

# 09: Continuous RVs

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Lisa Yan

April 24, 2020

# Quick slide reference

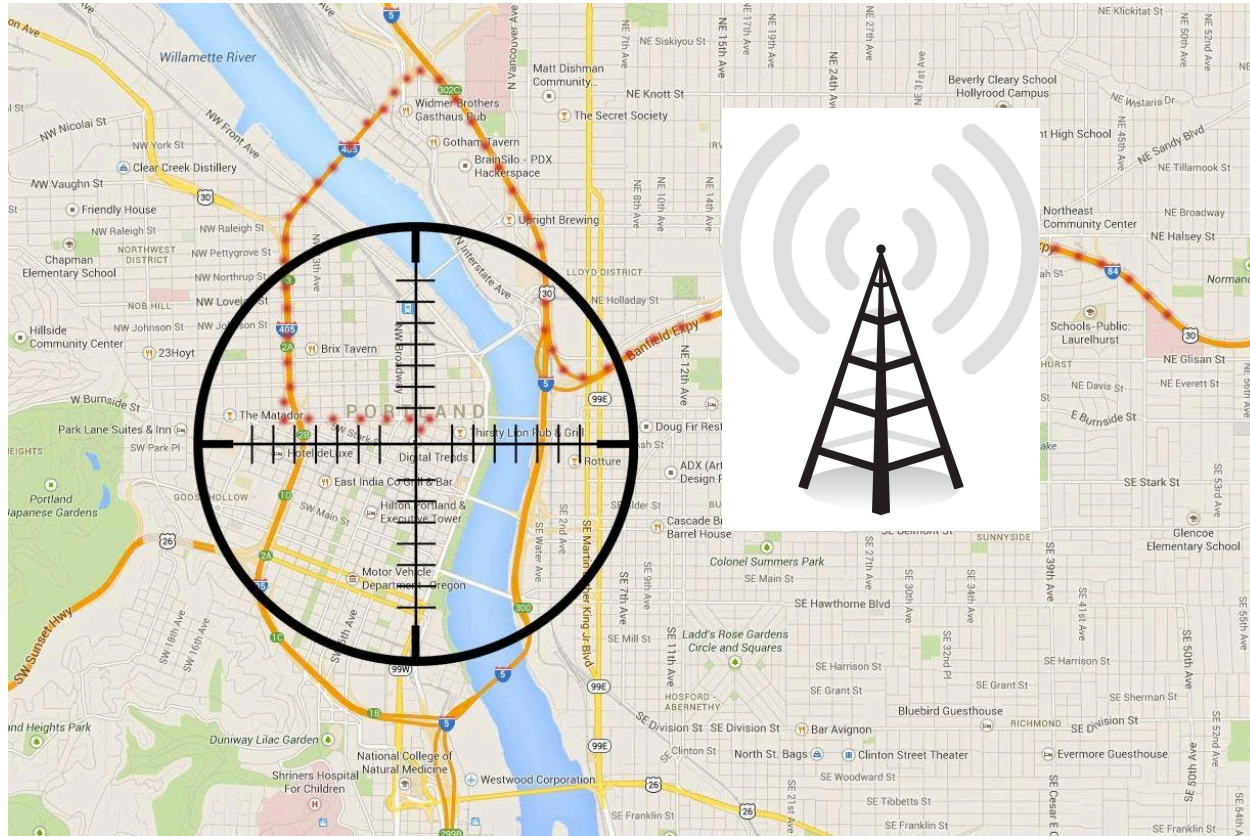
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|    |                |                    |
|----|----------------|--------------------|
| 3  | Continuous RVs | 09a_continuous_rvs |
| 15 | Uniform RV     | 09b_uniform        |
| 22 | Exponential RV | 09c_exponential    |
| 31 | Exercises, CDF | LIVE               |
| 55 | Extra material | 09e_extra          |

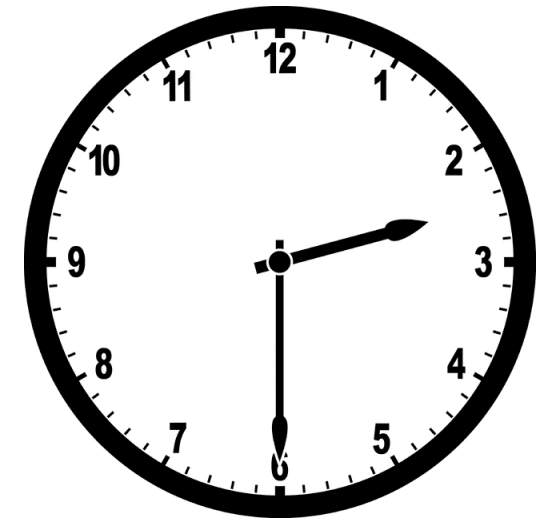


# Continuous RVs

# Not all values are discrete



`import numpy as np`  
`np.random.random()` ?





# People heights

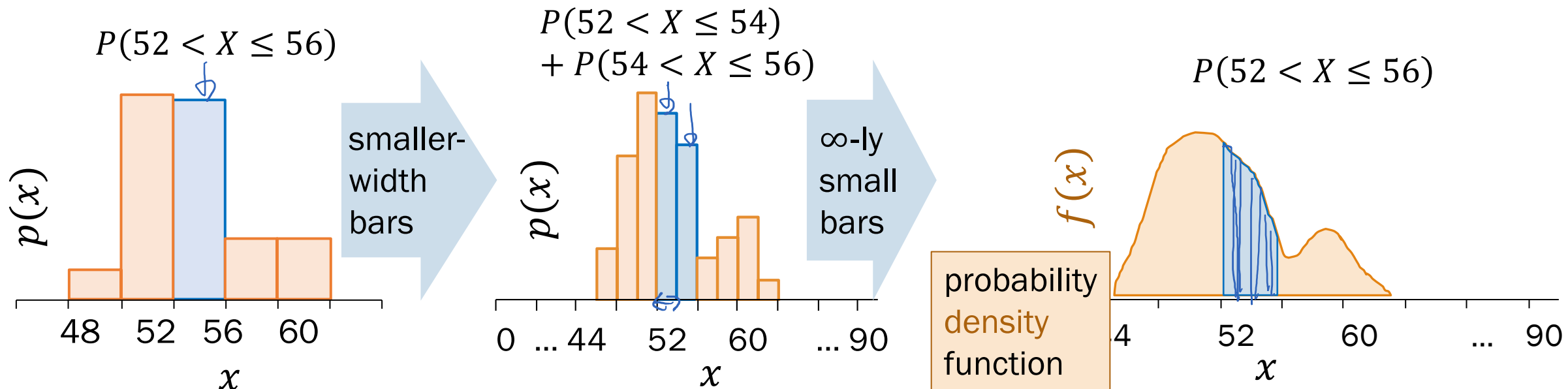
You are volunteering at the local elementary school.

- To choose a t-shirt for your new buddy Jordan, you need to know their height.

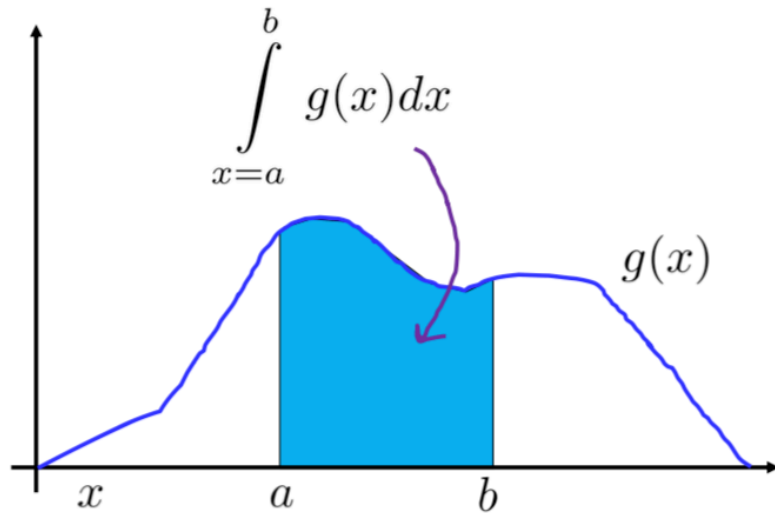
1. What is the probability that your buddy is 54.0923857234 inches tall?

Essentially 0

2. What is the probability that your buddy is between 52–56 inches tall?



# Integrals



Integral = area under a curve



Loving, not scary

# Continuous RV definition

A random variable  $X$  is **continuous** if there is a **probability density function**  $f(x) \geq 0$  such that for  $-\infty < x < \infty$ : (PDF)

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

Integrating a PDF must always yield valid probabilities, and therefore the PDF must also satisfy

$$\int_{-\infty}^{\infty} f(x) dx = P(-\infty < X < \infty) = 1$$

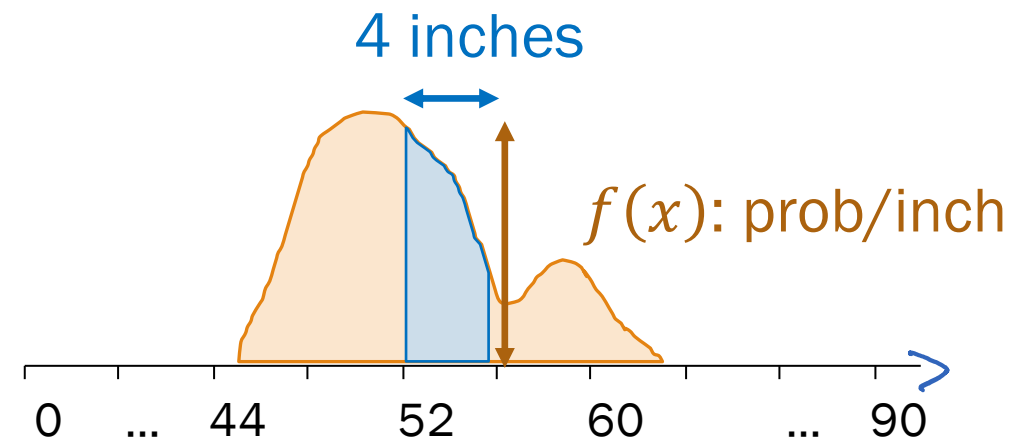
Also written as:  $f_X(x)$

↑ PDF    ↑ random variable

# Today's main takeaway, #1

Integrate  $f(x)$  to get probabilities.

PDF Units: probability per units of  $X$



$$P(52 \leq X \leq 56) = \int_{52}^{56} f(x) dx$$

*probability*

Discrete      Continuous  
PMF vs PDF

**Discrete** random variable  $X$

Probability mass function (PMF):

$$p(x)$$

To get probability:

$$P(X = x) = p(x)$$
$$P(a \leq X \leq b) = \sum_{x=a}^b p(x)$$

**Continuous** random variable  $X$

Probability density function (PDF):

$$f(x)$$

To get probability:

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

Both are measures of how **likely**  $X$  is to take on a value.

