19: Sampling and the Bootstrap

Lisa Yan

May 18, 2020

Quick slide reference

Sampling definitions

Motivating example

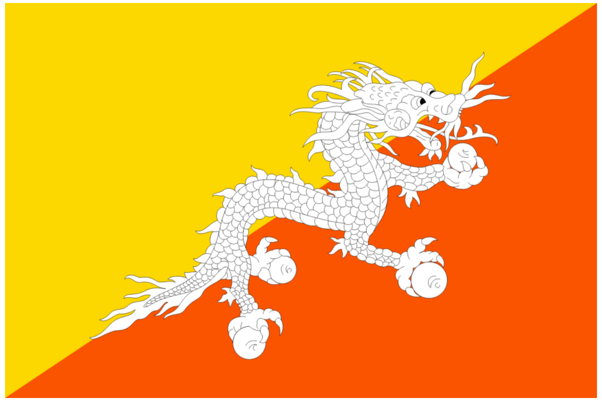
You want to know the true mean and variance of happiness in Bhutan.

- But you can’t ask everyone.
- You poll 200 random people.
- Your data looks like this:

  Happiness = {72, 85, 79, 91, 68, …, 71}

- The mean of all these numbers is 83.

Is this the **true mean happiness** of Bhutanese people?



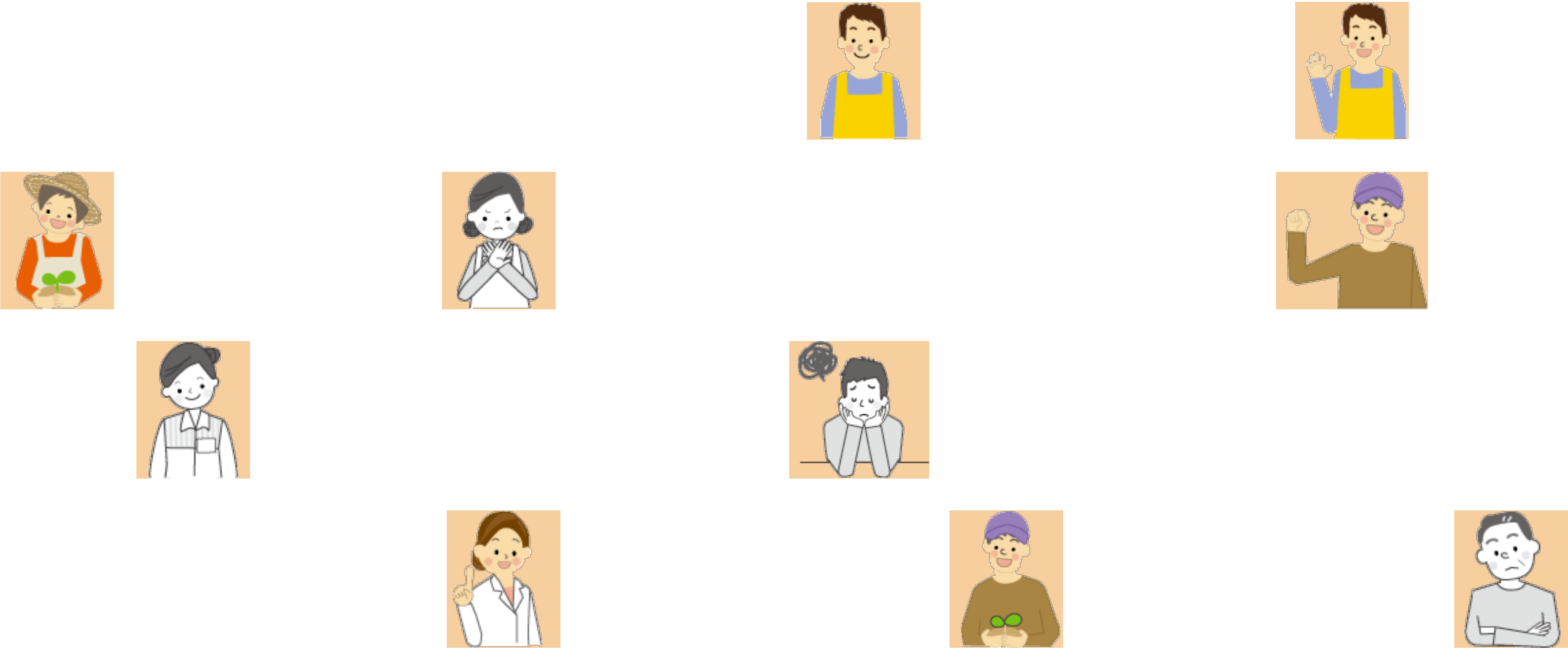
Population

This is a population.

Sample

A sample is selected from a population.

Sample



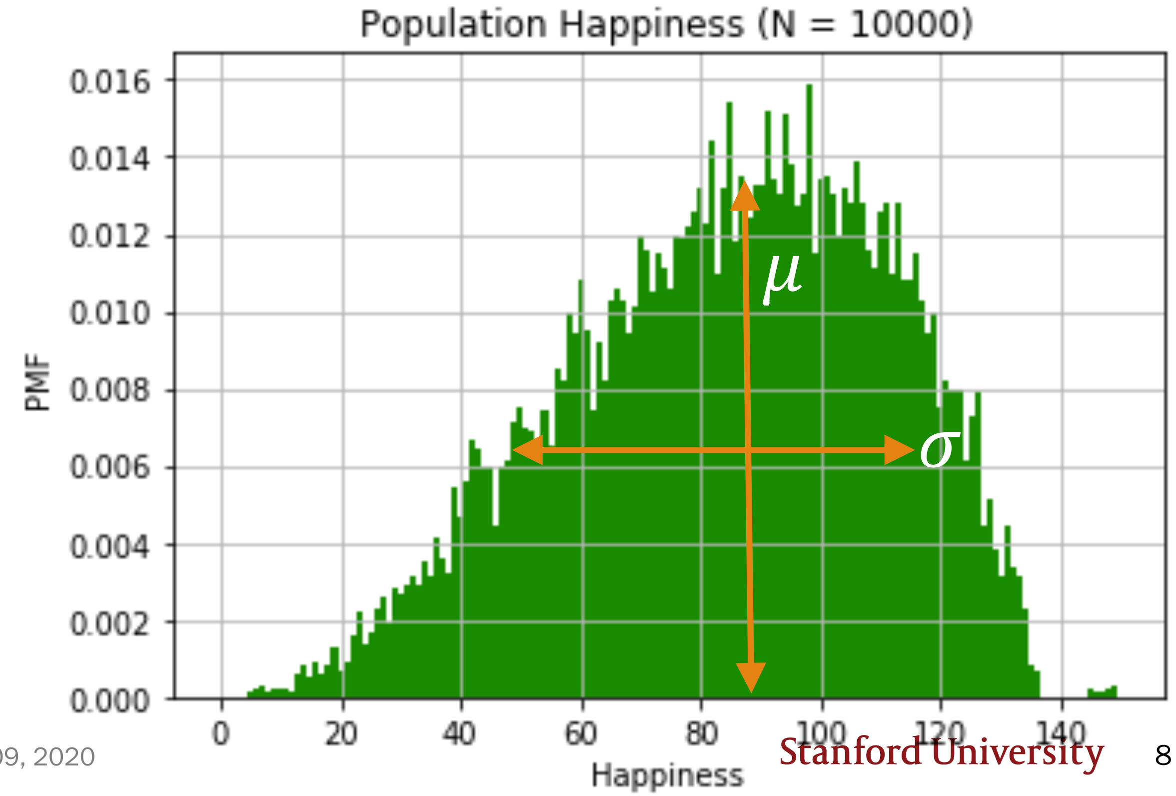
A **sample** is selected from a population.

A sample, mathematically

Consider n random variables $X_1, X_2, \dots, X_n$.

The sequence $X_1, X_2, \dots, X_n$ is a **sample** from distribution F if:

- X_i are all independent and identically distributed (i.i.d.)
- X_i all have same distribution function F (the **underlying distribution**), where $E[X_i] = \mu$, $\text{Var}(X_i) = \sigma^2$



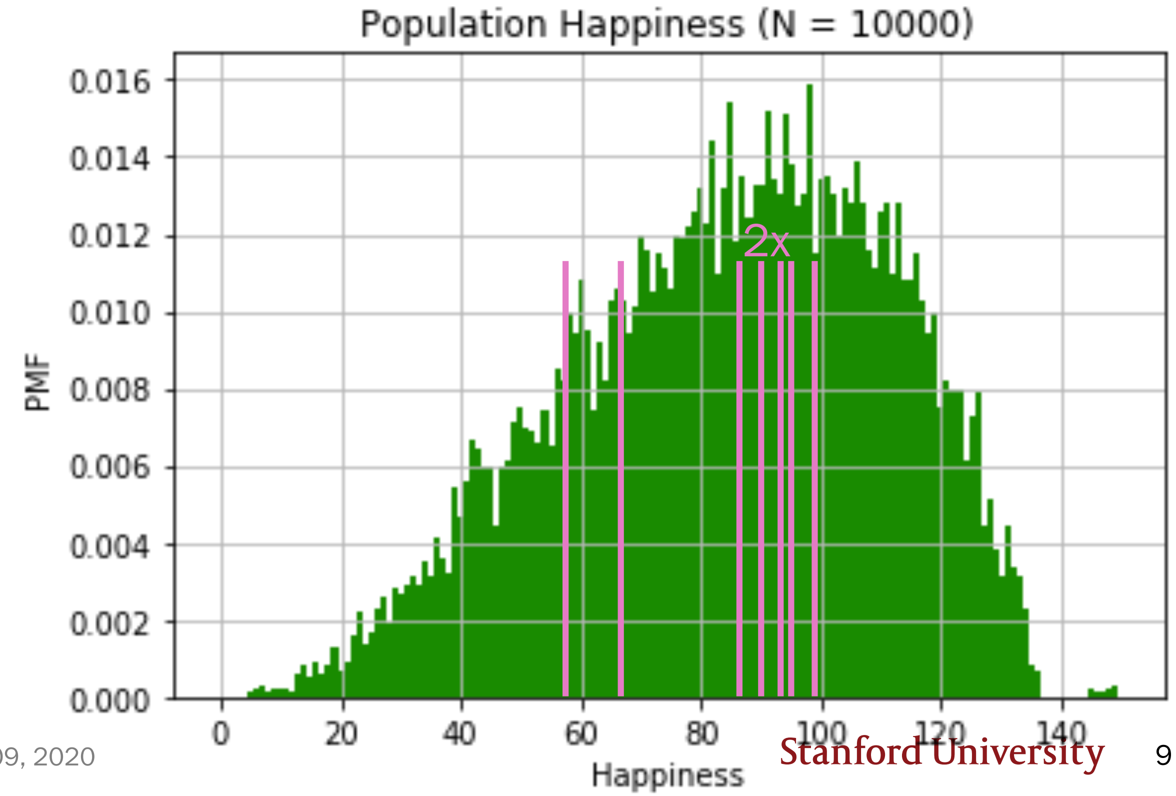
A sample, mathematically

A sample of sample size 8:

$$(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$$

A realization of a sample of size 8:

$$(59, 87, 94, 99, 87, 78, 69, 91)$$



A single sample

A happy
Bhutanese person

If we had a distribution F of our entire population, we could compute exact statistics about about happiness.

But we only have 200 people (a sample).

Today: If we only have a single sample,

- How do we report *estimated* statistics?
- How do we report estimated error of these estimates?
- How do we perform hypothesis testing?

Unbiased estimators

A single sample

A happy
Bhutanese person

If we had a distribution F of our entire population, we could compute exact statistics about about happiness.

But we only have 200 people (a sample).

So these population statistics are unknown:

- μ, the population mean
- σ^2, the population variance

A single sample

A happy
Bhutanese person

If we had a distribution F of our entire population, we could compute exact statistics about about happiness.

But we only have 200 people (a sample).

- From these 200 people, what is our best estimate of population mean and population variance?
- How do we define best estimate?

