - p_1$

WTF (not-real acronym for Want To Find):

$$P(E) = P(S_1 \cup S_2 \cup \cdots \cup S_m)$$

$$= 1 - P\left((S_1 \cup S_2 \cup \cdots \cup S_m)^C\right)$$ Complement

$$= 1 - P\left(S_1^C S_2^C \cdots S_m^C\right)$$ De Morgan's Law

$$= 1 - P\left(S_1^C\right)P\left(S_2^C\right)\cdots P\left(S_m^C\right) = 1 - \left(P\left(S_1^C\right)\right)^m$$ S_i independent trials

$$= 1 - (1 - p_1)^m$$

More hash table fun: Possible approach?

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?

Define F_i = bucket i has at least one string in it

$$
\begin{aligned}
P(E) &= P(F_1 \cup F_2 \cup \cdots \cup F_k) \\
&= 1 - P\left((F_1 \cup F_2 \cup \cdots \cup F_k)^C\right) \\
&= 1 - P\left(F_1^C F_2^C \cdots F_k^C\right) \\
&?= 1 - P\left(F_1^C\right) P\left(F_2^C\right) \cdots P\left(F_k^C\right)
\end{aligned}
$$

⚠ F_i bucket events are *dependent*!

So we cannot approach with complement.

More hash table fun

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?

Define F_i = bucket i has at least one string in it

$$P(E) = P(F_1 \cup F_2 \cup \cdots \cup F_k)$$
$$= 1 - P\left((F_1 \cup F_2 \cup \cdots \cup F_k)^C\right)$$
$$= 1 - P\left(F_1^C F_2^C \cdots F_k^C\right)$$

$= P(\text{no strings hashed to buckets 1 to } k)$
$= \left(P(\text{string hashed outside bkts 1 to } k)\right)^m$
$= (1 - p_1 - p_2 \ldots - p_k)^m$

$$= 1 - (1 - p_1 - p_2 \ldots - p_k)^m$$

The fun never stops with hash tables

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it? ✅

Looking for a challenge? ☺

The fun never stops with hash tables

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?

3. E = each of of buckets 1 to k has $\geq$ 1 string hashed into it?

Hint: Use Part 2's event definition:

Define F_i = bucket i has at least one string in it

Check out the Lecture Notes for a solution!