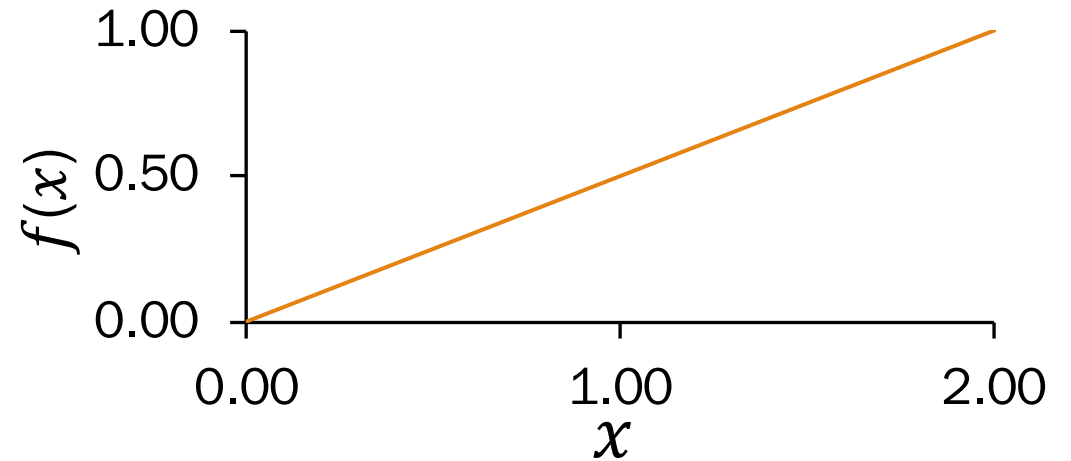
# Computing probability

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

Let  $X$  be a continuous RV with PDF:

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

What is  $P(X \geq 1)$ ?



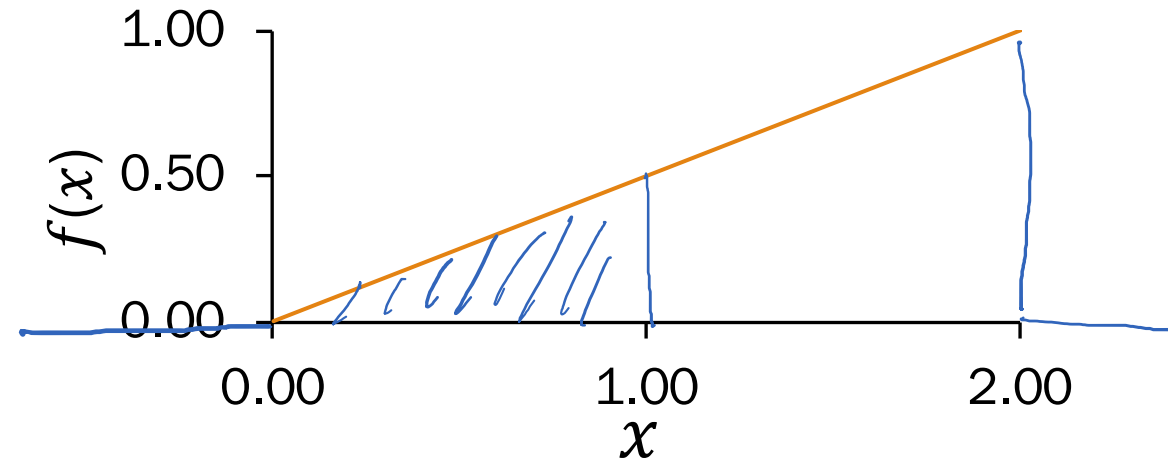


# Computing probability

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

Let  $X$  be a continuous RV with PDF:

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$



What is  $P(X \geq 1)$ ?

Strategy 1: Integrate

$$\begin{aligned} P(1 \leq X < \infty) &= \int_1^{\infty} f(x) dx = \int_1^2 \frac{1}{2}x dx \\ &= \frac{1}{2} \left( \frac{1}{2}x^2 \right) \Big|_1^2 = \frac{1}{2} \left[ 2 - \frac{1}{2} \right] = \frac{3}{4} \end{aligned}$$

Strategy 2: Know triangles  $\frac{1}{2}bh = A$

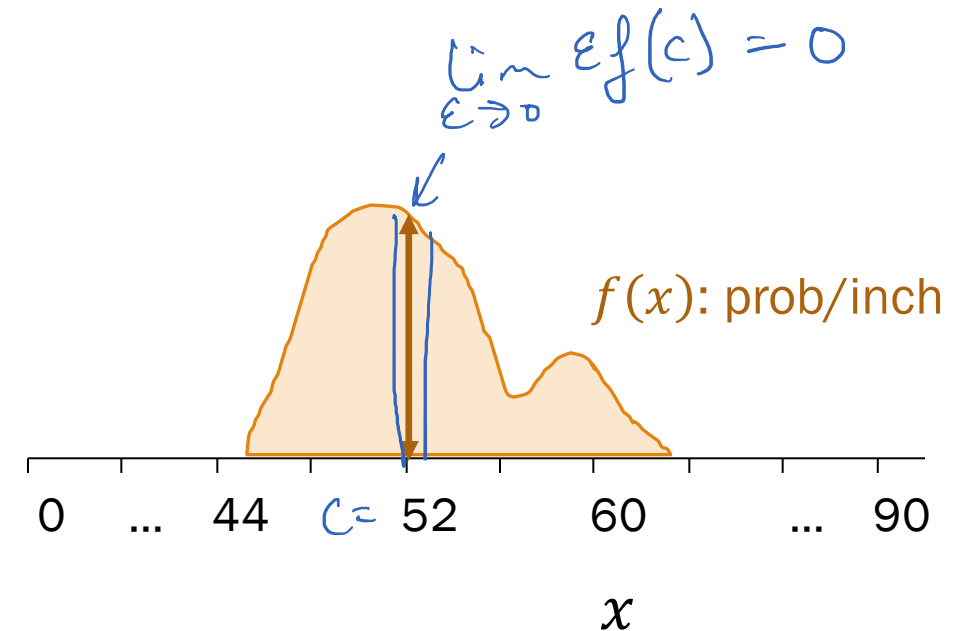
$$1 - \frac{1}{2} \left( \frac{1}{2} \right) = \frac{3}{4} = P(1 \leq X < \infty)$$

Wait...is this even legal?

$$P(0 \leq X < 1) = \int_0^1 f(x) dx ??$$

# Today's main takeaway, #2

For a continuous random variable  $X$  with PDF  $f(x)$ ,  
$$P(X = c) = \int_c^c f(x) dx = 0.$$

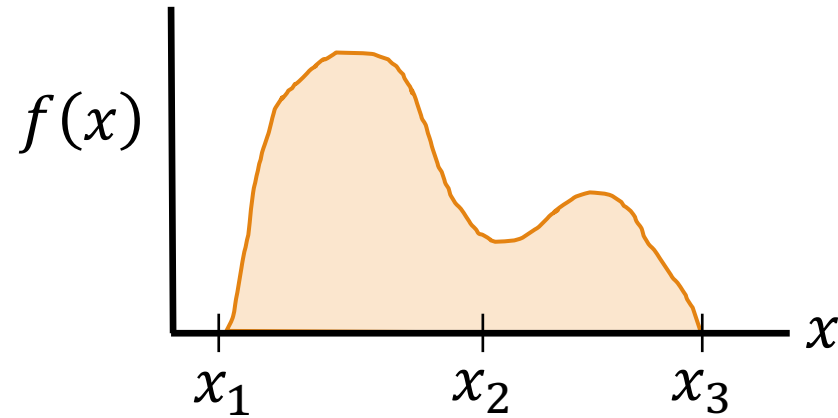


Contrast with PMF in discrete case:  $P(X = c) = p(c)$

# PDF Properties

For a continuous RV  $X$  with PDF  $f$ ,

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$



support: set of  $x$  where  $f(x) > 0$

True/False:

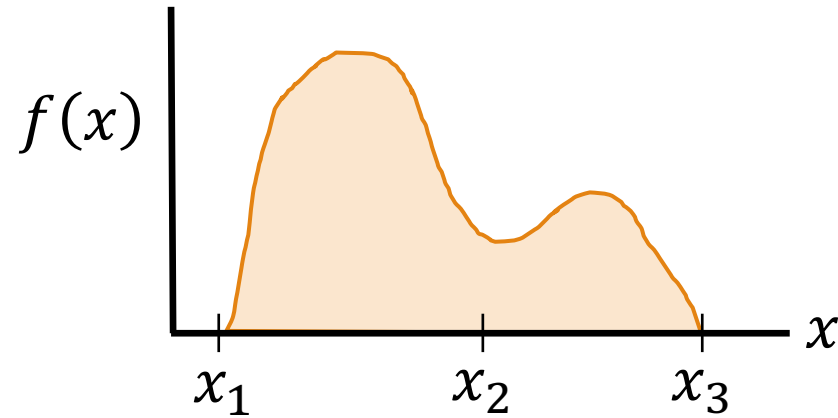
1.  $P(X = c) = 0$
2.  $P(a \leq X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a < X \leq b)$
3.  $f(x)$  is a probability
4. In the graphed PDF above,  
 $P(x_1 \leq X \leq x_2) > P(x_2 \leq X \leq x_3)$



# PDF Properties

For a continuous RV  $X$  with PDF  $f$ ,

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$



support: set of  $x$  where  $f(x) > 0$   
Interval width  $dx \rightarrow 0$

True/False:

1.  $P(X = c) = 0$

★ 2.  $P(a \leq X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a < X \leq b)$

✗ 3.  $f(x)$  is a probability

4. In the graphed PDF above,  
 $P(x_1 \leq X \leq x_2) > P(x_2 \leq X \leq x_3)$

Compare area under the curve  $f$

# Uniform RV

# Uniform Random Variable

def An **Uniform** random variable  $X$  is defined as follows:

$$X \sim \text{Uni}(\alpha, \beta)$$

Support:  $[\alpha, \beta]$   
(sometimes defined  
over  $(\alpha, \beta)$ )

PDF

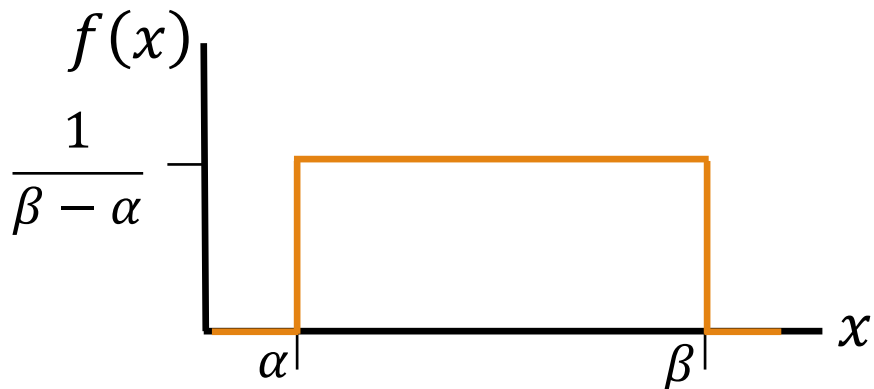
$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