Estimating the population mean

1. What is our best estimate of μ, the **mean happiness** of Bhutanese people?

If we only have a sample, $(X_1, X_2, \dots, X_n)$:

The best estimate of μ is the **sample mean**:

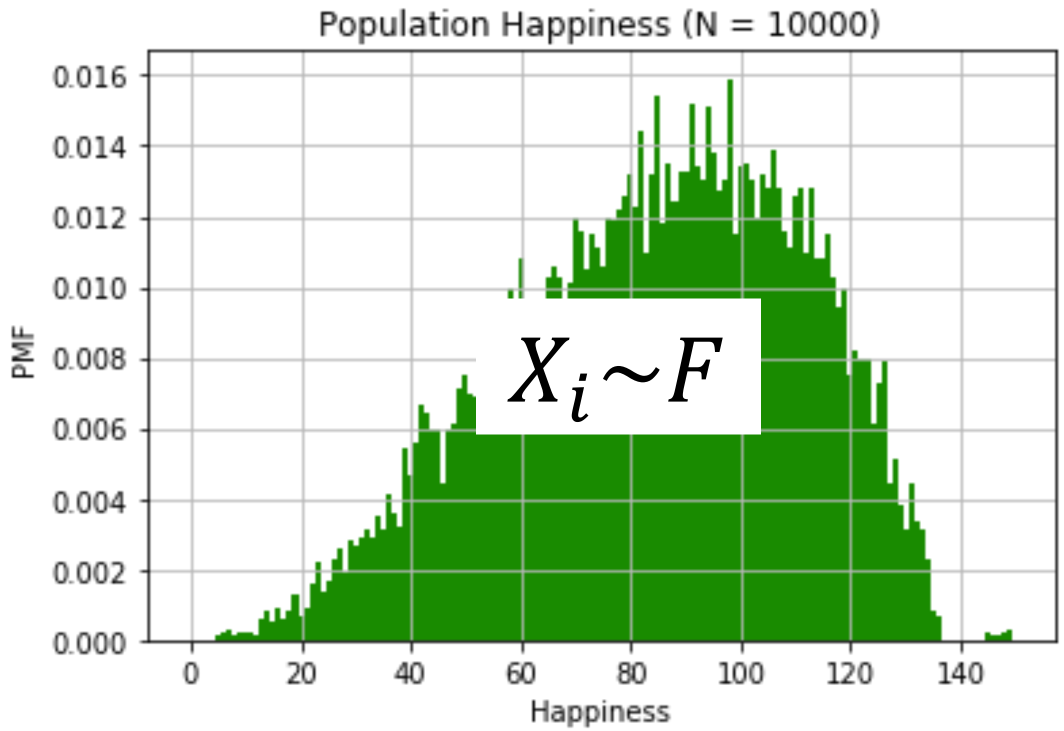
$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

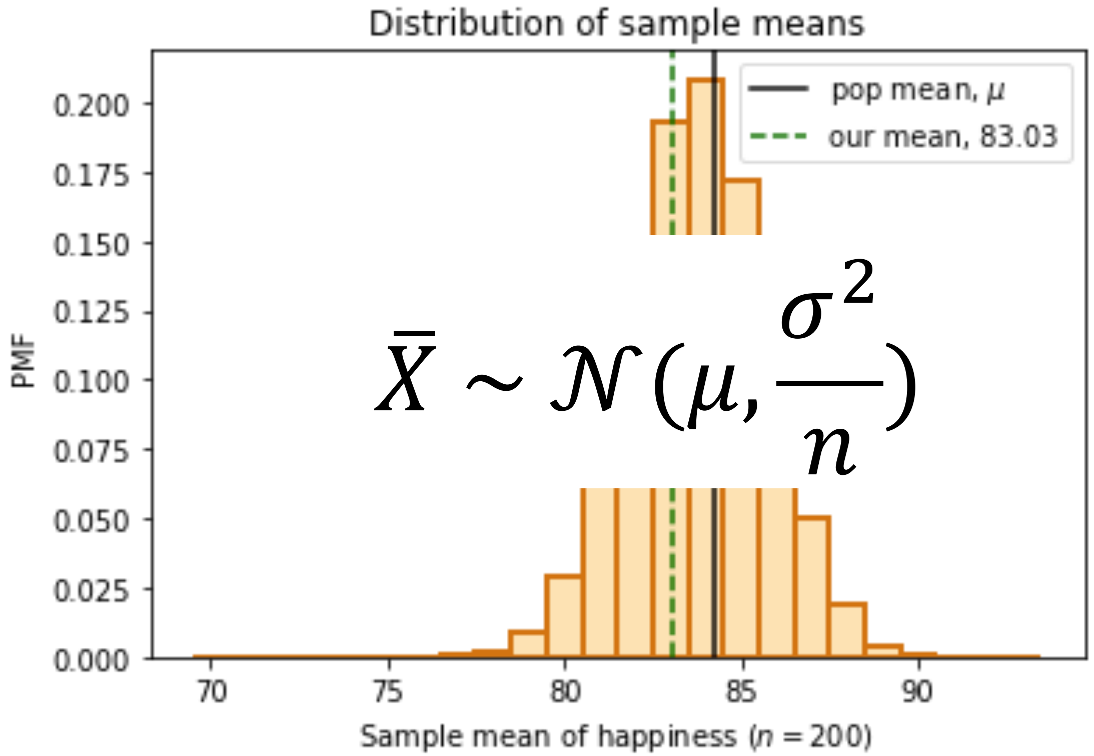
$\bar{X}$ is an <u>**unbiased estimator**</u> of the population mean μ. $\quad E[\bar{X}] = \mu$

Intuition: By the CLT, $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$

If we could take *multiple* samples of size n:
1. For each sample, compute sample mean
2. On average, we would get the population mean

Sample mean





Even if we can't report μ, we can report our sample mean 83.03, which is an unbiased estimate of μ.

Estimating the population variance

2. What is σ^2, the variance of happiness of Bhutanese people?

If we knew the entire population $(x_1, x_2, \dots, x_N)$:

population mean

population variance

$$\sigma^2 = E[(X-\mu)^2] = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$$

If we only have a sample, $(X_1, X_2, \dots, X_n)$:

sample mean

sample variance

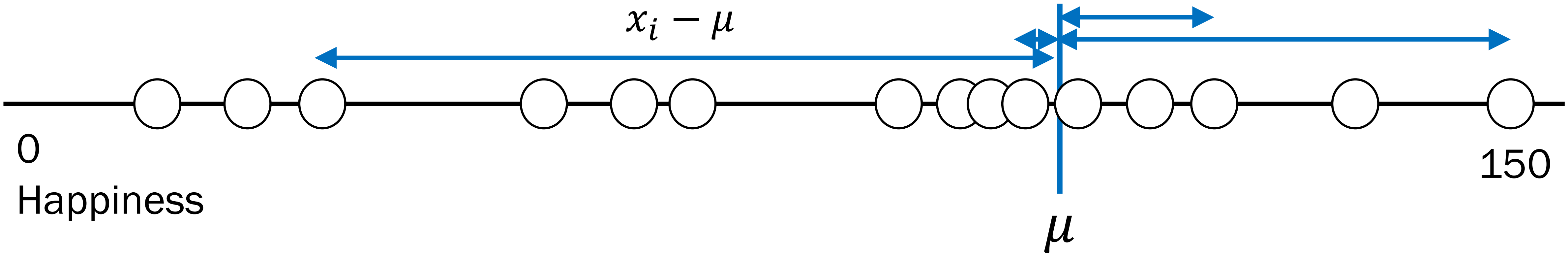
$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Intuition about the sample variance, S^2

<u>Actual, σ^2</u>

population mean

population variance $$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$$



Population size, N

Calculating population statistics <u>exactly</u> requires us knowing all N datapoints.

Intuition about the sample variance, S^2

Actual, σ^2

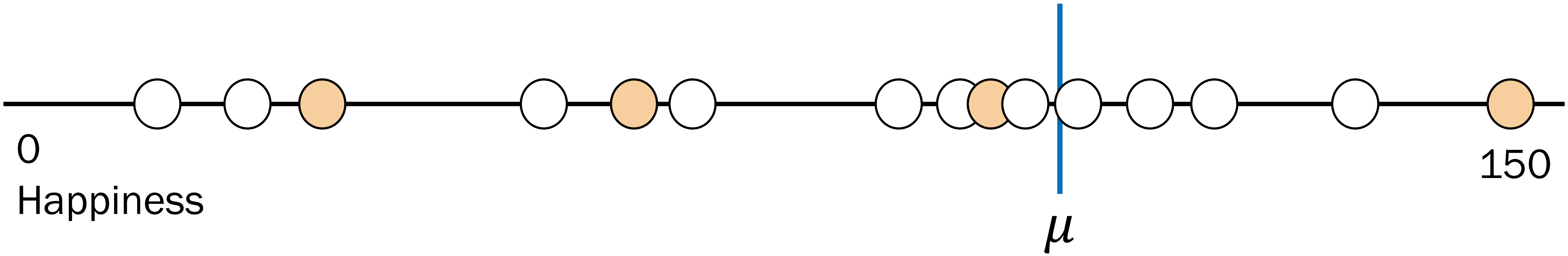
population mean

population variance $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$

Estimate, S^2

sample mean

sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$



Population size, N

Intuition about the sample variance, S^2

Actual, σ^2

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population variance $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$

Estimate, S^2

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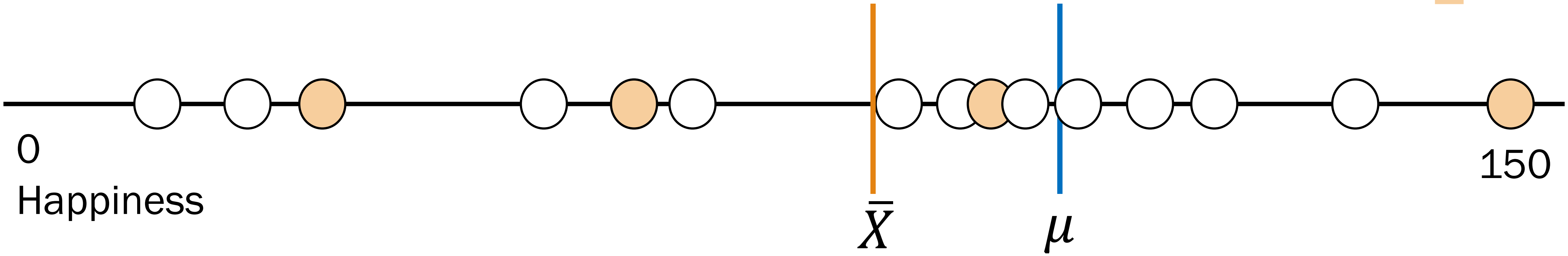


Population size, N

Intuition about the sample variance, S^2

Actual, σ^2

population variance

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$$

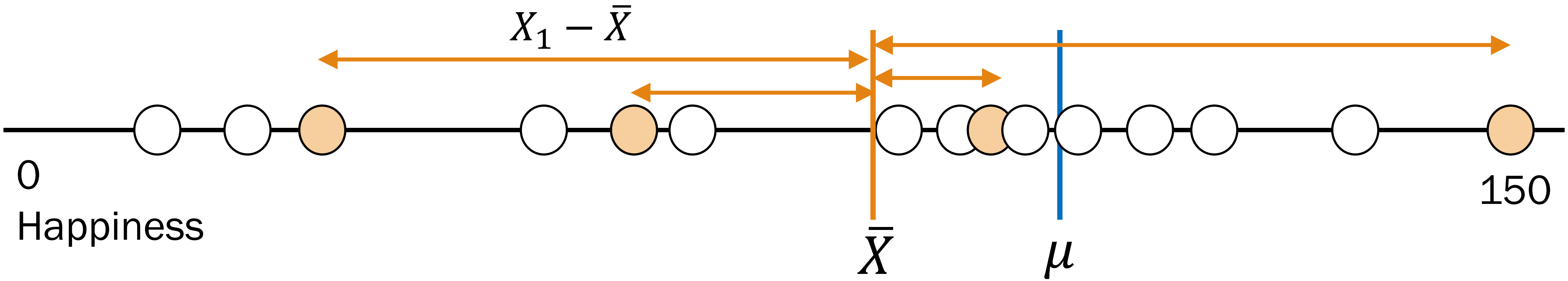
population mean

Estimate, S^2

sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

sample mean



Population size, N

Sample variance is an estimate using an estimate, so it needs additional scaling.

Estimating the population variance

2. What is σ^2, the variance of happiness of Bhutanese people?

If we only have a sample, $(X_1, X_2, \ldots, X_n)$:

The best estimate of σ^2 is the **sample variance:**

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

S^2 is an **unbiased estimator** of the population variance, σ^2.

$$E[S^2] = \sigma^2$$

Proof that S^2 is unbiased (just for reference)

$$E[S^2] = \sigma^2$$

$$E[S^2] = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] \quad \Rightarrow \quad (n-1)E[S^2] = E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right]$$

$$(n-1)E[S^2] = E\left[\sum_{i=1}^{n}\big((X_i - \mu) + (\mu - \bar{X})\big)^2\right]$$

(introduce $\mu - \mu$)

$$= E\left[\sum_{i=1}^{n}(X_i - \mu)^2 + \sum_{i=1}^{n}(\mu - \bar{X})^2 + 2\sum_{i=1}^{n}(X_i - \mu)(\mu - \bar{X})\right]$$

$$2(\mu - \bar{X})\sum_{i=1}^{n}(X_i - \mu)$$

$$2(\mu - \bar{X})\left(\sum_{i=1}^{n}X_i - n\mu\right)$$

$$2(\mu - \bar{X})n(\bar{X} - \mu)$$

$$-2n(\mu - \bar{X})^2$$

$$= E\left[\sum_{i=1}^{n}(X_i - \mu)^2 + n(\mu - \bar{X})^2 - 2n(\mu - \bar{X})^2\right]$$

$$= E\left[\sum_{i=1}^{n}(X_i - \mu)^2 - n(\mu - \bar{X})^2\right] = \sum_{i=1}^{n}E[(X_i - \mu)]^2 - nE[(\bar{X} - \mu)^2]$$

$$= n\sigma^2 - n\text{Var}(\bar{X}) = n\sigma^2 - n\frac{\sigma^2}{n} = n\sigma^2 - n\sigma^2 = (n-1)\sigma^2$$

Therefore $E[S^2] = \sigma^2$

Standard error

Estimating population statistics

A particular outcome

1. Collect a sample, $X_1, X_2, \dots, X_n$.

   $(72, 85, 79, 79, 91, 68, \dots, 71)$
   $n = 200$

2. Compute **sample mean**, $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

   $\bar{X} = 83$

3. Compute sample deviation, $X_i - \bar{X}$.

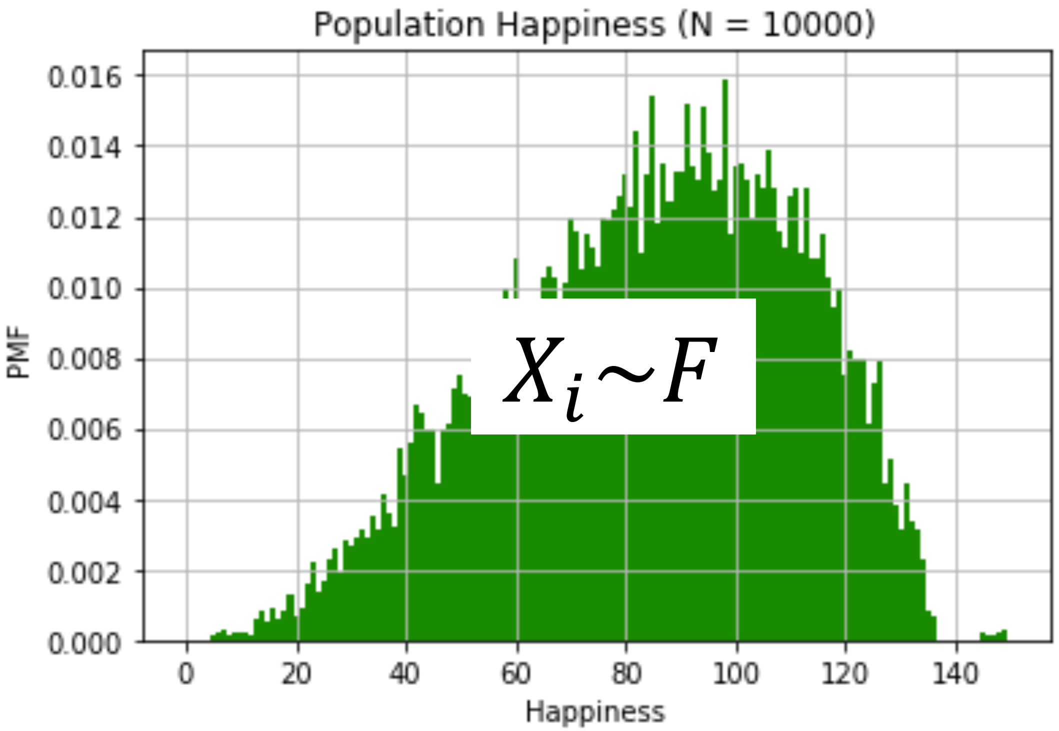
   $(-11, 2, -4, -4, 8, -15, \dots, -12)$