Expectation

$$E[X] = \frac{\alpha + \beta}{2}$$

Variance

$$\text{Var}(X) = \frac{(\beta - \alpha)^2}{12}$$

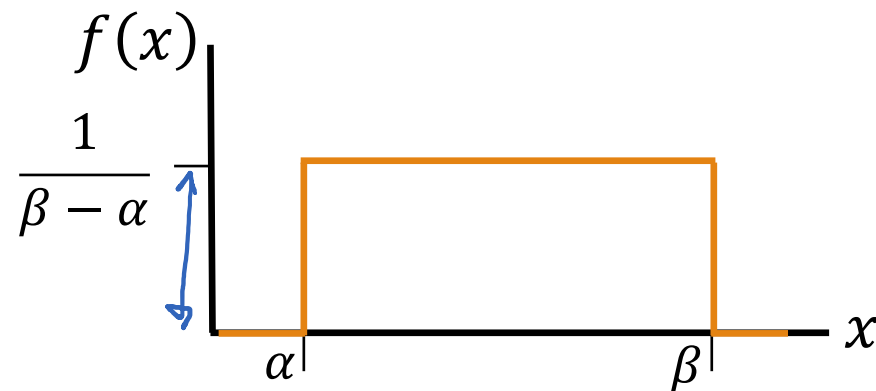




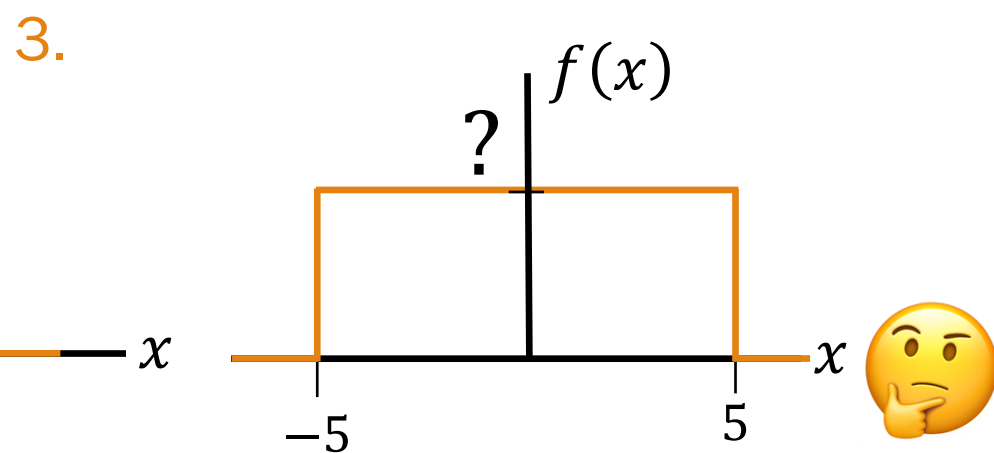
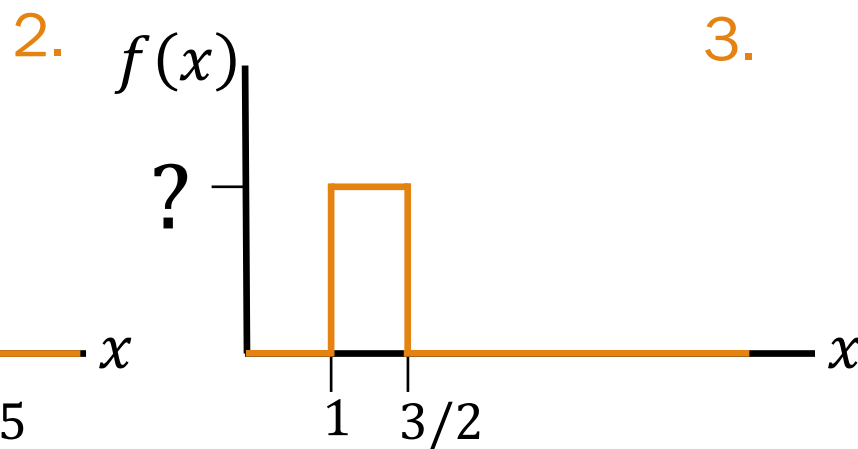
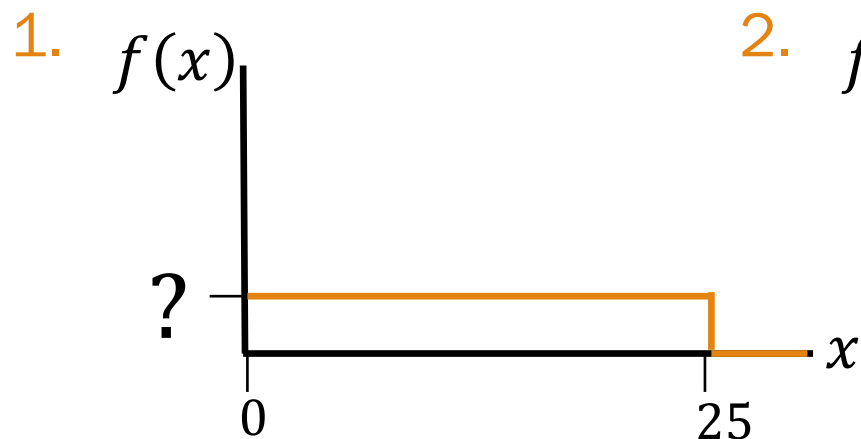
# Quick check

If  $X \sim \text{Uni}(\alpha, \beta)$ , the PDF of  $X$  is:

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$



What is  $\frac{1}{\beta - \alpha}$  if the following graphs are PDFs of Uniform RVs  $X$ ?

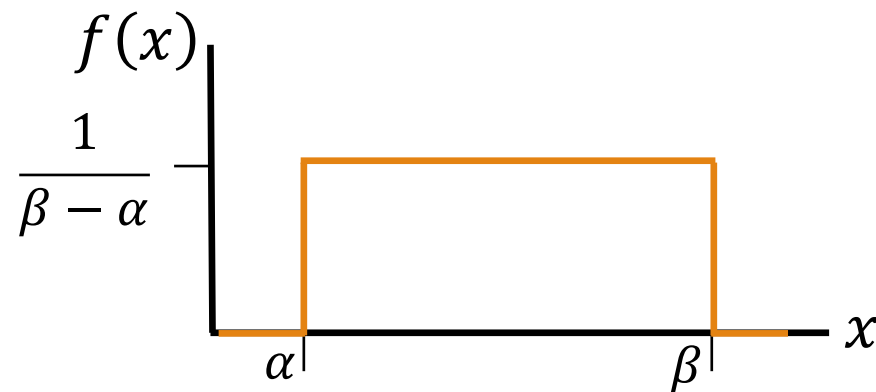


# Quick check

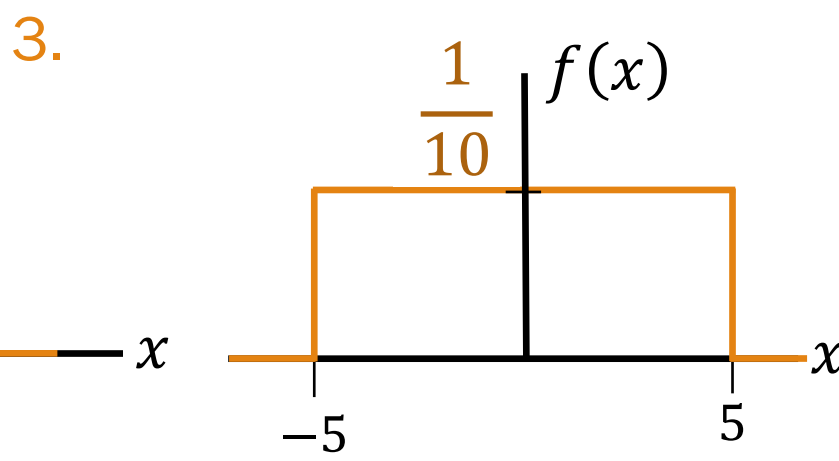
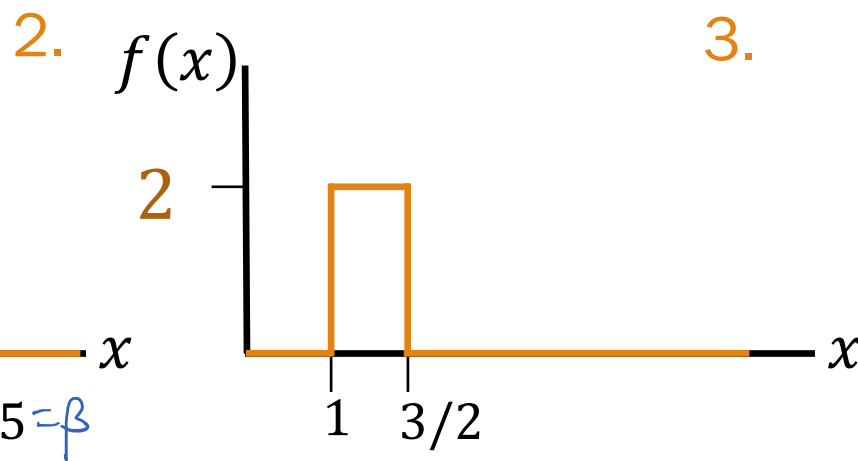
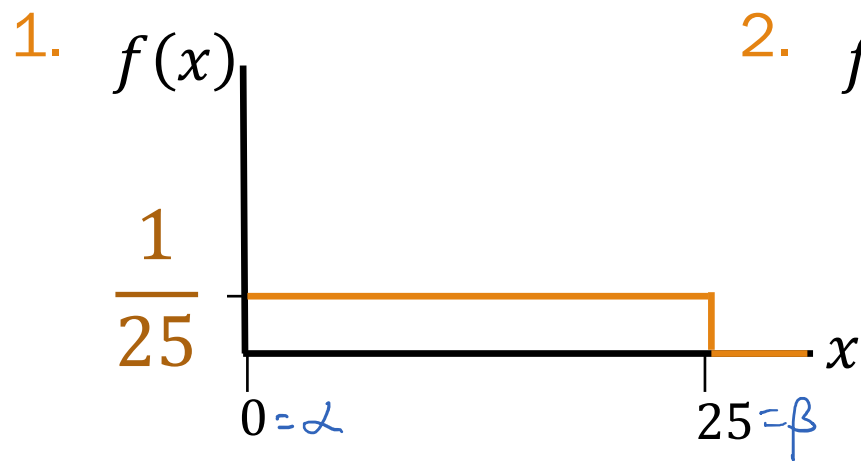
If  $X \sim \text{Uni}(\alpha, \beta)$ , the PDF of  $X$  is:

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x) dx = 1$$
$$\int_{\alpha}^{\beta} \frac{1}{\beta - \alpha} = 1$$



What is  $\frac{1}{\beta - \alpha}$  if the following graphs are PDFs of Uniform RVs  $X$ ?



# Expectation and Variance

Discrete RV  $X$

$$E[X] = \sum_x x p(x)$$

LOTUS

$$E[g(X)] = \sum_x g(x) p(x)$$

Continuous RV  $X$

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

Both continuous and discrete RVs

$$E[aX + b] = aE[X] + b$$

$$\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

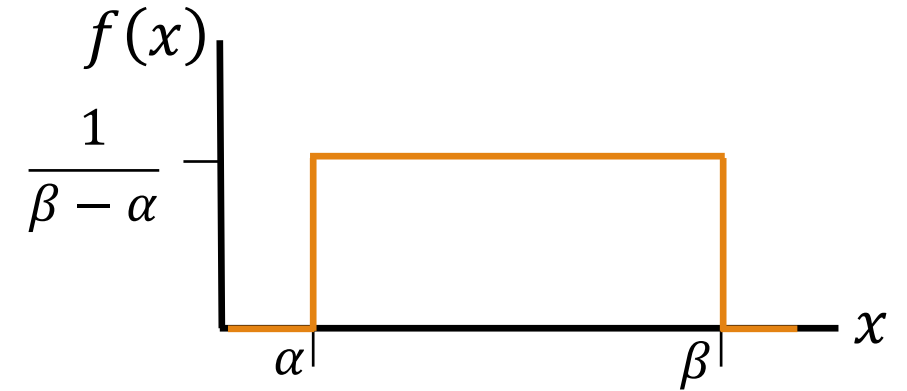
$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

} Linearity of  
Expectation  
} Properties of  
variance

$$\text{TL;DR: } \sum_{x=a}^b \Rightarrow \int_a^b dx$$

# Uniform RV expectation

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x \cdot f(x) dx \\ &= \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta - \alpha} dx \\ &= \frac{1}{\beta - \alpha} \cdot \frac{1}{2} x^2 \Big|_{\alpha}^{\beta} \\ &= \frac{1}{\beta - \alpha} \cdot \frac{1}{2} (\beta^2 - \alpha^2) \\ &= \frac{1}{2} \cdot \frac{(\beta + \alpha)(\cancel{\beta - \alpha})}{\cancel{\beta - \alpha}} = \frac{\alpha + \beta}{2} \end{aligned}$$



Interpretation:  
Average the start & end

# Uniform Random Variable

def An Uniform random variable  $X$  is defined as follows:

$$X \sim \text{Uni}(\alpha, \beta)$$

Support:  $[\alpha, \beta]$   
(sometimes defined  
over  $(\alpha, \beta)$ )

PDF

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

Expectation

$$E[X] = \frac{\alpha + \beta}{2}$$

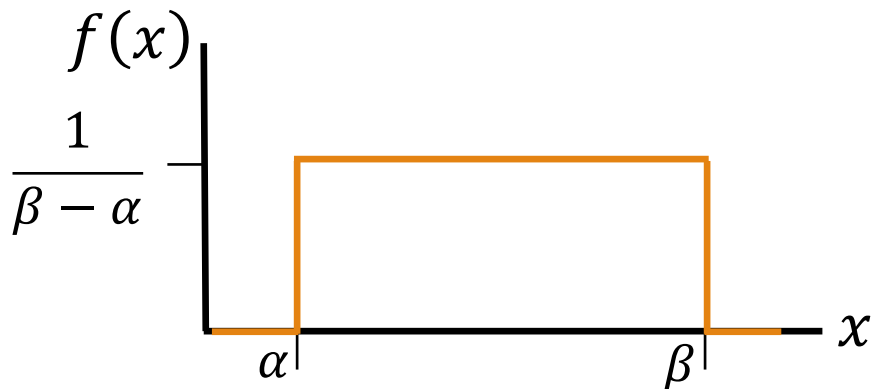
Just now

Variance

$$\text{Var}(X) = \frac{(\beta - \alpha)^2}{12}$$



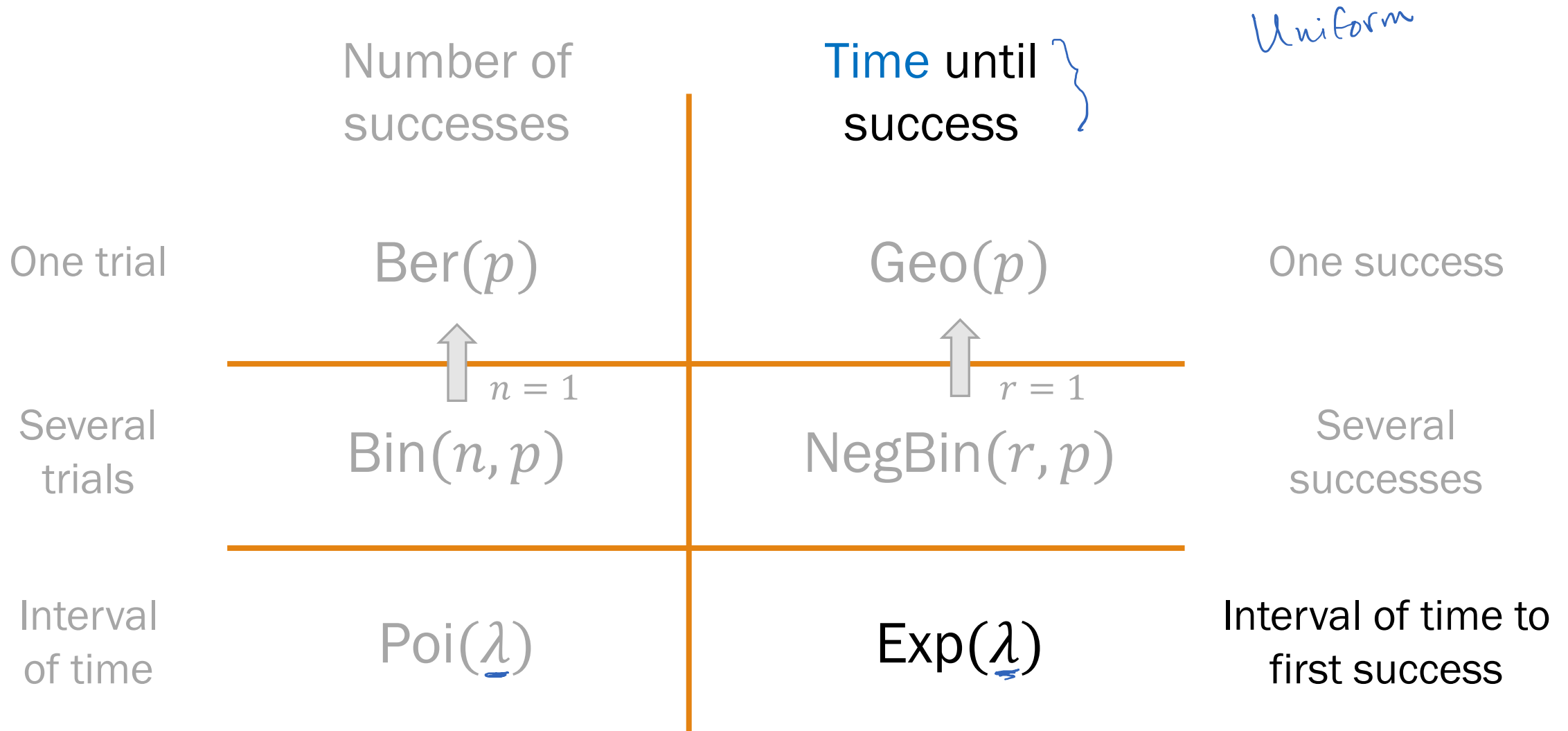
On your own time



# Exponential RV



# Grid of random variables



# Exponential Random Variable

Consider an experiment that lasts a duration of time until <sup>first</sup> success occurs.  
def An **Exponential** random variable  $X$  is the amount of time until <sup>first</sup> success.

$$X \sim \text{Exp}(\lambda)$$

Support:  $[0, \infty)$

PDF

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Expectation

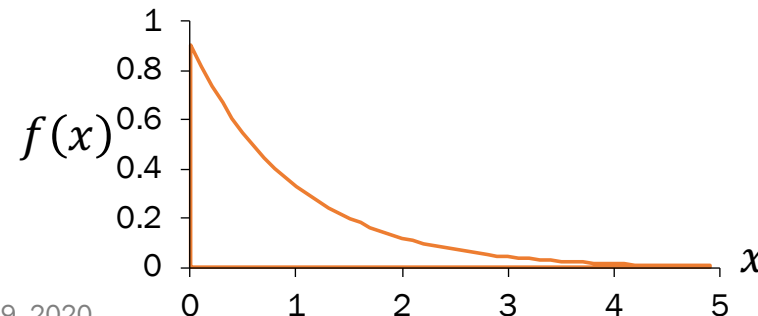
$$E[X] = \frac{1}{\lambda} \quad (\text{in extra slides})$$

Variance

$$\text{Var}(X) = \frac{1}{\lambda^2} \quad (\text{on your own})$$

## Examples:

- Time until next earthquake
- Time for request to reach web server
- Time until end of cell phone contract



# Interpreting $\text{Exp}(\lambda)$

---

def An **Exponential** random variable  $X$  is the amount of time until success.

$$X \sim \text{Exp}(\lambda)$$

Expectation

$$E[X] = \frac{1}{\lambda}$$

Based on the expectation  $E[X]$ , what are the units of  $\lambda$ ?



# Interpreting $\text{Exp}(\lambda)$

def An **Exponential** random variable  $X$  is the amount of time until success.

$$X \sim \text{Exp}(\lambda) \quad \text{Expectation} \quad E[X] = \frac{1}{\lambda}$$

Based on the expectation  $E[X]$ , what are the units of  $\lambda$ ?

$$\frac{1}{\lambda} = E[X]: \frac{\text{time}}{\text{event}} \Rightarrow \lambda: \frac{\text{event}}{\text{time}}$$

$$Y \sim \text{Poi}(\lambda), E[Y] = \lambda \frac{\text{event}}{\text{time}}$$

e.g., average # of successes per second

For both Poisson and Exponential RVs,  
 $\lambda = \# \text{ successes/time}$ .

# Earthquakes



ILL. No. 65. MEMORIAL ARCH, WITH CHURCH IN BACKGROUND, STANFORD UNIVERSITY, SHOWING TYPES OF CARVED WORK WITH THE SANDSTONE.

1906 Earthquake  
Magnitude 7.8

# Earthquakes

$$X \sim \text{Exp}(\lambda) \quad \begin{array}{l} E[X] = 1/\lambda \\ f(x) = \lambda e^{-\lambda x} \quad \text{if } x \geq 0 \end{array}$$

Major earthquakes (magnitude 8.0+) occur once every 500 years.\*

1. What is the probability of a major earthquake in the next 30 years?

We know on average:

$$E[X] = \frac{1}{\lambda} = 500 \frac{\text{years}}{\text{earthquake}}$$

$$\frac{1}{500} = 0.002 \frac{\text{earthquakes}}{\text{year}}$$

$$1 \frac{\text{earthquakes}}{500 \text{ years}}$$

$$X \sim \text{Exp}(\lambda = 0.002)$$

$$E[X] = \frac{1}{\lambda}$$

$\geq 1$  event within 30 yrs  $\iff$  first event within 30 years

$$P(X < 30)$$





# Earthquakes

$$X \sim \text{Exp}(\lambda) \quad \begin{array}{l} E[X] = 1/\lambda \\ f(x) = \lambda e^{-\lambda x} \quad \text{if } x \geq 0 \end{array}$$

Major earthquakes (magnitude 8.0+) occur once every 500 years.\*

1. What is the probability of a major earthquake in the next 30 years?

Define events/  
RVs & state goal

Solve

$X$ : when next  
earthquake happens (years)

$X \sim \text{Exp}(\lambda = 0.002)$

$\lambda$ : year<sup>-1</sup> = 1/500

Want:  $P(X < 30)$

$$P(X < 30) = P(0 < X < 30)$$

$$= \int_0^{30} \lambda e^{-\lambda x} dx$$

$$= \lambda \left[ \frac{1}{-\lambda} e^{-\lambda x} \right]_0^{30}$$

$$= \left[ -e^{-30\lambda} - (-e^{-0\lambda}) \right]$$

$$= 1 - e^{-30\lambda} \approx 0.658$$

Recall

$$\int e^{cx} dx = \frac{1}{c} e^{cx}$$

$c = -\lambda$

\*In California, according to historical data from USGS, 2015

# Earthquakes

$$X \sim \text{Exp}(\lambda) \quad \begin{array}{l} E[X] = 1/\lambda \\ f(x) = \lambda e^{-\lambda x} \quad \text{if } x \geq 0 \end{array}$$

Major earthquakes (magnitude 8.0+) occur once every 500 years.\*  $= E[X]$

1. What is the probability of a major earthquake in the next 30 years?
2. What is the **standard deviation** of years until the next earthquake?