4. Compute **sample variance**, $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

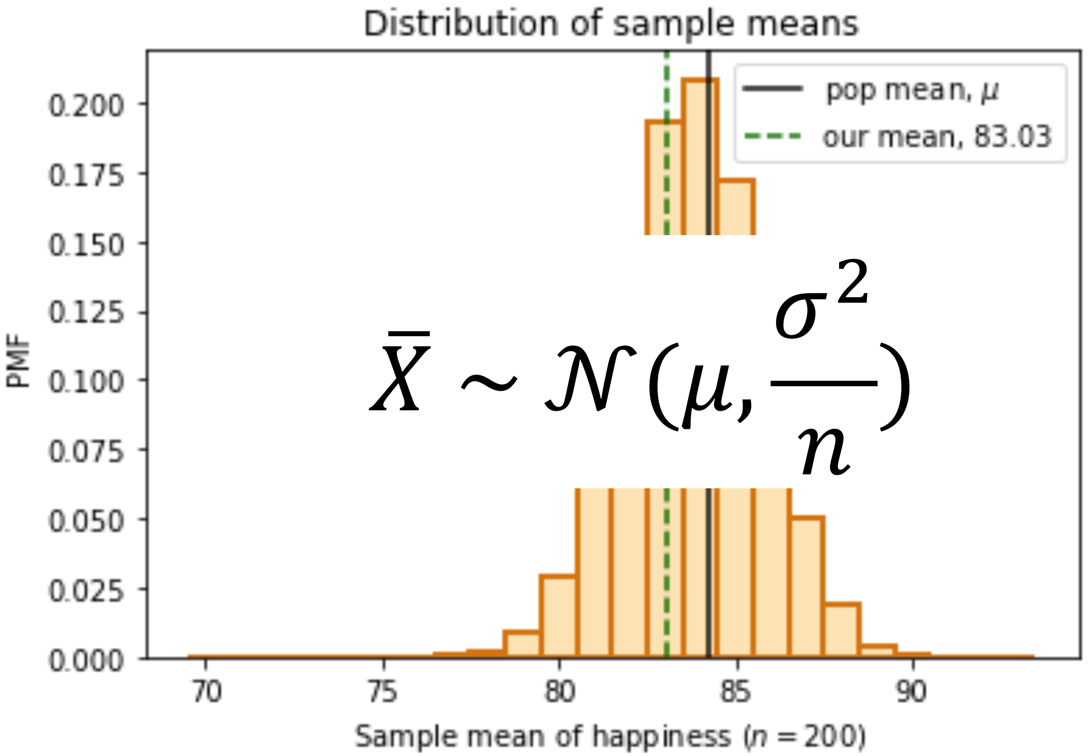
   $S^2 = 793$

How "close" are our estimates $\bar{X}$ and S^2?

Sample mean



$X_i \sim F$



$\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$

- $\text{Var}(\bar{X})$ is a measure of how "close" $\bar{X}$ is to μ.
- **How do we estimate $\text{Var}(\bar{X})$?**

How “close” is our estimate $\bar{X}$ to μ?

$$E[\bar{X}] = \mu$$

$$\boxed{\text{Var}(\bar{X}) = \frac{\sigma^2}{n}}$$

We want to estimate this

<u>def</u> The **standard error** of the mean is an estimate of the standard deviation of $\bar{X}$.

$$SE = \sqrt{\frac{S^2}{n}}$$

Intuition:

- S^2 is an unbiased estimate of σ^2
- S^2/n is an unbiased estimate of $\sigma^2/n = \text{Var}(\bar{X})$
- $\sqrt{S^2/n}$ can estimate $\sqrt{\text{Var}(\bar{X})}$

Errata (5/18)

More info on bias of standard error: wikipedia

Standard error of the mean

1. Mean happiness:

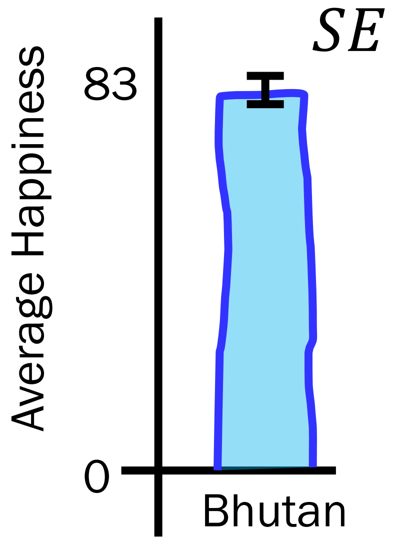
Claim: The average happiness of Bhutan is 83, with a standard error of 1.99.

this is our best estimate of μ

this is our estimate of how "close" we are

Closed form: $SE = \sqrt{\dfrac{S^2}{n}}$

error bars



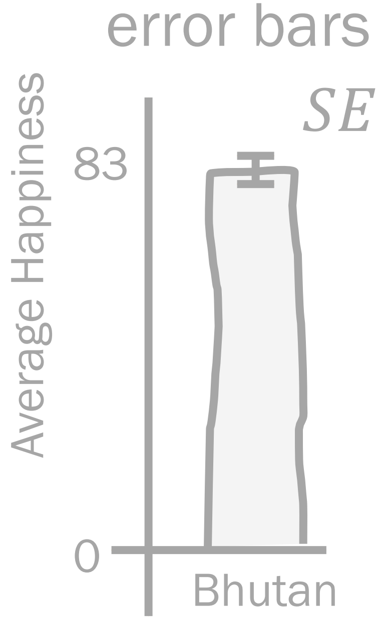
These 2 statistics give a sense of how the sample mean random variable $\bar{X}$ behaves.

Standard error of variance?

1. Mean happiness:

   Claim: The average happiness of Bhutan is 83, with a standard error of 1.99.

   Closed form: $SE = \sqrt{\frac{S^2}{n}}$



2. Variance of happiness:

   this is our best estimate of σ^2

   Claim: The variance of happiness of Bhutan is 793.

   Closed form: Not covered in CS109

   But how close are we?

Up next: Compute Statistics with code!

Bootstrap: Sample mean

Bootstrap

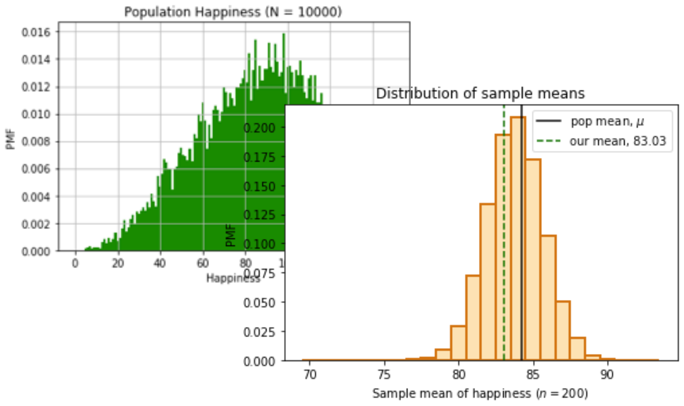
The Bootstrap:

Probability for Computer Scientists

Computing statistic of sample mean

What is the standard deviation of the sample mean $\bar{X}$? (sample size $n = 200$)

Population distribution
(we don't have this)



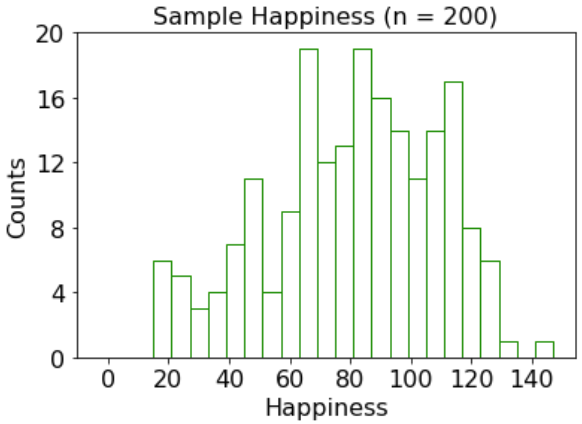
$$\frac{\sigma}{\sqrt{n}} = 1.886$$

Exact statistic
(we don't have this)

1.869

Simulated statistic
(we don't have this)

Sample distribution
(we do have this)



$$SE = \frac{S}{\sqrt{n}} = 1.992$$

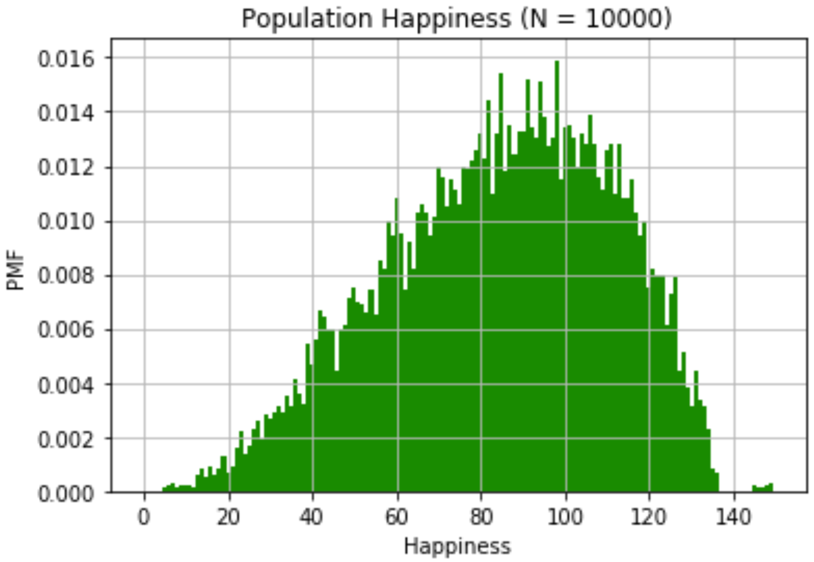
Estimated statistic, by formula, **standard error**

???

Simulated estimated statistic

Note: We don't have access to the population.
But Lisa is sharing the exact statistic with you.

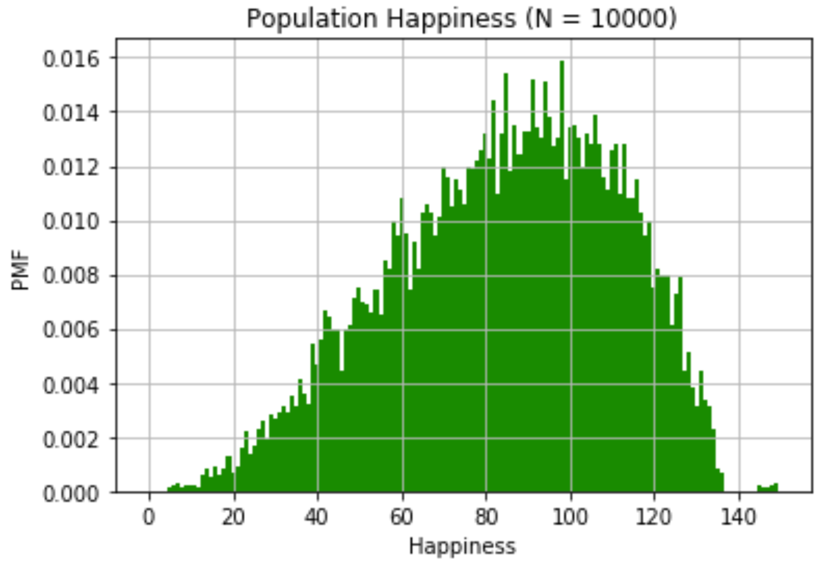
Bootstrap insight 1: Estimate the true distribution



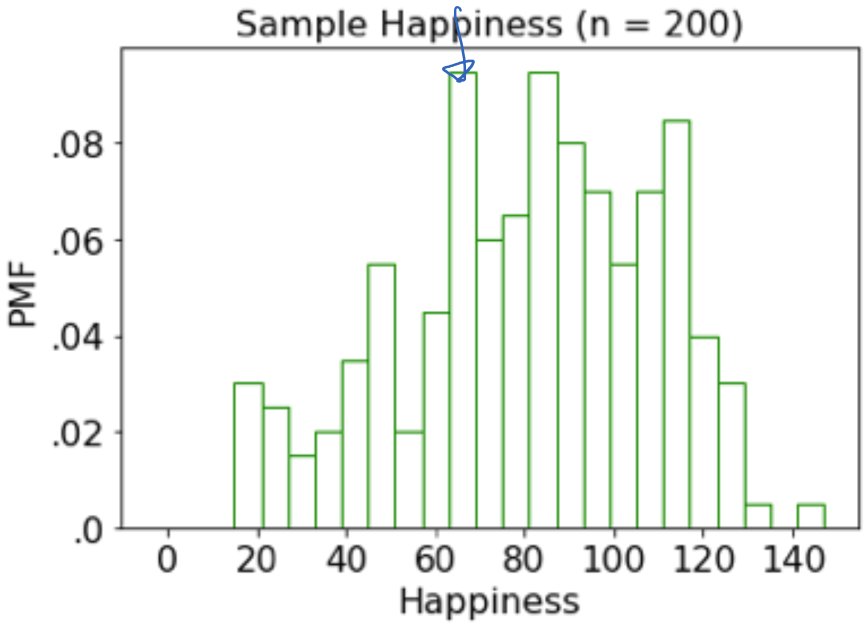
$\approx$

Bootstrap insight 1: Estimate the true distribution

You can estimate the PMF of the underlying distribution, using your sample.*



$\approx$



The underlying distribution

$$F \approx \hat{F}$$

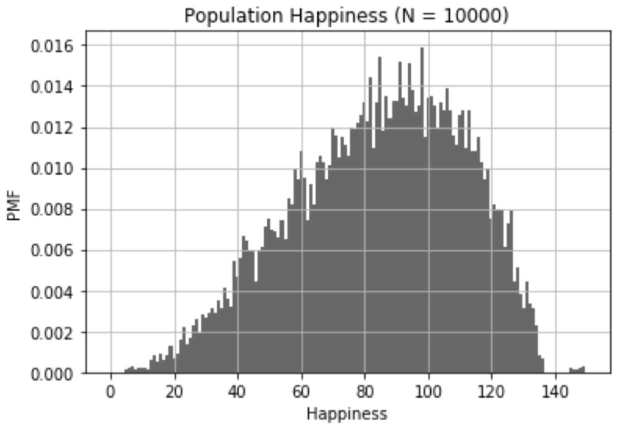
the sample distribution (aka the histogram of your data)

*This is just a histogram of your data!