Define events/  
RVs & state goal

$X$ : when next  
earthquake happens

$X \sim \text{Exp}(\lambda = 0.002)$

$\lambda$ : year<sup>-1</sup>

Want:  $P(X < 30)$

Solve

$$\begin{aligned} \frac{1}{\lambda} &= 500 \\ \text{SD}(X) &= \sqrt{\text{Var}(X)} = \sqrt{1/\lambda^2} \\ &= \frac{1}{\lambda} = 500 \text{ years} \end{aligned}$$

# 09: Continuous RVs (live)

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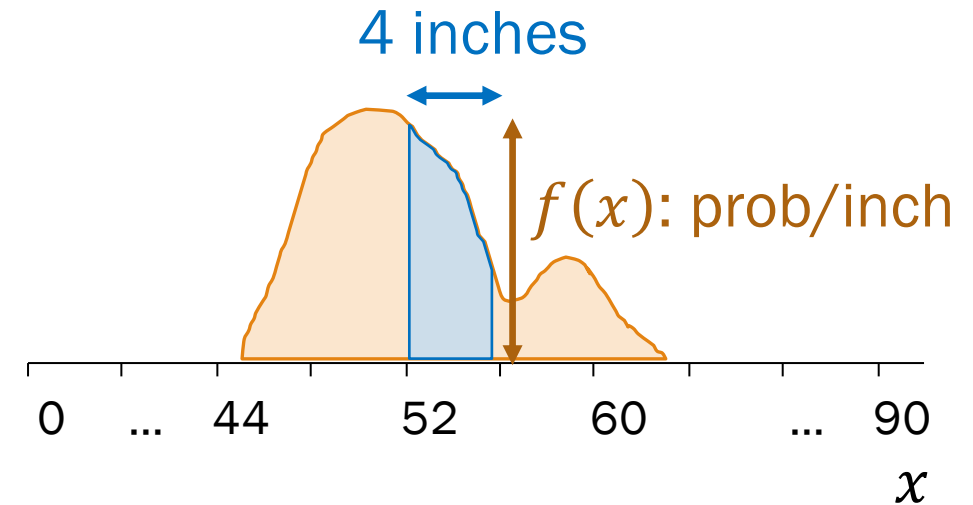
Lisa Yan

April 24, 2020

# Today's main takeaway, #1

PDF  
↓

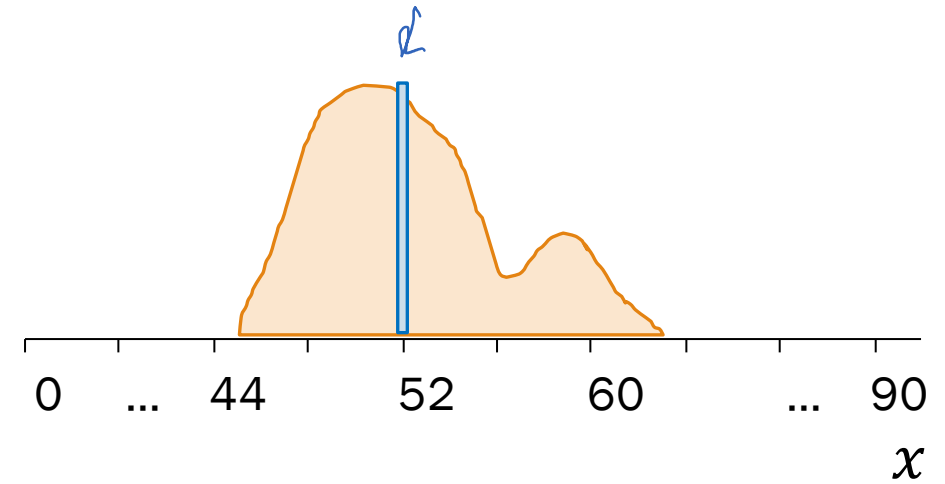
Integrate  $f(x)$  to get probabilities.



$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

For a continuous random variable  $X$  with PDF  $f(x)$ ,

$$P(X = c) = \int_c^c f(x) dx = 0.$$



Implication:  $P(a \leq X \leq b) = P(a < X < b)$

# Think

Slide 35 has a matching question to go over by yourself. We'll go over it together afterwards.

Post any clarifications here!

<https://us.edstem.org/courses/109/discussion/39083>

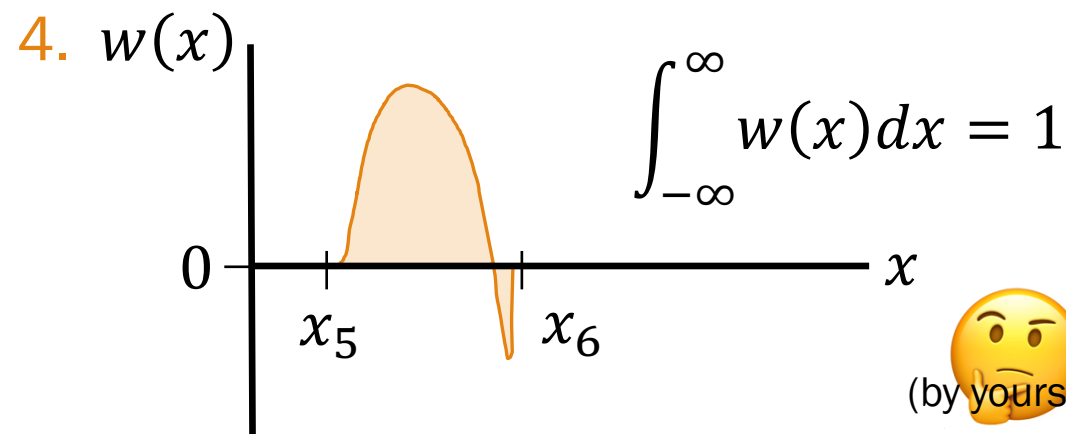
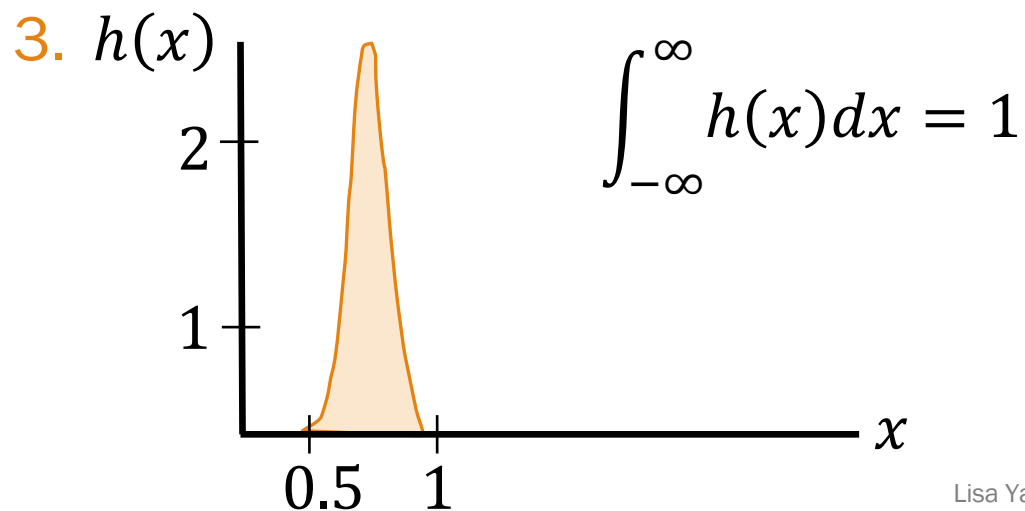
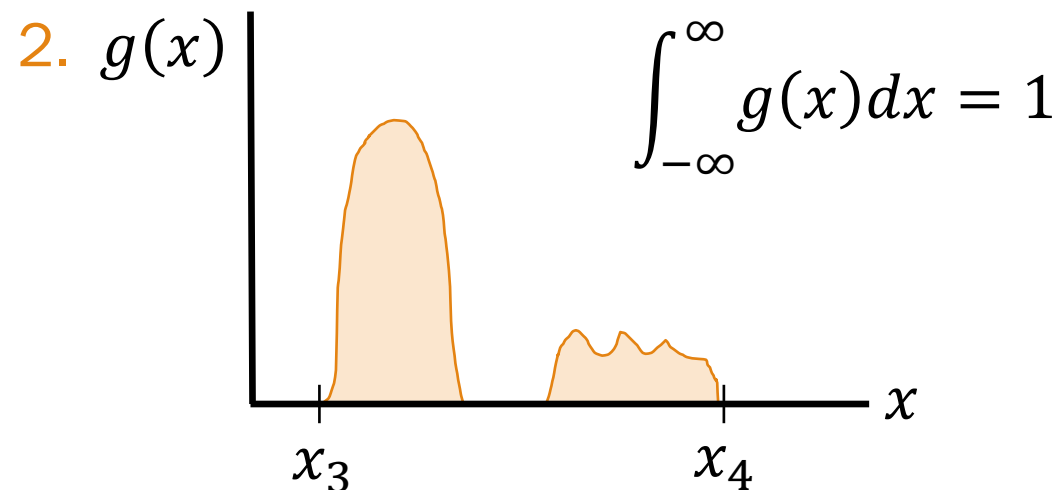
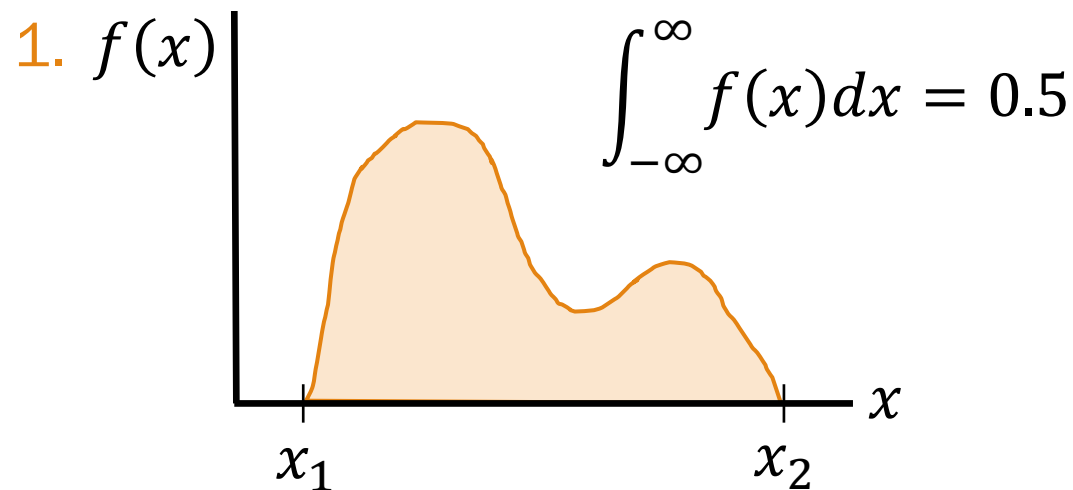
Think by yourself: 2 min



# Determining valid PDFs

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

Which of the following functions are valid PDFs?

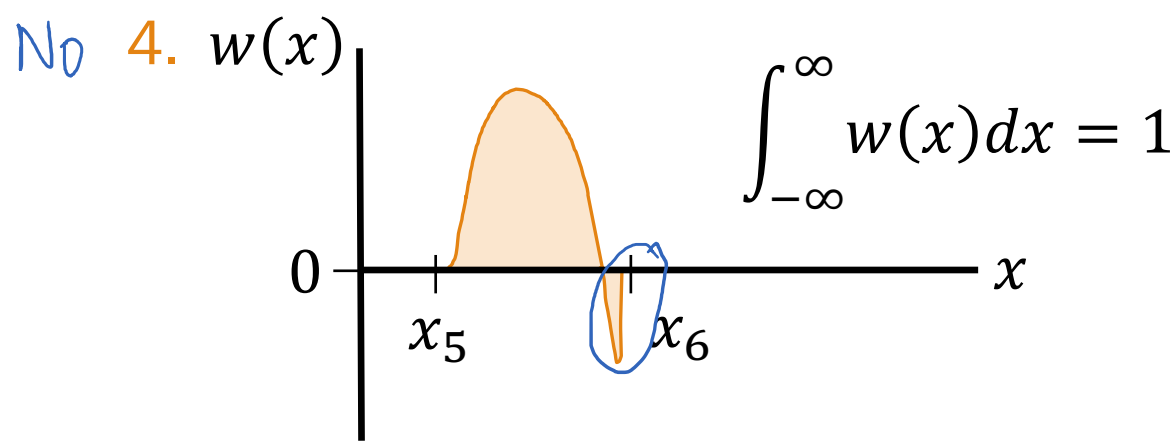
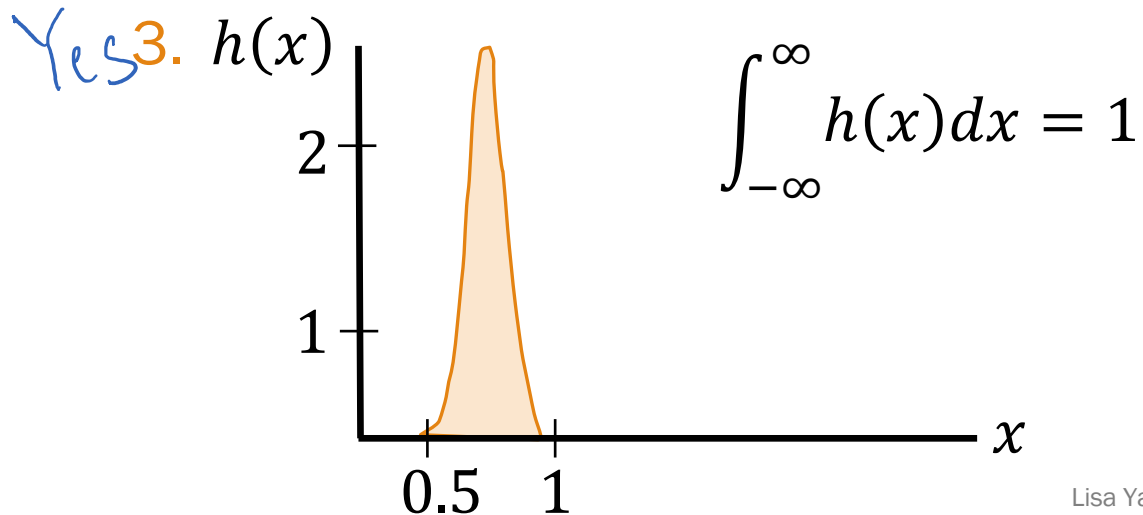
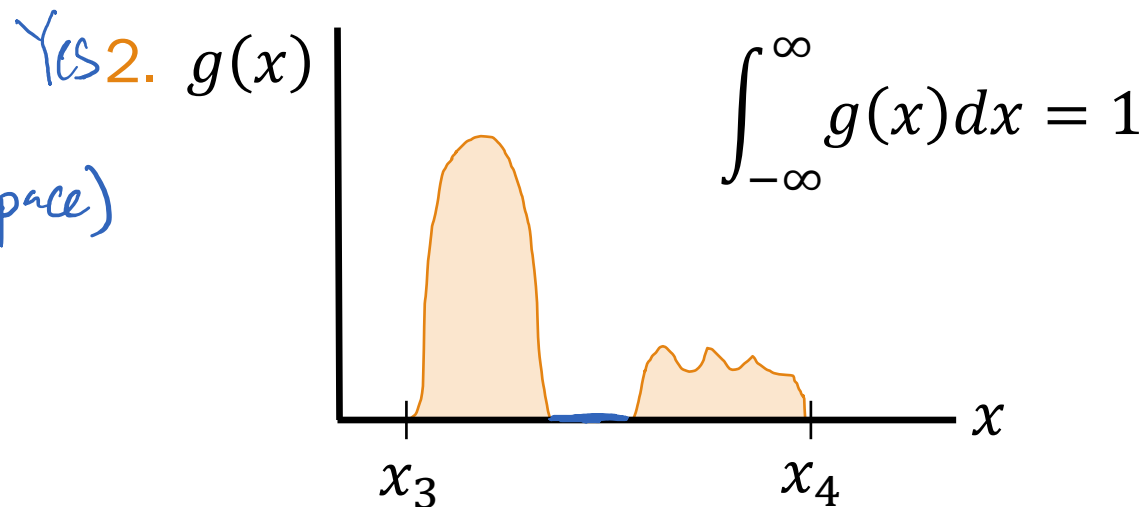
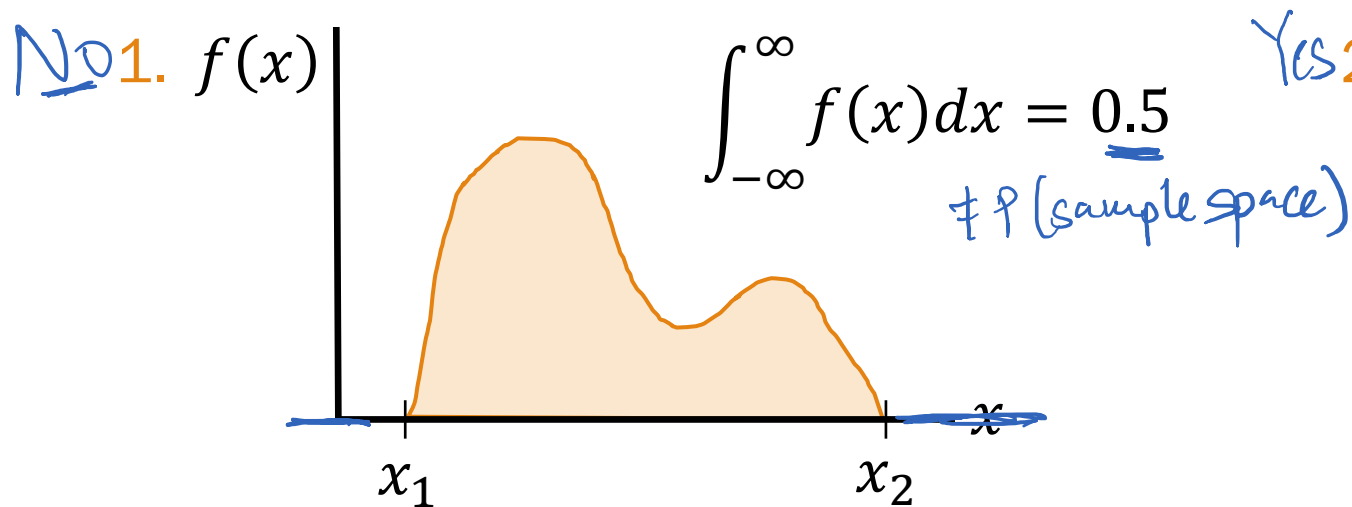


 (by yourself)

# Determining valid PDFs

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

Which of the following functions are valid PDFs?





# Breakout Rooms

Check out the question on the next slide (Slide 38). Post any clarifications here!

<https://us.edstem.org/courses/109/discussion/39083>

Breakout rooms: 4 min. Introduce yourself!



# Riding the Marguerite Bus

You want to get on the Marguerite bus.

- The bus stops at the Gates building at 15-minute intervals (2:00, 2:15, etc.).
- You arrive at the stop uniformly between 2:00-2:30pm.

$P(\text{you wait} < 5 \text{ minutes for bus})?$



# Riding the Marguerite Bus

You want to get on the Marguerite bus.

- The bus stops at the Gates building at 15-minute intervals (2:00, 2:15, etc.).
- You arrive at the stop uniformly between 2:00-2:30pm.

P(you wait < 5 minutes for bus)?

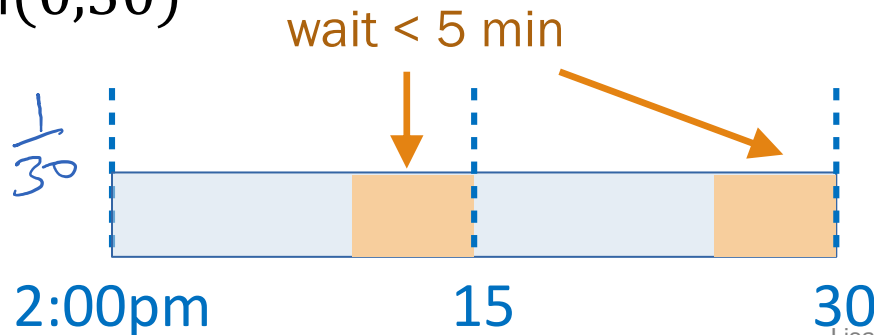


1. Define events/  
RVs & state goal

$X$ : time passenger  
arrives after 2:00

$X \sim \text{Uni}(0,30)$

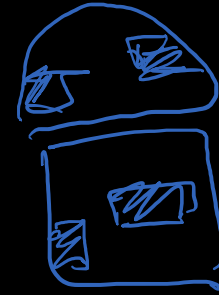
Want:



2. Solve

$$\begin{aligned} &P(10 < X \leq 15) + P(25 < X \leq 30) \\ &= \int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx \\ &= \frac{5}{30} + \frac{5}{30} = \frac{1}{3} \end{aligned}$$

# Interlude for jokes/announcements



R2D2  
↓  
R2Detour

# Announcements

---

## Quiz #1

Time frame: Thursday 4/30 12:00am-11:59pm PT

Covers: Up to end of Week 3 (including Lecture 9)

Review session (Tim): Tuesday 4/28 12-2pm PT

<https://stanford.zoom.us/j/92275547392>

Info and practice: <https://web.stanford.edu/class/cs109/exams/quizzes.html>

## Python tutorial #2 (Sandra)

When: today 4/24 5:00-6:00PT

<https://stanford.zoom.us/j/621852324>

Recorded Notes online

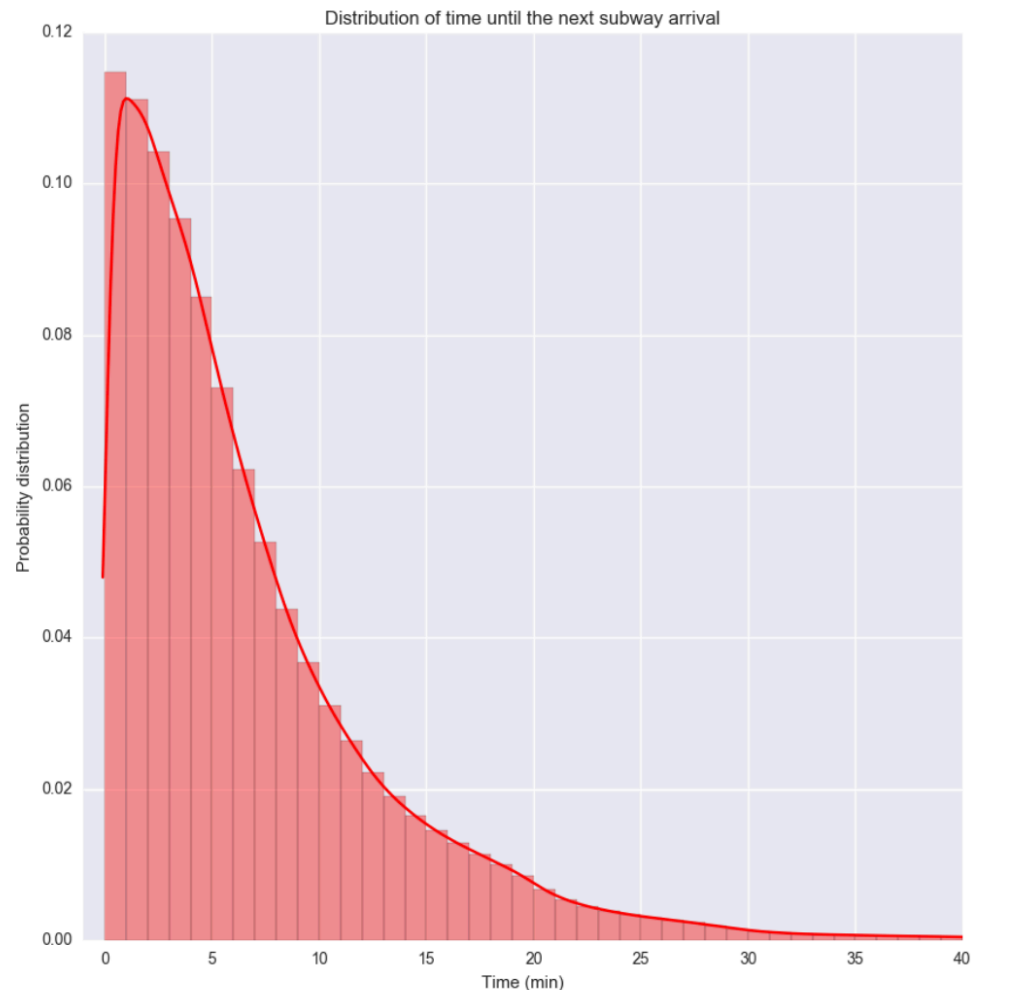
Useful for: pset2, pset3

**Note:** If you have an emergency situation during the quiz, please contact Lisa and Cooper. We will try our best to accommodate.