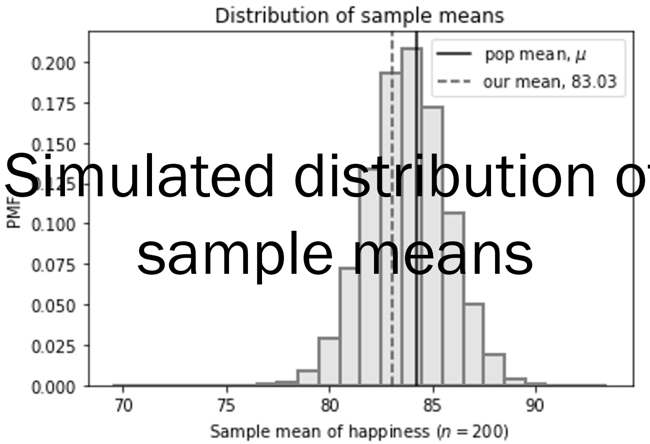
Bootstrap insight 2: Simulate a distribution

Approximate the procedure of simulating a distribution of a statistic, e.g., $\bar{X}$.

Population distribution
(we don't have this)



Distribution of $\bar{X}$

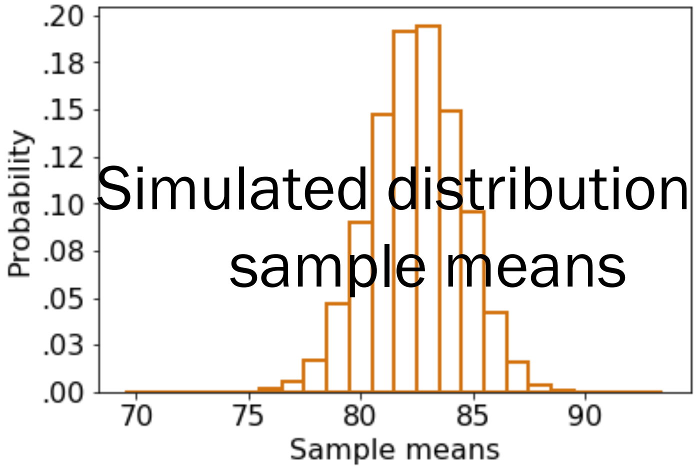


Simulated distribution of sample means

$\approx$

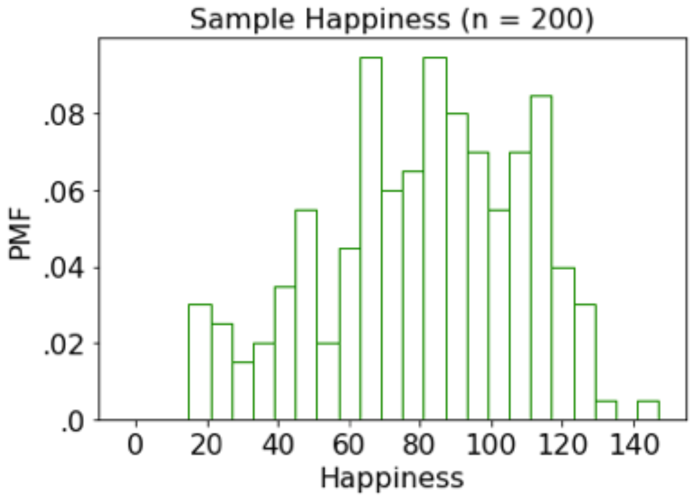
Sample distribution
(we do have this)

Bootstrap means



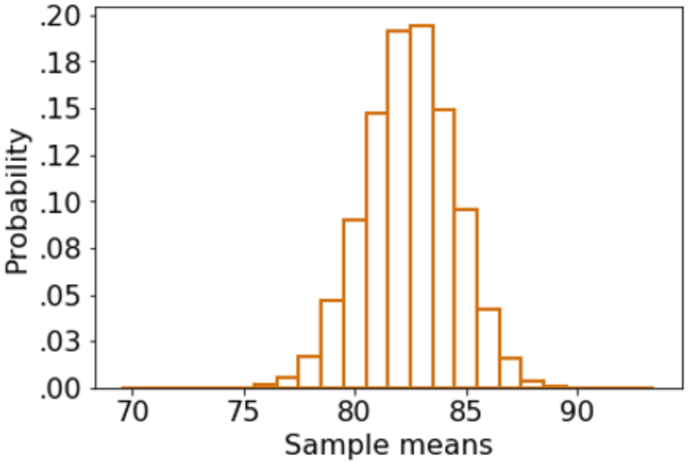
Simulated distribution of sample means

Bootstrapped sample means



Estimate the true PMF using our "PMF" (histogram) of our sample.

```
means = [84.7,
83.9, 80.6, 79.8,
90.3, …, 85.2]
```



...generate a whole bunch of sample means of this estimated distribution...

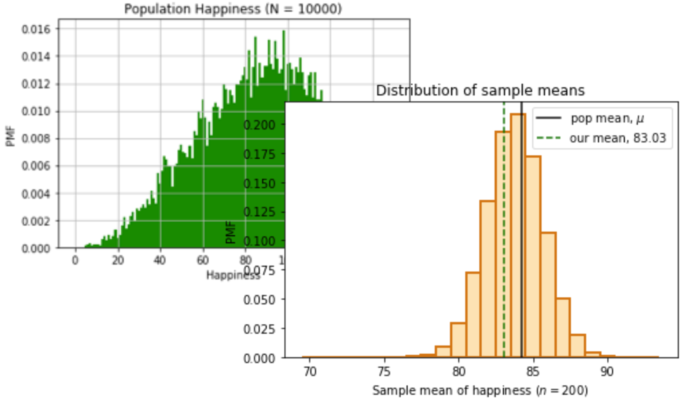
```
np.std(means)
2.003
```

...and compute the standard deviation of this distribution.

Computing statistic of sample mean

What is the standard deviation of the sample mean $\bar{X}$? (sample size $n = 200$)

Population distribution
(we don't have this)



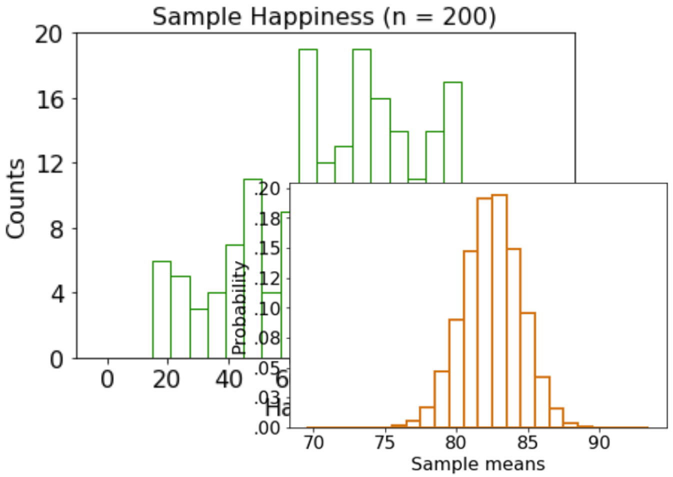
$\frac{\sigma}{\sqrt{n}} = 1.886$

Exact statistic
(we don't have this)

1.869

Simulated statistic
(we don't have this)

Sample distribution
(we do have this)



$SE = \frac{S}{\sqrt{n}} = 1.992$

Estimated statistic, by formula, **standard error**

2.003

Simulated estimated statistic, **bootstrapped standard error**

Bootstrap algorithm

```
Bootstrap Algorithm (sample):
1. Estimate the PMF using the sample
2. Repeat 10,000 times:
      a. Resample sample.size() from PMF
      b. Recalculate the sample mean on the resample
3. You now have a distribution of your sample mean
```

What is the distribution of your sample mean?

We'll talk about this algorithm in detail during live lecture!

Bootstrap algorithm

Bootstrap Algorithm (sample):

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
   a. Resample **sample.size**() from PMF
   b. Recalculate the **statistic** on the resample
3. You now have a **distribution of your statistic**

What is the distribution of your statistic?

Bootstrapped sample variance

Bootstrap Algorithm (sample):

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
   a. Resample **sample.size**() from PMF
   b. Recalculate the sample variance on the resample
3. You now have a **distribution of your sample variance**

What is the distribution of your sample variance?

Even if we don't have a closed form equation,
we estimate statistics of sample variance with bootstrapping!

(live)

19: Sampling and the Bootstrap

Lisa Yan

May 18, 2020

Quick check

1. μ, the population mean

2. $(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$, a sample

3. σ^2, the population variance

4. $\bar{X}$, the sample mean

5. $\bar{X} = 83$

6. $(X_1 = 59, X_2 = 87, X_3 = 94, X_4 = 99,$
   $X_5 = 87, X_6 = 78, X_7 = 69, X_8 = 91)$

A. Random variable(s)
B. Value
C. Event

Quick check

1. μ, the population mean

2. $(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$, a sample

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4. $\bar{X}$, the sample mean

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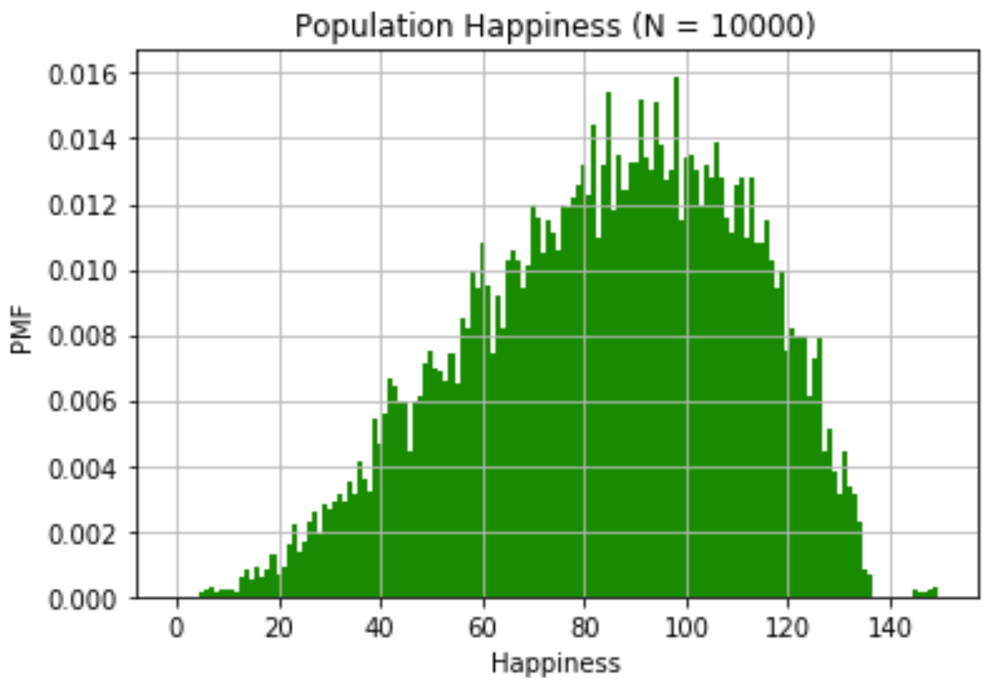
6. $(X_1 = 59, X_2 = 87, X_3 = 94, X_4 = 99,$
   $X_5 = 87, X_6 = 78, X_7 = 69, X_8 = 91)$

A. Random variable(s)
B. Value
C. Event

These are outcomes from your collected data.

Questions we want answers to



Hypothetical questions:

- What is the probability that a Bhutanese peep is just straight up loving life?
- What is the probability that the mean of a subsample of 200 people is within the range 81 to 85?
- What is the variance of the sample variance of subsamples of 200 people?

Bootstrap

The Bootstrap:

Probability for Computer Scientists

Allows you to do the following:

- Calculate distributions over statistics
- Calculate p values

How “close” is our estimate $\bar{X}$ to μ?