# Interesting probability news

## NYC subway math

<https://erikbern.com/2016/04/04/nyc-subway-math.html>



Distribution of time  
until the next subway arrival  
Probably Beta RV (Week 8)

# Cumulative Distribution Function (CDF)

For a random variable  $X$ , the **cumulative distribution function** (CDF) is defined as

$$F(a) = F_X(a) = P(X \leq a), \text{ where } -\infty < a < \infty$$

For a discrete RV  $X$ , the CDF is:

$$F(a) = P(X \leq a) = \sum_{\text{all } x \leq a} p(x)$$

# Cumulative Distribution Function (CDF)

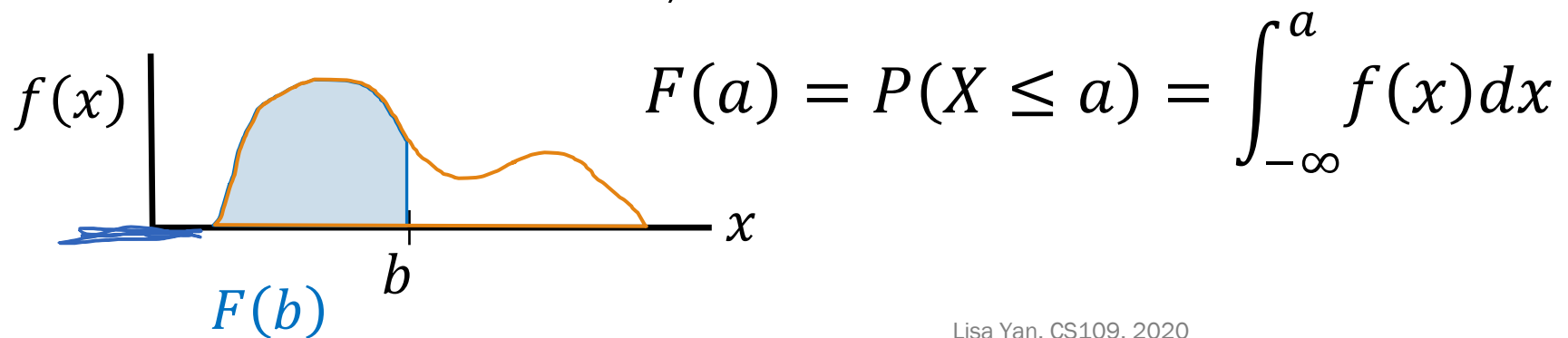
For a random variable  $X$ , the cumulative distribution function (CDF) is defined as

$$F(a) = F_X(a) = P(X \leq a), \text{ where } -\infty < a < \infty$$

For a discrete RV  $X$ , the CDF is:

$$F(a) = P(X \leq a) = \sum_{\text{all } x \leq a} p(x)$$

For a continuous RV  $X$ , the CDF is:



CDF is a probability, though PDF is not.

If you learn to use CDFs, you can avoid integrating the PDF.



# Think

Slide 46 has a matching question to go over by yourself. We'll go over it together afterwards.

Post any clarifications here!

<https://us.edstem.org/courses/109/discussion/27279>

Think by yourself: 1 min



(by yourself)

# Using the CDF for continuous RVs

For a continuous random variable  $X$  with PDF  $f(x)$ , the CDF of  $X$  is

$$P(X \leq a) = \boxed{F(a)} = \int_{-\infty}^a f(x) dx$$

Matching (choices are used 0/1/2 times)

1.  $P(X < a)$

2.  $P(X > a)$

3.  $P(X \geq a)$

4.  $P(a \leq X \leq b)$

A.  $F(a)$

B.  $1 - F(a)$

C.  $F(a) - F(b)$

D.  $F(b) - F(a)$



# Using the CDF for continuous RVs

For a continuous random variable  $X$  with PDF  $f(x)$ , the CDF of  $X$  is

$$P(X \leq a) = F(a) = \int_{-\infty}^a f(x) dx$$

Matching (choices are used 0/1/2 times)

- |    |                      |       |    |                                |
|----|----------------------|-------|----|--------------------------------|
| 1. | $P(X < a)$           | ————— | A. | $F(a) = \mathcal{P}(X \leq a)$ |
| 2. | $P(X > a)$           | ————— | B. | $1 - F(a)$                     |
| 3. | $P(X \geq a)$        | ————— | C. | $F(a) - F(b)$                  |
| 4. | $P(a \leq X \leq b)$ | ————— | D. | $F(b) - F(a)$ (next slide)     |

# Using the CDF

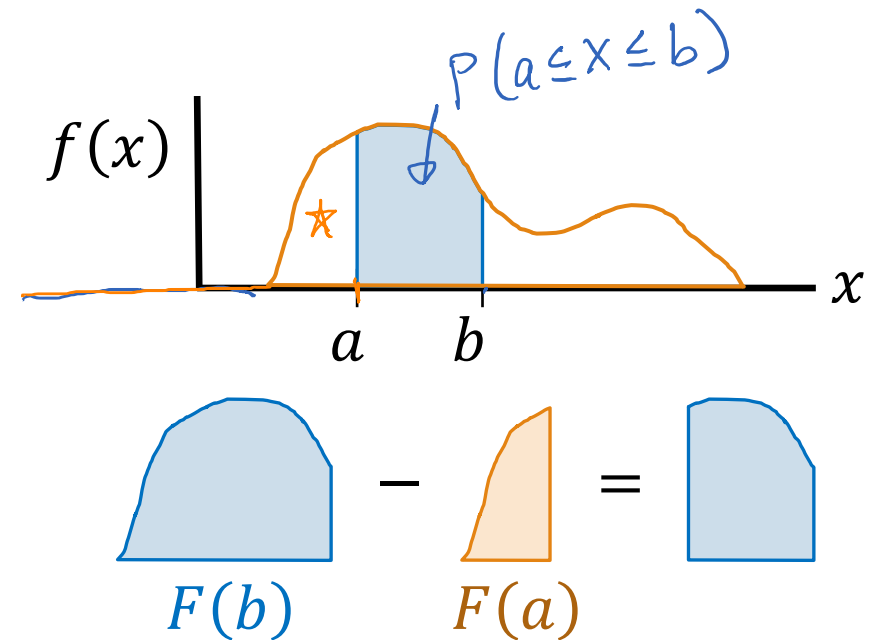
For a continuous random variable  $X$  with PDF  $f(x)$ , the CDF of  $X$  is

$$F(a) = \int_{-\infty}^a f(x) dx$$

4.  $P(a \leq X \leq b) = F(b) - F(a)$

Proof:

$$\begin{aligned} F(b) - F(a) &= \int_{-\infty}^b f(x) dx - \int_{-\infty}^a f(x) dx \\ &= \left( \int_{-\infty}^a f(x) dx + \int_a^b f(x) dx \right) - \int_{-\infty}^a f(x) dx \\ &= \int_a^b f(x) dx = P(a \leq X \leq b) \end{aligned}$$



# CDF of an Exponential RV

$$X \sim \text{Exp}(\lambda) \quad f(x) = \lambda e^{-\lambda x} \quad \text{if } x \geq 0$$

$$X \sim \text{Exp}(\lambda) \quad F(x) = 1 - e^{-\lambda x} \quad \text{if } x \geq 0$$

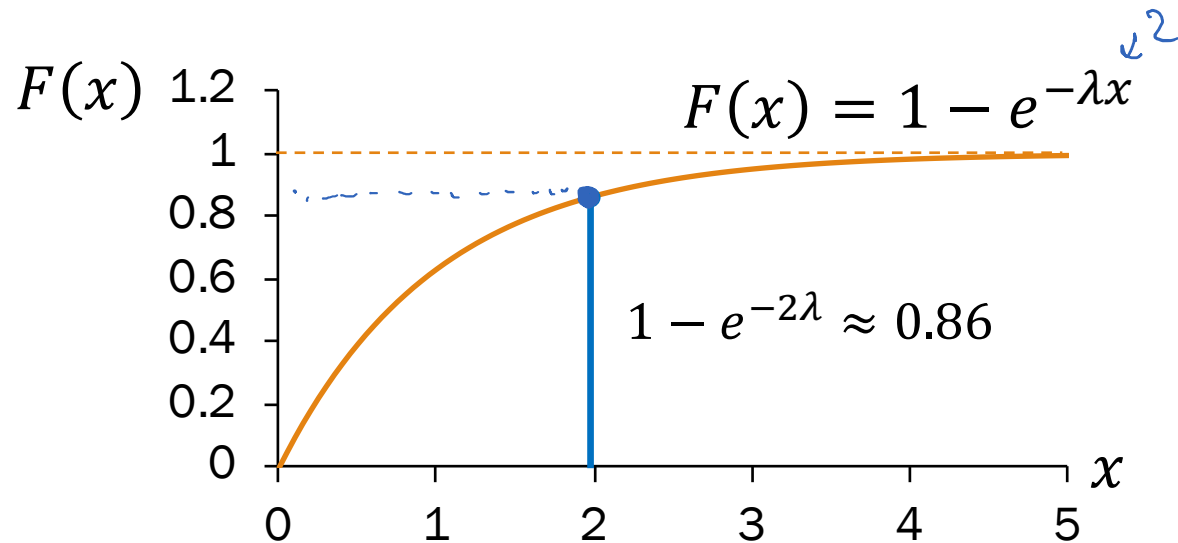
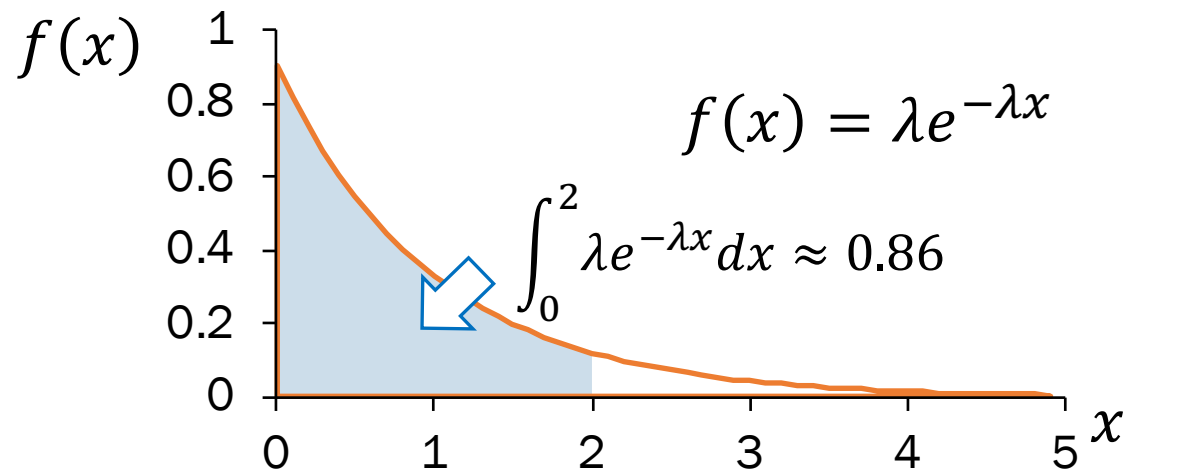
Proof:

$$\begin{aligned} F(x) &= P(X \leq x) = \int_{y=-\infty}^x f(y) dy = \int_{y=0}^x \lambda e^{-\lambda y} dy \\ &= \lambda \frac{1}{-\lambda} e^{-\lambda y} \Big|_0^x \\ &= -1(e^{-\lambda x} - e^{-\lambda 0}) \\ &= 1 - e^{-\lambda x} \end{aligned}$$