Errata

$$E[\bar{X}] = \mu$$

$$\boxed{\text{Var}(\bar{X}) = \frac{\sigma^2}{n}}$$

We want to estimate this

def The **standard error** of the mean is an estimate of the standard deviation of $\bar{X}$.

$$SE = \sqrt{\frac{S^2}{n}}$$

Intuition:

- S^2 is an unbiased estimate of σ^2
- S^2/n is an unbiased estimate of $\sigma^2/n = \text{Var}(\bar{X})$
- $\sqrt{S^2/n}$ is ~~an unbiased~~ estimate of $\sqrt{\text{Var}(\bar{X})}$

Errata (5/18)

More info on bias of standard error: wikipedia

Standard error

1. Mean happiness:

Claim: The average happiness of Bhutan is 83, with a standard error of 1.99.

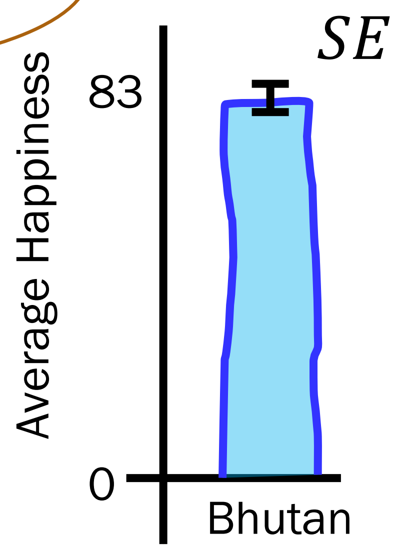
this is our best estimate of μ

this is how close we are

Closed form: $SE = \sqrt{\dfrac{S^2}{n}}$

✅ Verified via bootstrap:

```
np.std(means)
  = 2.003
```



Standard error

1. Mean happiness:

   Claim: The average happiness of Bhutan is 83, with a standard error of 1.99.

   Closed form: $SE = \sqrt{\frac{S^2}{n}}$

2. Variance of happiness:

   this is our best estimate of σ^2

   Claim: The variance of happiness of Bhutan is 793.

   Closed form: Not covered in CS109

   But how close are we?

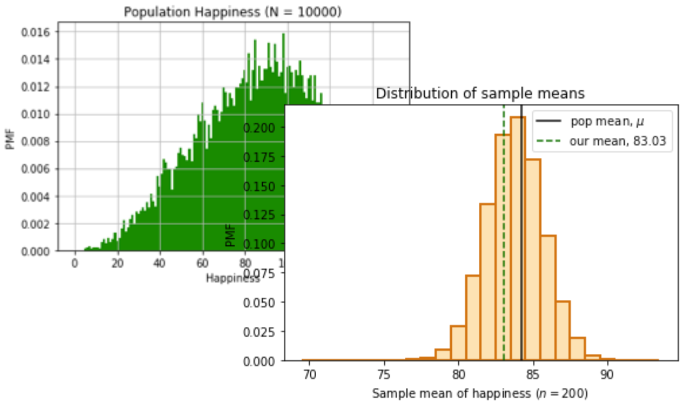
We can bootstrap for standard error of sample variance—a statistic of a statistic.

Computing statistic of sample mean

Review

What is the standard deviation of the sample mean $\bar{X}$? (sample size $n = 200$)

Population distribution
(we don't have this)



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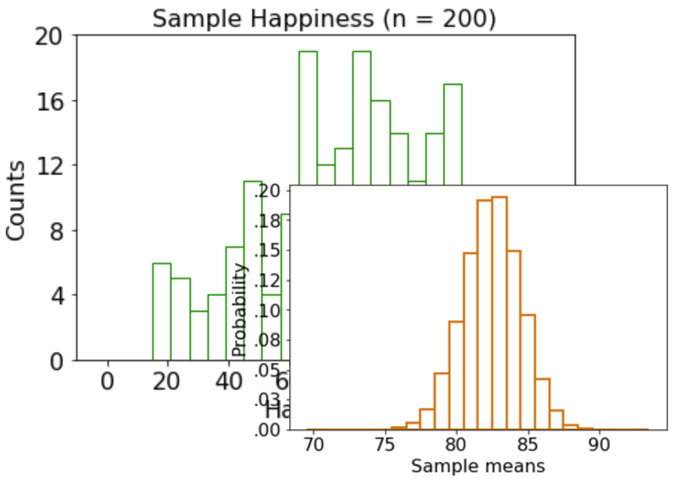
Simulated statistic
(we don't have this)

Sample distribution
(we do have this)



$$SE = \frac{S}{\sqrt{n}} = 1.992$$

Estimated statistic, by formula, **standard error**

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Simulated estimated statistic, **bootstrapped standard error**

Bootstrapped sample variance

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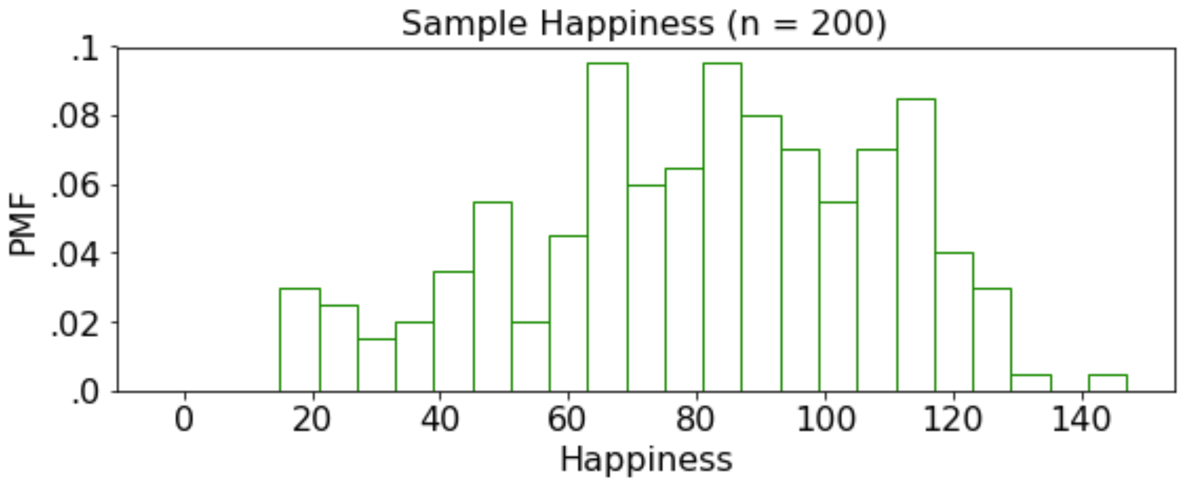
3. You now have a **distribution of your sample variance**

What is the distribution of your sample variance?

Bootstrapped variance



1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
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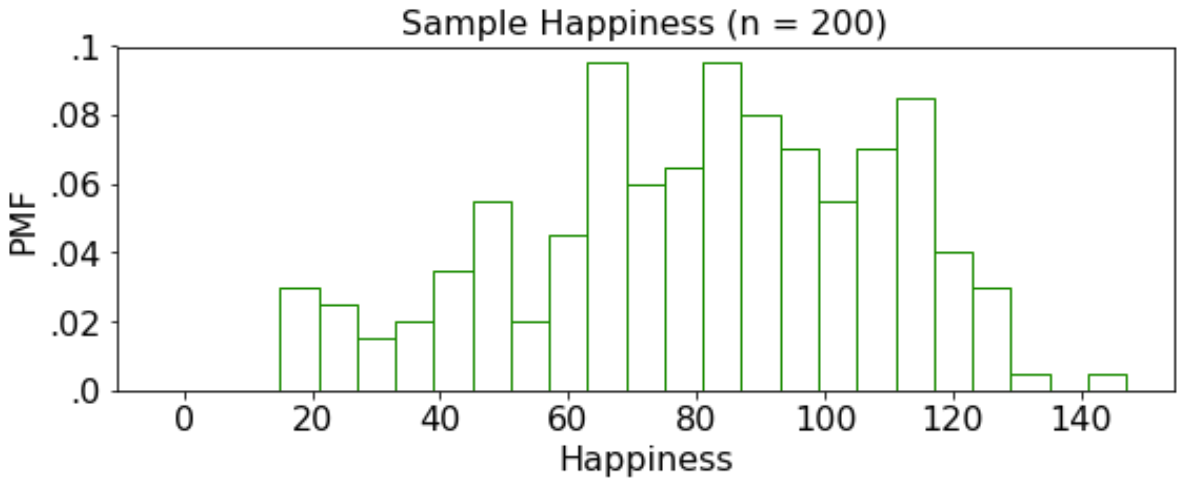
Bootstrapped variance



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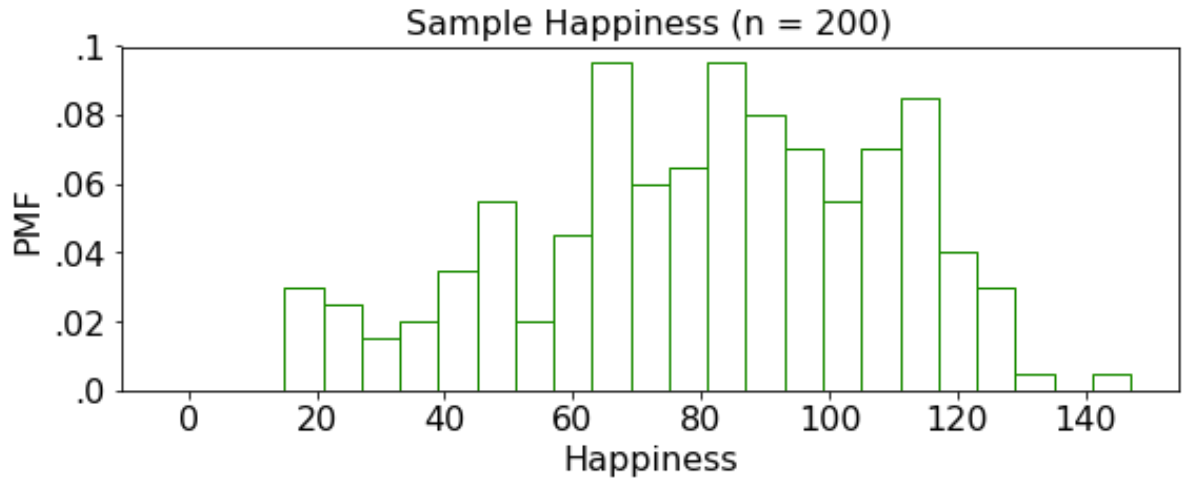
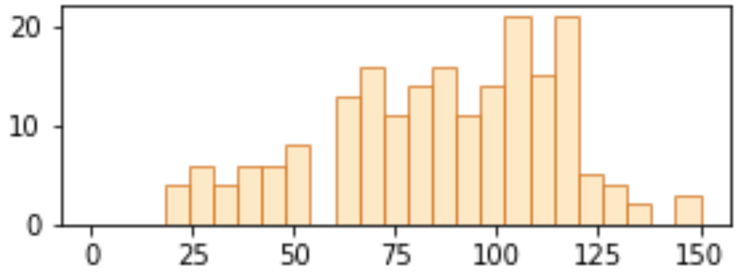


[52, 38, 98, 107, ..., 94]



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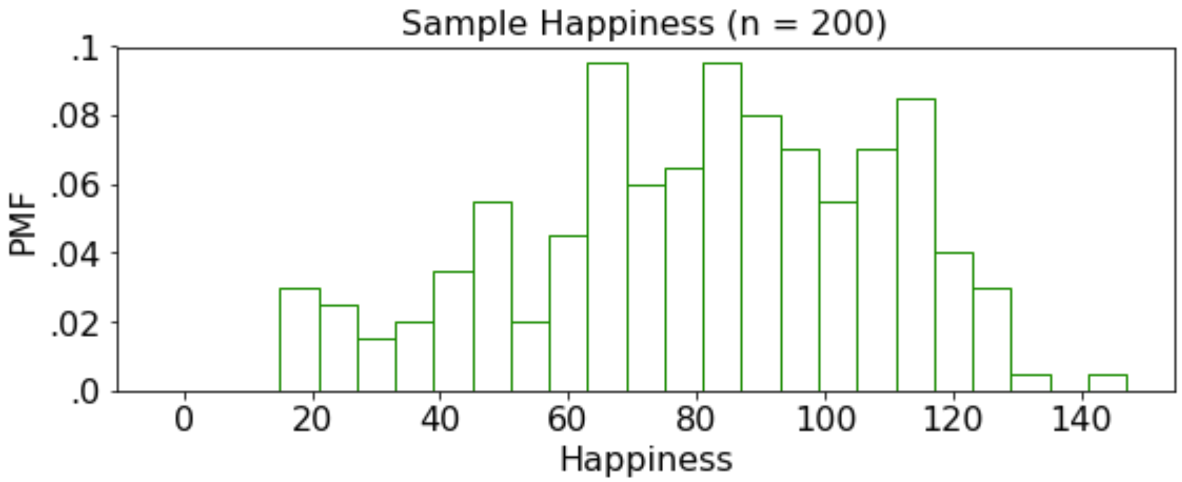
3. You now have a **distribution of your**

Why are these samples **different?**

This resampled sample is generated **with replacement.**

Bootstrapped variance



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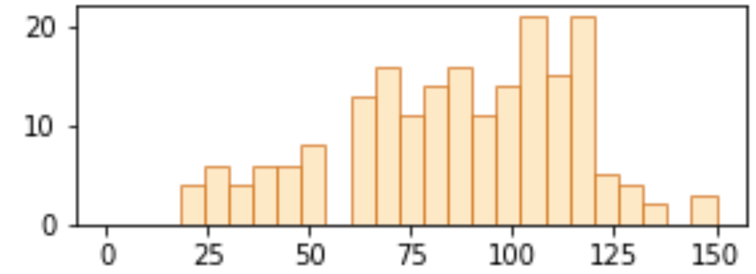

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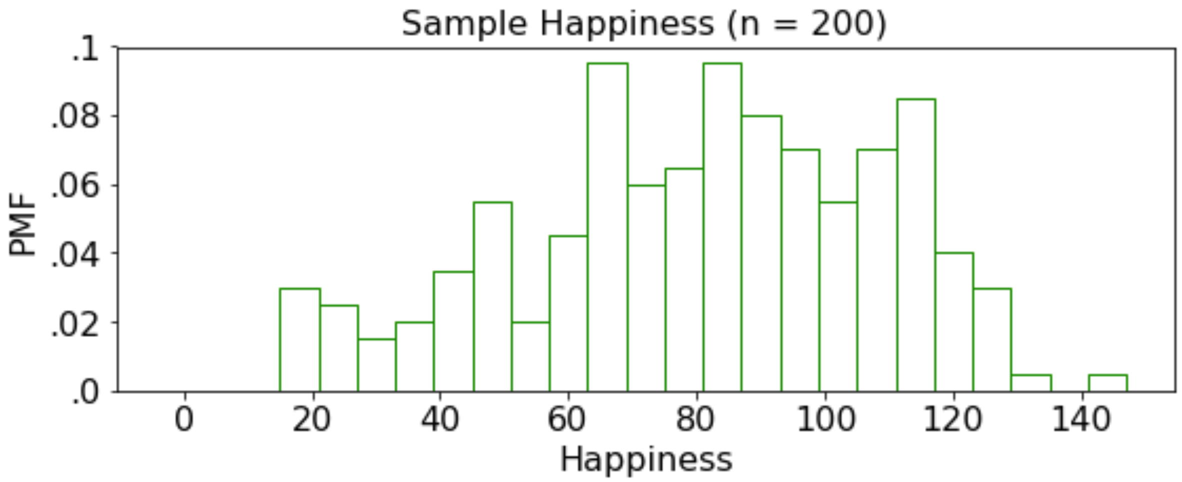
```
variances = [827.4]
```

Bootstrapped variance



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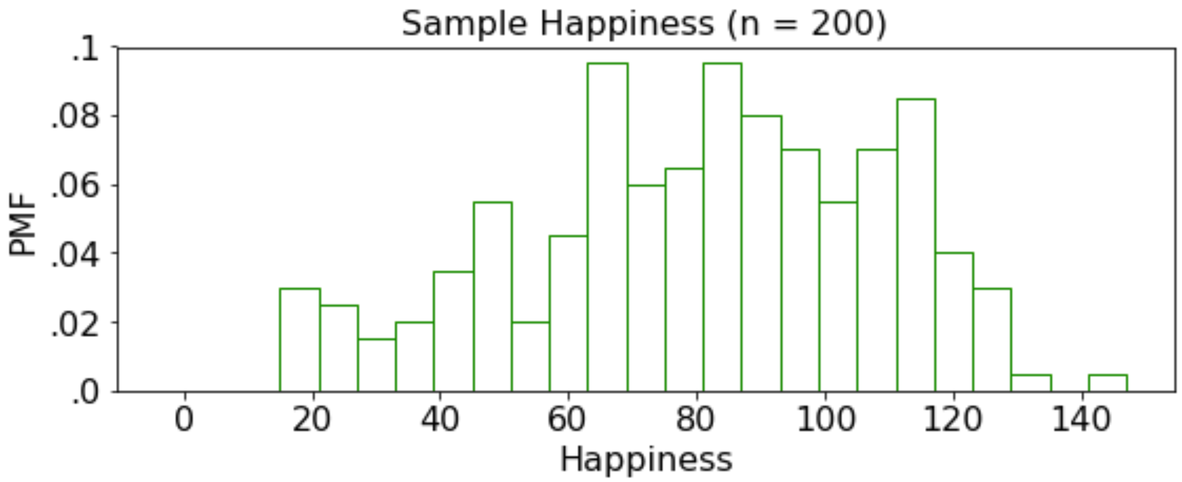
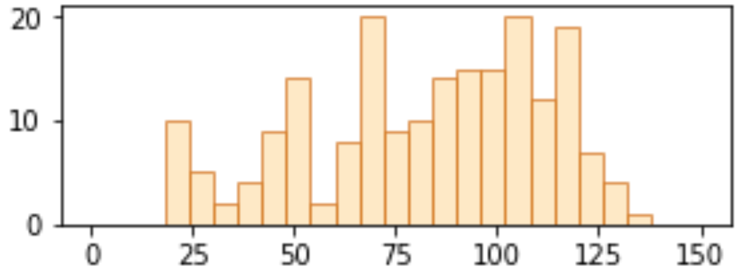


[116, 76, 132, 85, ..., 78]



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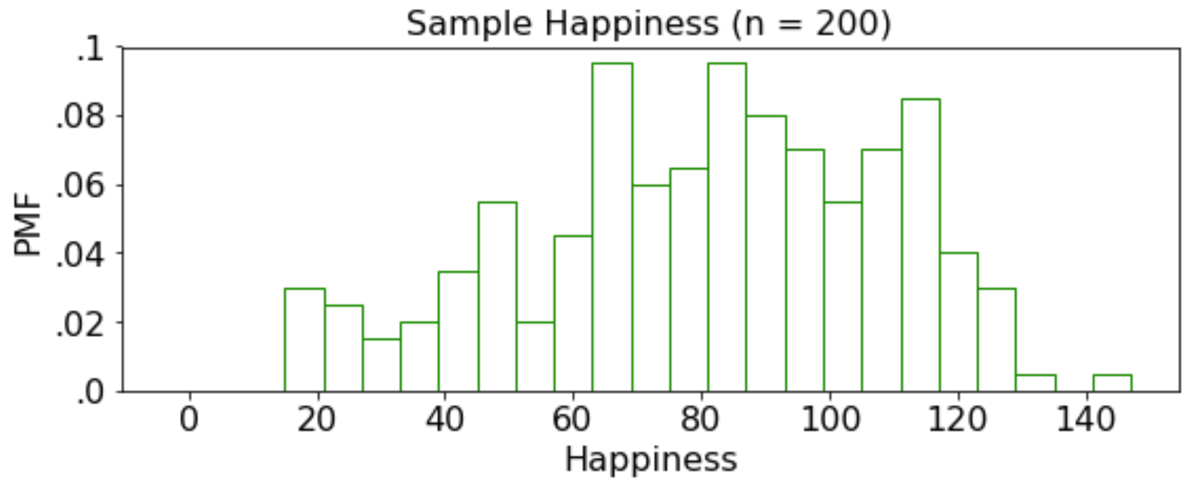
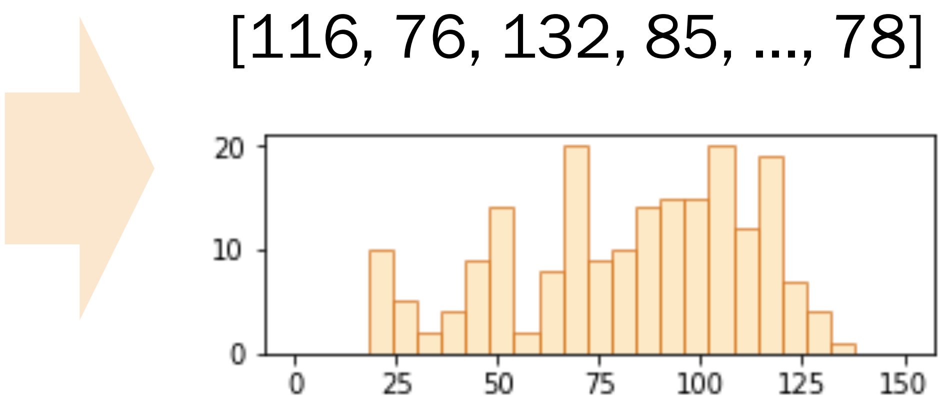
Bootstrapped variance





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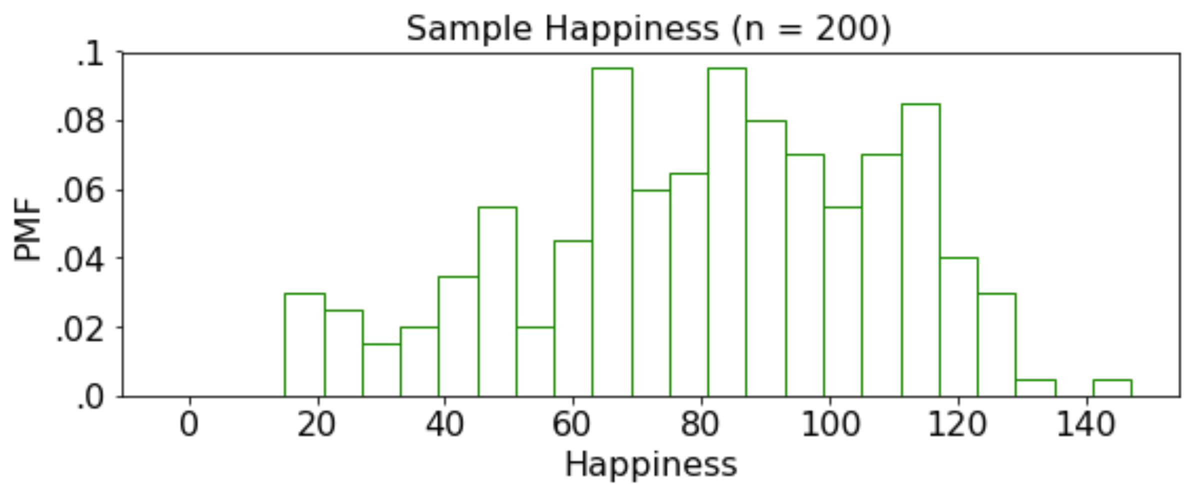
```
variances = [827.4, 846.1]
```

Bootstrapped variance



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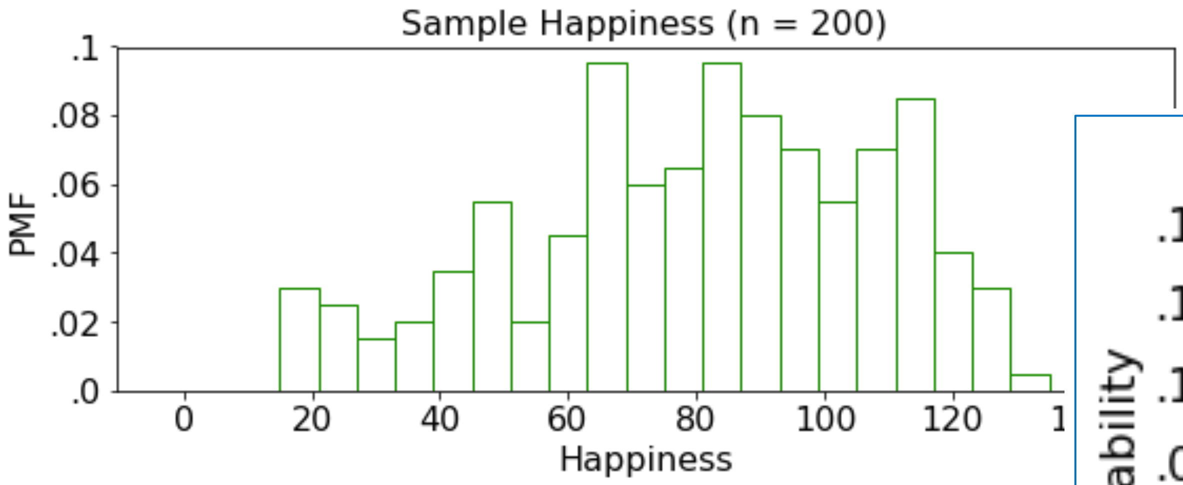
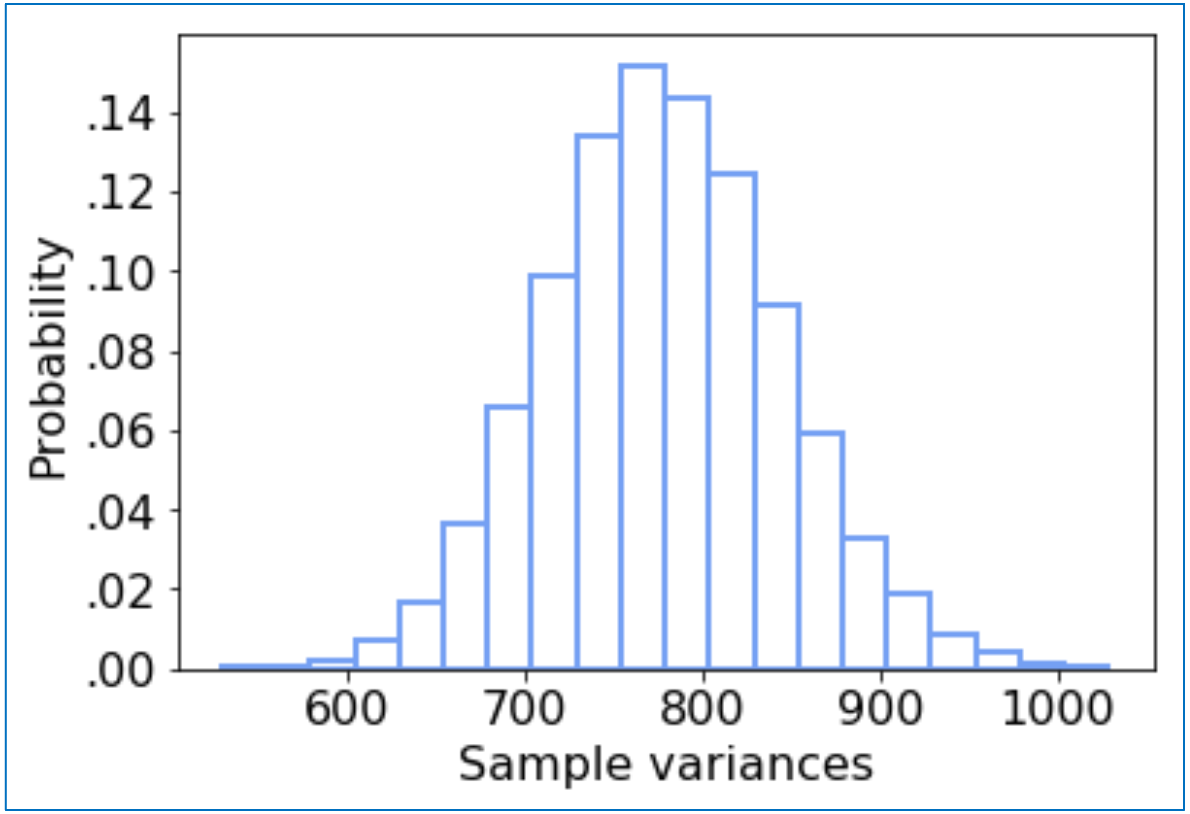
variances = [827.4, 846.1]
```

Bootstrapped variance





1. Estimate the **PMF** using the
2. Repeat **10,000** times:
   a. Resample **sample.size**()
   b. Recalculate the **sample**
3. You now have a **distribution of your sample variance**

variances = [827.4, 846.1, 726.0, …, 860.7]

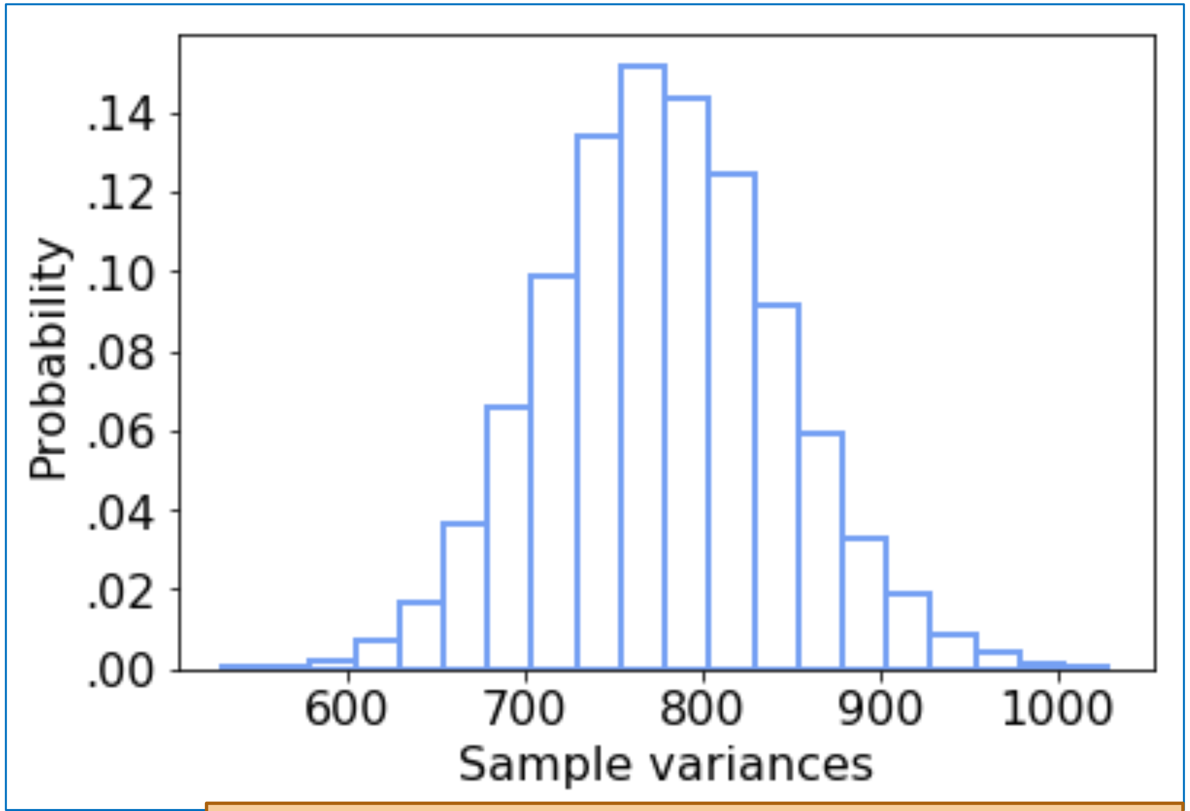
Bootstrapped variance

3. You now have a **distribution of your** sample variance