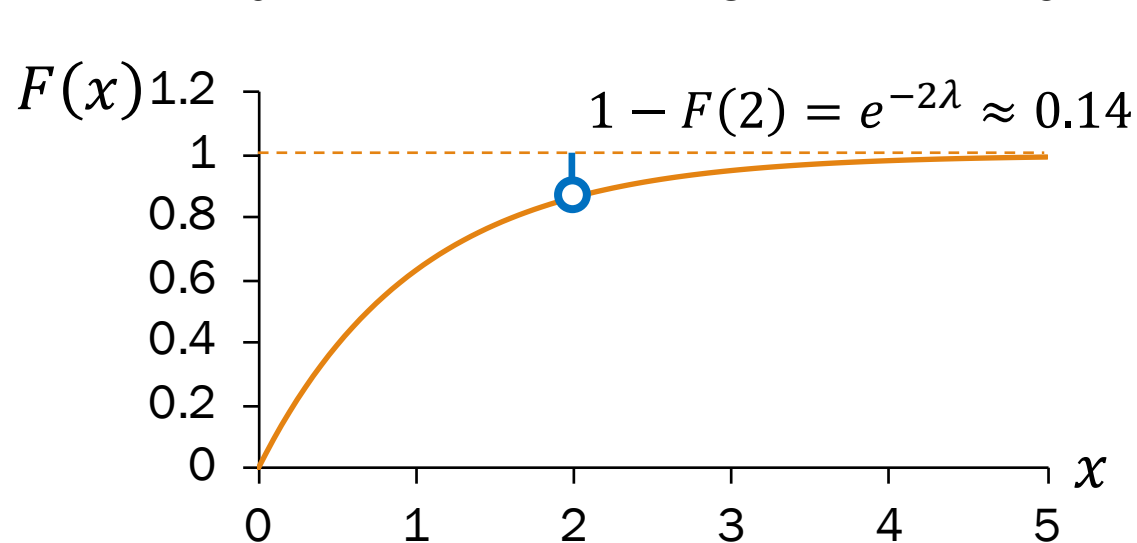
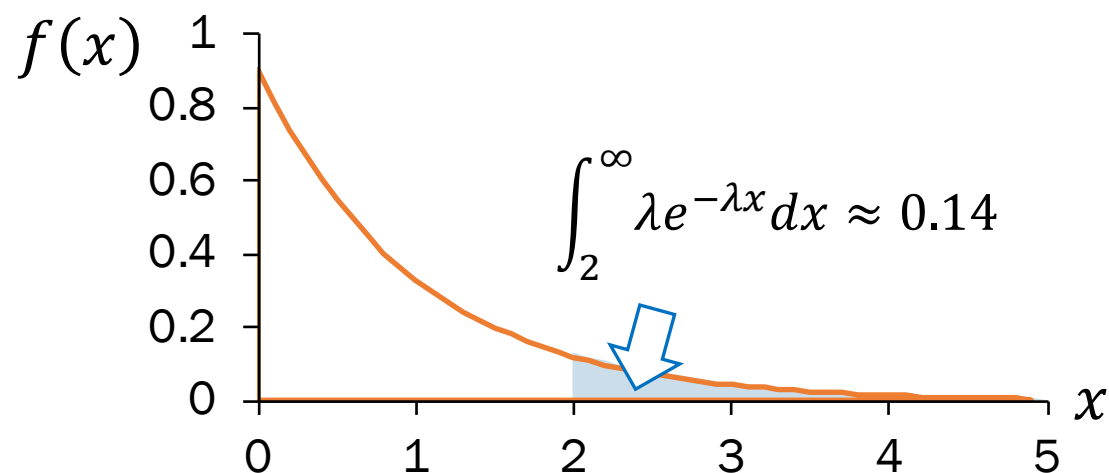
Recall

$$\int e^{cx} dx = \frac{1}{c} e^{cx}$$

# PDF/CDF $X \sim \text{Exp}(\lambda = 1)$



$$P(X \leq 2)$$



$$P(X > 2)$$

# Breakout Rooms

Check out the question on the next slide (Slide 52). Post any clarifications here!

<https://us.edstem.org/courses/109/discussion/39083>

Breakout rooms: 4 min. Introduce yourself!



# Earthquakes

Major earthquakes (magnitude 8.0+) occur <sup>on average</sup> once every 500 years.\*

What is the probability of zero major earthquakes next year?

average:  $\frac{\# \text{ successes}}{\text{year}}$

$$\lambda = \frac{1}{500} \text{ per year}$$

$X \sim \text{Poisson RV}$ : wait 1 yr, observe # events:  $P(X=0)$

$Y \sim \text{Exponential RV}$ : wait until 1<sup>st</sup> event,  
observe duration of time

$$P(Y > 1)$$





# Earthquakes

Major earthquakes (magnitude 8.0+) occur once every 500 years.\*

What is the probability of **zero major earthquakes next year**?

Strategy 1: Exponential RV

Define events/RVs & state goal

$X$ : when first earthquake happens

$X \sim \text{Exp}(\lambda = 0.002)$

Want:  $P(X > 1) = 1 - F(1)$

Solve

$$P(X > 1) = 1 - (1 - e^{-\lambda \cdot 1}) = e^{-\lambda}$$



\*In California, according to historical data form USGS, 2015

# Earthquakes

$$X \sim \text{Poi}(\lambda) \quad p(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Major earthquakes (magnitude 8.0+) occur once every 500 years.\*

What is the probability of zero major earthquakes next year?

Strategy 1: Exponential RV

Define events/RVs & state goal

$X$ : when first earthquake happens

$X \sim \text{Exp}(\lambda = 0.002)$

Want:  $P(X > 1) = 1 - F(1)$

Solve

$$P(X > 1) = 1 - (1 - e^{-\lambda \cdot 1}) = e^{-\lambda}$$

Strategy 2: Poisson RV

Define events/RVs & state goal

$X$ : # earthquakes next year

$X \sim \text{Poi}(\lambda = 0.002)$

Want:  $P(X = 0)$

$\lambda$ :  $\frac{\text{earthquakes}}{\text{year}}$

Solve

$$P(X = 0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda} \approx 0.998$$

# Extra

$$X \sim \text{Exp}(\lambda)$$

- ①  $\mathbb{E}[X] = \frac{1}{\lambda}$
- ② after lecture  
CDF  $\xi \text{Exp}$

# Expectation of the Exponential

$$X \sim \text{Exp}(\lambda) \quad f(x) = \lambda e^{-\lambda x} \quad \text{if } x \geq 0$$

$$X \sim \text{Exp}(\lambda)$$

Expectation

$$E[X] = \frac{1}{\lambda}$$

Proof:

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} x \lambda e^{-\lambda x} dx$$

$$= -x e^{-\lambda x} \Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx$$

$$= -x e^{-\lambda x} \Big|_0^{\infty} - \frac{1}{\lambda} e^{-\lambda x} \Big|_0^{\infty}$$

$$= [0 - 0] + \left[ 0 - \left( \frac{-1}{\lambda} \right) \right]$$

$$= \frac{1}{\lambda}$$

$\lim_{x \rightarrow \infty} \frac{-x}{e^{\lambda x}} = \lim_{x \rightarrow \infty} \frac{-1}{\lambda e^{\lambda x}} = 0$

Integration by parts

$$\int x \lambda e^{-\lambda x} dx = \int u \cdot dv$$

$$u = x \quad dv = \lambda e^{-\lambda x} dx$$
$$du = dx \quad v = -e^{-\lambda x}$$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$-x e^{-\lambda x} - \int -e^{-\lambda x} dx$$

# Website visits

(please watch after lecture if you want)

$$X \sim \text{Exp}(\lambda) \quad \begin{aligned} E[X] &= 1/\lambda \\ F(x) &= 1 - e^{-\lambda x} \end{aligned}$$

Suppose a visitor to your website leaves after  $X$  minutes.

- On average, visitors leave the site after 5 minutes.
- The length of stay,  $X$ , is exponentially distributed.

1.  $P(X > 10)$ ?

2.  $P(10 < X < 20)$ ?



# Website visits

$$X \sim \text{Exp}(\lambda) \quad \begin{aligned} E[X] &= 1/\lambda \\ F(x) &= 1 - e^{-\lambda x} \end{aligned}$$

Suppose a visitor to your website leaves after  $X$  minutes.

- On average, visitors leave the site after 5 minutes.
- The length of stay,  $X$ , is exponentially distributed.

$$E[X] = 5 = \frac{1}{\lambda} \rightarrow \lambda = 1/5$$

$X$ : minutes

1.  $P(X > 10)$ ?

Define

$X$ : when visitor leaves  
 $X \sim \text{Exp}(\lambda = 1/5 = 0.2)$

Solve

alternate strat:  $\int_{10}^{\infty} \lambda e^{-\lambda x} dx$

$$\begin{aligned} P(X > 10) &= 1 - F(10) \\ &= 1 - (1 - e^{-10/5}) = e^{-2} \approx \mathbf{0.1353} \end{aligned}$$

2.  $P(10 < X < 20)$ ?

Solve

$\int_{10}^{20} \lambda e^{-\lambda x} dx$

$$\begin{aligned} P(10 < X < 20) &= F(20) - F(10) \\ &= (1 - e^{-4}) - (1 - e^{-2}) \approx \mathbf{0.1170} \end{aligned}$$

# Replacing your laptop

$$X \sim \text{Exp}(\lambda) \quad \begin{array}{l} E[X] = 1/\lambda \\ F(x) = 1 - e^{-\lambda x} \end{array}$$

Let  $X = \#$  hours of use until your laptop dies.

- $X$  is distributed as an Exponential RV, where
- On average, laptops die after 5000 hours of use.
- You use your laptop 5 hours a day.

What is  $P(\text{your laptop lasts 4 years})$ ?



# Replacing your laptop

$$X \sim \text{Exp}(\lambda) \quad \begin{aligned} E[X] &= 1/\lambda \\ F(x) &= 1 - e^{-\lambda x} \end{aligned}$$

Let  $X = \#$  hours of use until your laptop dies.

- $X$  is distributed as an Exponential RV, where
- On average, laptops die after 5000 hours of use.
- You use your laptop 5 hours a day.

$$E[X] = 5000 \text{ hours} = \frac{1}{\lambda}$$

What is  $P(\text{your laptop lasts 4 years})$ ?

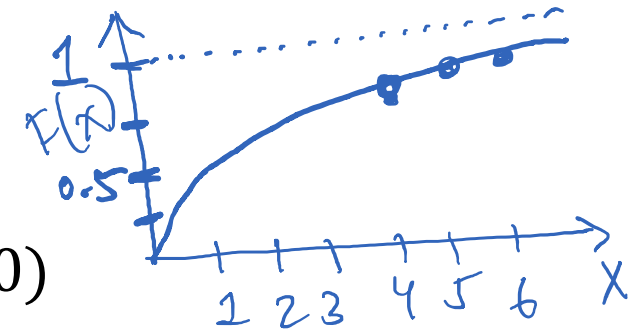
Define

$X$ : # hours until laptop death  
 $X \sim \text{Exp}(\lambda = 1/5000)$

Want:  $P(X > 5 \cdot 365 \cdot 4)$   
 $\begin{matrix} \uparrow & \uparrow & \uparrow \\ \text{hrs} & \text{days} & \text{yr} \\ \text{day} & \text{yr} & \end{matrix}$

Solve

$$\begin{aligned} P(X > 7300) &= 1 - F(7300) \\ &= 1 - (1 - e^{-7300/5000}) = e^{-1.46} \approx 0.2322 \end{aligned}$$



Better plan ahead if you're co-termining!

- 5-year plan:

$$P(X > 9125) = e^{-1.825} \approx 0.1612$$

- 6-year plan:

$$P(X > 10950) = e^{-2.19} \approx 0.1119$$