```
variances = [827.4,
     846.1, 726.0, …,
     860.7]
```

What is the bootstrapped standard error?

```
np.std(variances)
```



Bootstrapped standard error: 66.16

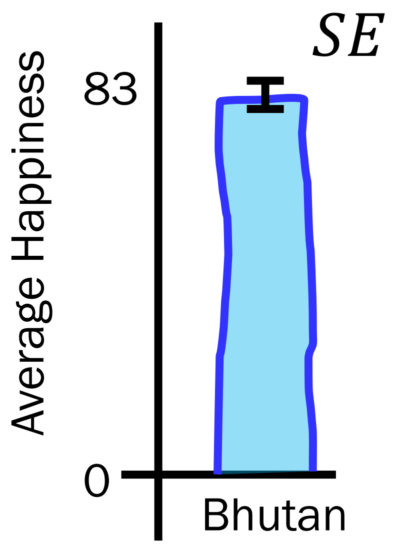
- Simulate a distribution of sample variances
- Compute standard deviation

Standard error

1. Mean happiness:

   Claim: The average happiness of Bhutan is 83, with a standard error of 1.99.

   Closed form: $SE = \sqrt{\frac{S^2}{n}}$



2. Variance of happiness:

   S^2 is our best estimate of σ^2

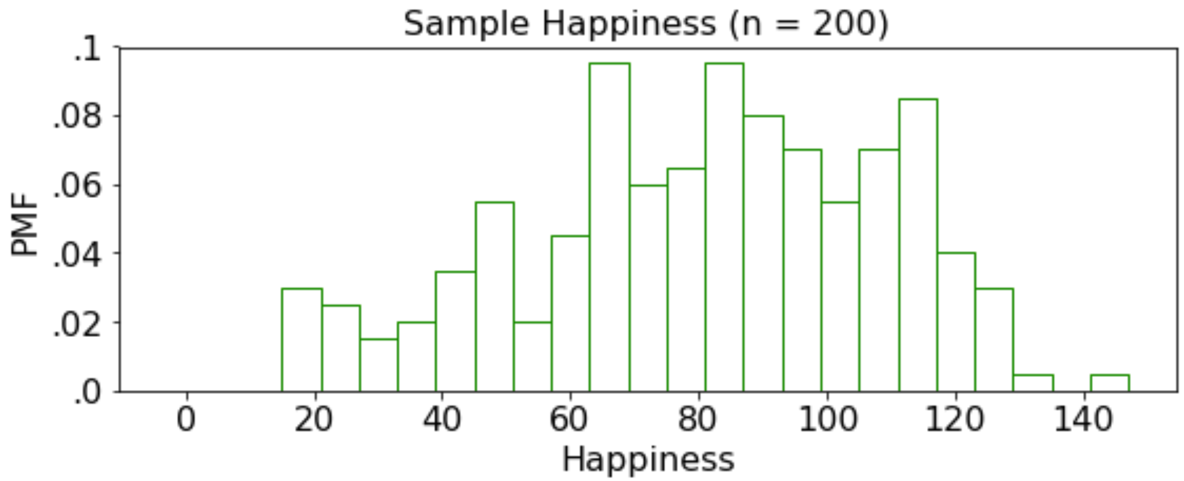
   Claim: The variance of happiness of Bhutan is 793, with a bootstrapped standard error of 66.16.

   this is how close we are, calculated by bootstrapping ✅

Algorithm in practice

?

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
   a. Resample **sample.size**() from PMF
   b. Recalculate the **statistic** on the resample
3. You now have a **distribution of your statistic**



$$P(X = k) = \frac{\text{\# values in sample equal to } k}{n}$$

Algorithm in practice

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
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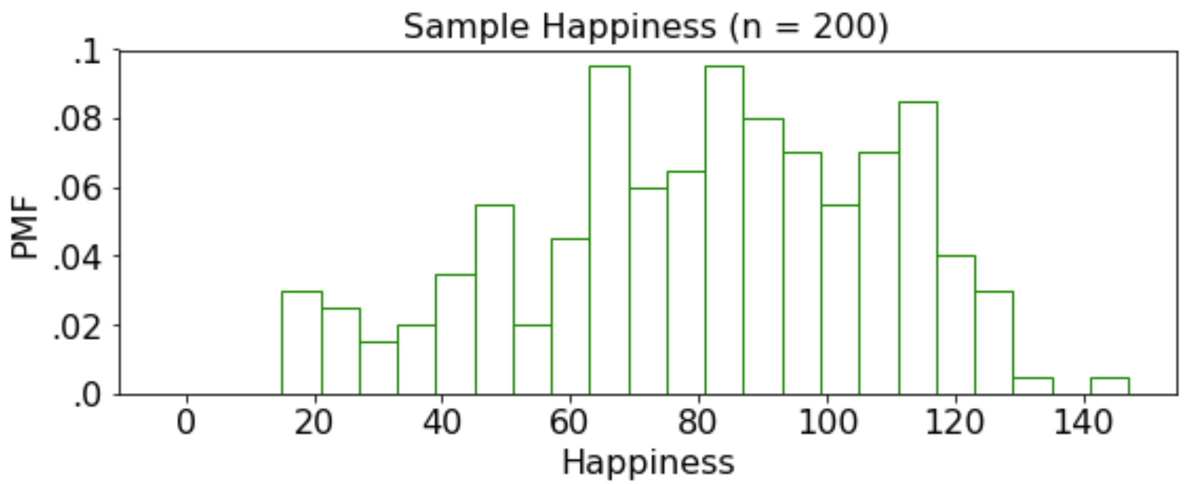
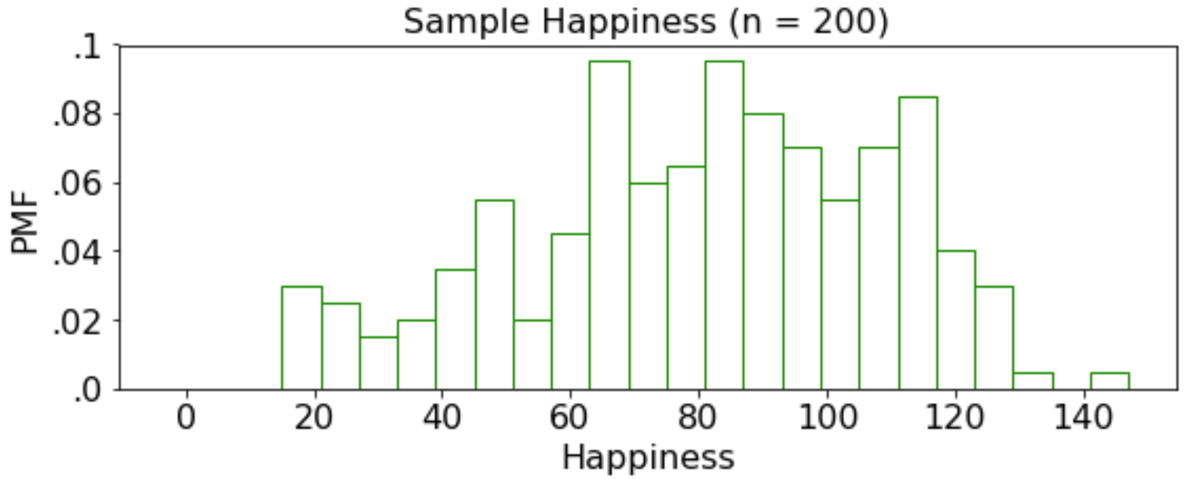


$$P(X = k) = \frac{\text{\# values in sample equal to } k}{n}$$

Algorithm in practice

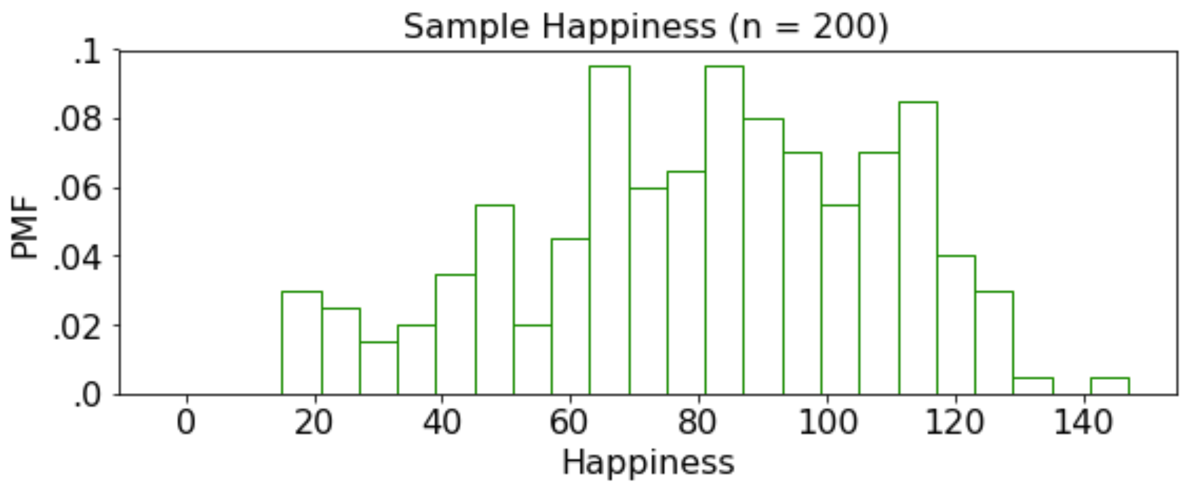
```python
def resample(sample, n):
    # estimate the PMF using the sample
    # draw n new samples from the PMF
    return np.random.choice(sample,
                            n, replace=True)
```



$$P(X = k) = \frac{\text{\# values in sample equal to } k}{n}$$

Algorithm in practice

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
   a. Resample **sample.size**() from PMF,
      with replacement
   b. Recalculate the **statistic** on the resample
3. You now have a **distribution of your statistic**

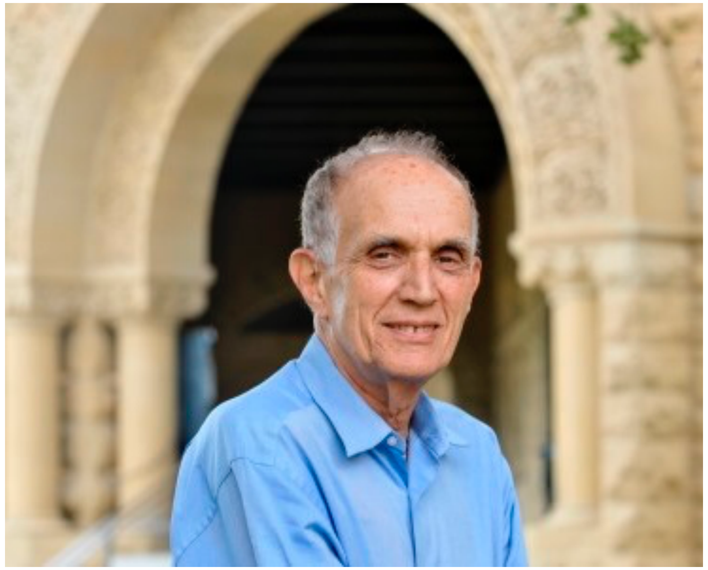


$$P(X = k) = \frac{\text{\# values in sample equal to } k}{n}$$

To the code!

Bootstrap provides a way to calculate probabilities of statistics using code.

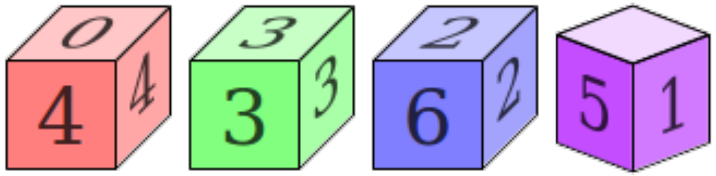
Bradley Efron



- Invented bootstrapping in 1979
- Still a professor at Stanford
- Won a National Science Medal

Efron's dice: 4 dice A, B, C, D such that

$$P(A > B) = P(B > C) = P(C > D) = P(D > A) = \frac{2}{3}$$



To the code!

Bootstrap provides a way to calculate probabilities of statistics using code.

Bootstrapping works for any statistic*

*as long as your sample is i.i.d. and the underlying distribution does not have a long tail

LIVE

Bootstrap: p-value

Null hypothesis test

Sample 1
4.44
3.36
5.87
2.31
…
3.70

Sample 2
4.44
3.36
5.87
2.31
…
3.70

$\bar{X}_1 = 3.1$

$\bar{X}_2 = 2.4$

Claim: Sample 1 and Sample 2 have a 0.7 difference of means.

Null hypothesis test

Nepal Happiness
4.44
3.36
5.87
2.31
…
3.70

$\bar{X}_1 = 3.1$

Bhutan Happiness
4.44
3.36
5.87
2.31
…
3.70

$\bar{X}_2 = 2.4$

Claim: The difference in mean happiness between Nepal and Bhutan is 0.7 happiness points, and **this is significant**.

Null hypothesis test

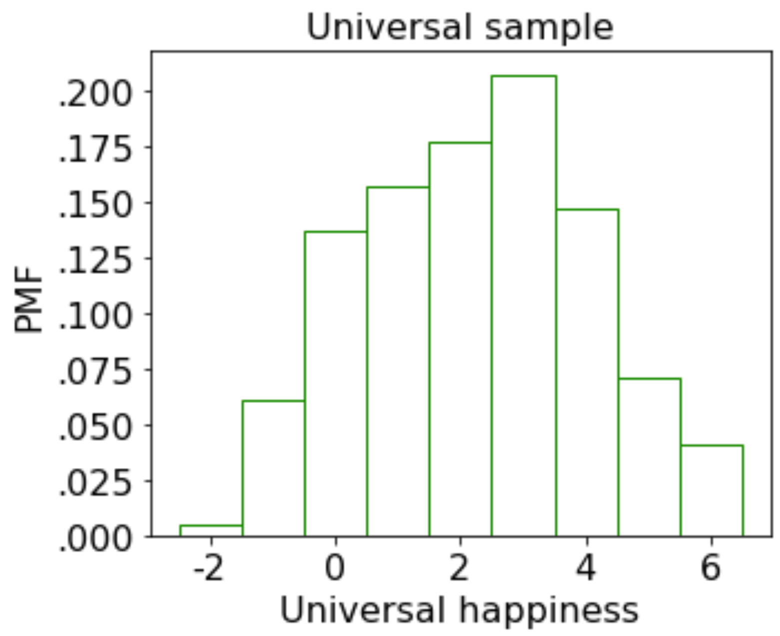
def **null hypothesis** – Even if there is no pattern (i.e., the two samples are from identical distributions), your claim might have arisen by chance.

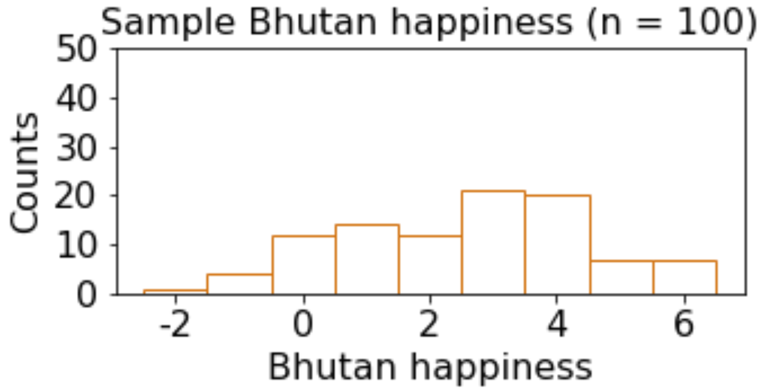
def **p-value** – What is the probability that, under the null hypothesis, the observed difference occurs?

Claim: The difference in mean happiness between Nepal and Bhutan is 0.7 happiness points, and **this is significant**.

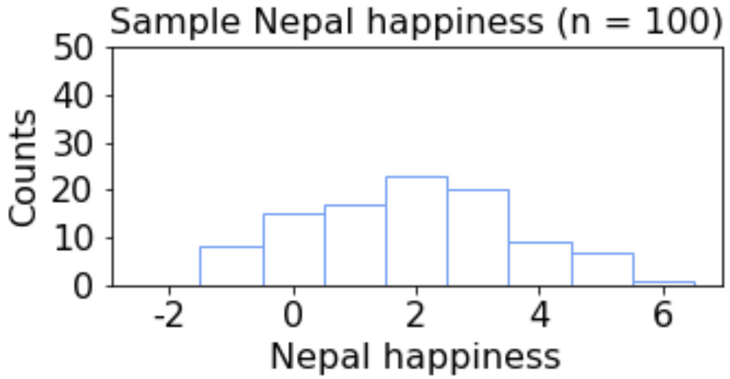
Universal sample

(this is what the null hypothesis assumes)





$\bar{X}_1 = 3.1$



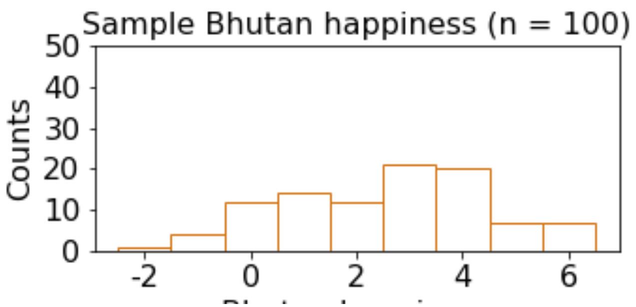
$\bar{X}_2 = 2.4$

Want p-value: probability $|\bar{X}_1 - \bar{X}_2| = |3.1 - 2.4|$ happens under null hypothesis

Bootstrap for p-values

1. Create a universal sample using your two samples

Recreate the null hypothesis



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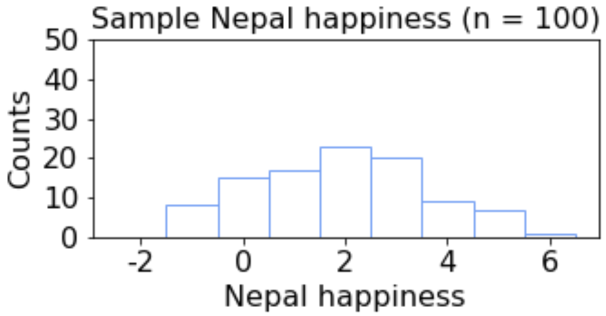

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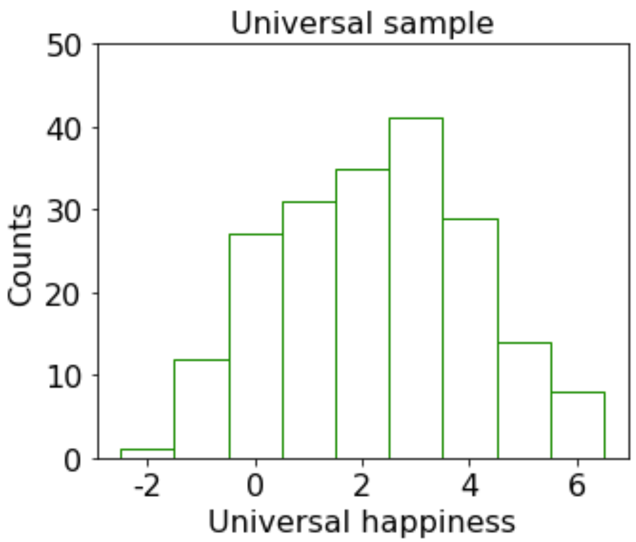

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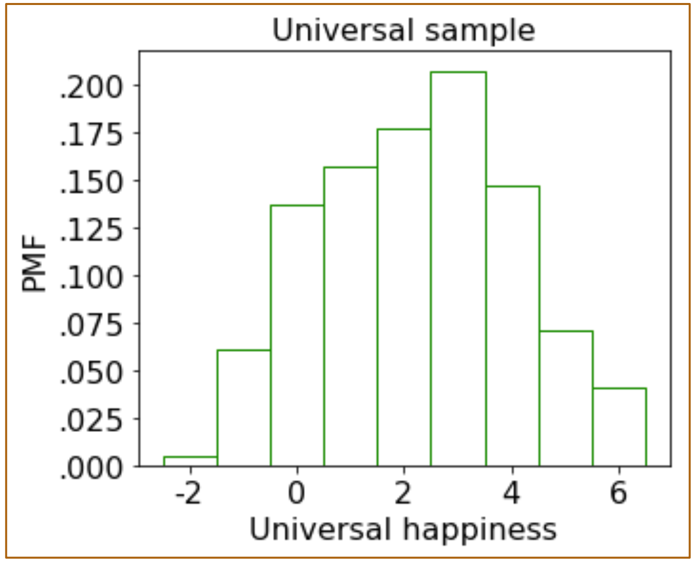

Bootstrap for p-values

1. Create a universal sample using your two samples
2. Repeat **10,000** times:
   a. Resample **both samples**
   b. Recalculate the **mean difference** between the resamples

3. **p-value** = $\frac{\text{\# (mean diffs >= observed diff)}}{n}$

Probability that observed difference arose by chance

Bootstrap for p-values

```
def pvalue_boot(bhutan_sample, nepal_sample):
    N = size of the bhutan_sample
    M = size of the nepal_sample
    observed_diff = |mean of bhutan_sample – mean of nepal_sample|

    uni_sample = combine bhutan_sample and nepal_sample
    count = 0

    repeat 10,000 times:
        bhutan_resample = draw N resamples from the uni_sample
        nepal_resample = draw M resamples from the uni_sample
        muBhutan = sample mean of the bhutan_resample
        muNepal = sample mean of the nepal_resample
        diff = |muNepal – muBhutan|
        if diff >= observed_diff:
            count += 1

pValue = count / 10,000
```

Bootstrap for p-values

1. Create a universal sample using your two samples

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pValue = count / 10,000
```

Bootstrap for p-values

2. a. Resample **both samples**

```
def pvalue_boot(bhutan_sample, nepal_sample):
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    M = size of the nepal_sample
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pValue = count / 10,000
```

Bootstrap for p-values

2. b. Recalculate the **mean difference** b/t resamples

```
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        diff = |muNepal – muBhutan|
        if diff >= observed_diff:
            count += 1

pValue = count / 10,000
```

Bootstrap for p-values

3. p-value = $\frac{\text{\# (mean diffs > observed diff)}}{n}$

```
def pvalue_boot(bhutan_sample, nepal_sample):
    N = size of the bhutan_sample
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```

Bootstrap for p-values

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        muNepal = sample mean of the nepal_resample
        diff = |muNepal – muBhutan|
        if diff >= observed_diff:
            count += 1

pValue = count / 10,000
```

with replacement!

Bootstrap

Let's try it!

Null hypothesis test

Nepal Happiness
4.44
3.36
5.87
2.31
…
3.70

$$\bar{X}_1 = 3.1$$

Bhutan Happiness
4.44
3.36
5.87
2.31
…
3.70

$$\bar{X}_2 = 2.4$$

Claim: The happiness of Nepal and Bhutan have a 0.7 difference of means, and this is significant ($p < 0.01$).

Cloud recording today

Click here to access:

https://stanford.zoom.us/rec/play/v5AIIun7qDo3GdDGsASDU6R5W427KPis0CEXrvIPnh29VXIEMQH0NeMTYrSirUNR6iyXB4P0MPVBj3yM?continueMode=true&_x_zm_rtaid=F2jTZc_2QQCu-okyhY9Krw.1589849628626.72cd8e06b89a0cf03b4ea163319c677a&_x_zm_rhtaid=460