Lisa Yan
CS109

Section #2
April 22-24, 2020

Section #2 Solutions

1 Linearity of Expectation: Hat-Check

Preamble: Typically, it is easier to use linearity of expectation for sums of random variables, then to manually compute the PMF and apply the definition.

Problem: n people go to a party and drop off their hats to a hat-check person. When the party is over, a different hat-check person is on duty, and returns the n hats randomly back to each person. Let X be the random variable representing the number of people who get their own hat back.

a. For $n = 3$, find $E[X]$ by first computing the probability mass function p_X, and then applying the definition of expectation.

b. Find a general formula for $E[X]$, for any positive integer n.

a. The number of people X who could get their hat back is in $\{0, 1, 3\}$ (why not 2?). One can enumerate the possibilities:

$$123 \to 3$$
$$132 \to 1$$
$$213 \to 1$$
$$231 \to 0$$
$$312 \to 0$$
$$321 \to 1$$

Hence, $p_X(0) = P(X = 0) = 1/3$, $p_X(1) = P(X = 1) = 1/2$, and $p_X(3) = P(X = 3) = 1/6$. We then have that

$$E[X] = \sum_x x p_X(x) = 0 \cdot (1/3) + 1 \cdot (1/2) + 3 \cdot (1/2) = 1.$$

b. For $i = 1, \ldots, n$, let X_i be the indicator variable of whether person i gets their hat back. That is, $X_i = 1$ if person i gets their hat back, and $X_i = 0$ otherwise. Then, $X = \sum_{i=1}^n X_i$. For a particular person i, the probability they get their hat back is exactly $1/n$ (why?), and so $E[X_i] = 1 \cdot (1/n) + 0 \cdot (1 - 1/n) = 1/n$.

By linearity of expectation,

$$E[X] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] = \sum_{i=1}^n \frac{1}{n} = n \cdot (1/n) = 1.$$

Now imagine finding the PMF for this random variable with n people/hats. There was no nice catch-all formula in part a) for $n = 3$, and so it would be extremely difficult/impossible to come up with one for general $P(X = k)$. Even if you could, evaluating the sum might be difficult. This is the power of linearity of expectation - though we don't know the PMF, we can still compute it easily by breaking it down into smaller pieces. Notice that people getting their hat backs are not independent events either!

2 Taking Expectation: Breaking Vegas

Preamble: When a random variable fits neatly into a family we've seen before (e.g. Binomial), we get its expectation for free. When it does not, we have to use the definition of expectation.

Problem: If you bet on "Red" in Roulette, there is $p = 18/38$ that you with win \$Y and a $(1 - p)$ probability that you lose \$Y. Consider this algorithm for a series of bets:

1. Let Y = \$1.
2. Bet Y.
3. If you win, then stop.
4. If you lose, set Y to be 2Y, goto step (2).

What are your expected winnings when you stop? It will help to recall that the sum of a geometric series $a^0 + a^1 + a^2 + \cdots = \frac{1}{1-a}$ if $0 < a < 1$. Vegas breaks you: Why doesn't everyone do this?

Let X be the number of dollars that your earn.
The possible values of x are from the outcomes of: winning on your first bet, winning on your second bet, and so on.

$$\begin{aligned}
E[X] &= \frac{18}{38} + \frac{20}{38}\frac{18}{38}(2-1) + \left(\frac{20}{38}\right)^2\frac{18}{38}(4-2-1) + \ldots \\
&= \sum_{i=0}^{\infty}\left(\frac{20}{38}\right)^i\left(\frac{18}{38}\right)\left(2^i - \sum_{j=0}^{i-1}2^j\right) \\
&= \left(\frac{18}{38}\right)\sum_{i=0}^{\infty}\left(\frac{20}{38}\right)^i \\
&= \left(\frac{18}{38}\right)\frac{1}{1-\frac{20}{38}} = 1
\end{aligned}$$

Real games have maximum bet amounts. You have finite money and casinos can kick you out. But, if you had no betting limits and infinite money, then go for it! (and tell me which planet you are living on).

3 Binomial Distribution: Sending Bits to Space

When sending binary data to satellites (or really over any noisy channel) the bits can be flipped with high probabilities. In 1947 Richard Hamming developed a system to more reliably send data. By using Error Correcting Hamming Codes, you can send a stream of 4 bits with 3 redundant bits. If zero or one of the seven bits are corrupted, using error correcting codes, a receiver can identify the original 4 bits.

Let's consider the case of sending a signal to a satellite where each bit is independently flipped with probability $p = 0.1$

a. If you send 4 bits, what is the probability that the correct message was received (i.e. none of the bits are flipped).

b. If you send 4 bits, with 3 Hamming error correcting bits, what is the probability that a correctable message was received?

c. Instead of using Hamming codes, you decide to send 100 copies of each of the four bits. If for every single bit, more than 50 of the copies are not flipped, the signal will be correctable. What is the probability that a correctable message was received?

Hamming codes are super interesting. It's worth looking up if you haven't seen them before! All these problems could be approached using a binomial distribution (or from first principles).

a. Let Y be the number of bits corrupted. $Y \sim \text{Bin}(n = 4, p = 0.1)$.

$$P(Y = 0) = \binom{4}{0} 0.9^4 = 0.656$$

b. Let Z be the number of bits corrupted. $Z \sim \text{Bin}(n = 7, p = 0.1)$. A correctable message is received if Z equals 0 or 1:

$$\begin{aligned} P(\text{correctable}) &= P(Z = 0) + P(Z = 1) \\ &= \binom{7}{0}(0.1)^0(0.9)^7 + \binom{7}{1}(0.1)^1(0.9)^6 = 0.850 \end{aligned}$$

That is a 30% improvement!

c. Let X_i be the number of copies of bit i which are not corrupted. We can represent each as a Binomial Random Variable: $X_i \sim \text{Bin}(n = 100, p = 0.9)$.

$$\begin{aligned} P(\text{correctable}) &= \prod_{i=1}^{4} P(X_i > 50) \\ &= \prod_{i=1}^{4} \sum_{j=51}^{100} P(X_i = j) \\ &= \prod_{i=1}^{4} \sum_{j=51}^{100} \binom{100}{j} (0.9)^j (0.1)^{100-j} \\ &= \Big(\sum_{j=51}^{100} \binom{100}{j} (0.9)^j (0.1)^{100-j} \Big)^4 > 0.999 \end{aligned}$$

But now you need to send 400 bites, instead of the 7 required by hamming codes :-).

4 Conditional Probabilities: Corrupt Hot-Dog-Eating Contest Judges

Preamble: We have three big tools for manipulating conditional probabilities:

- Definition of conditional probability: $P(EF) = P(E|F)P(F)$
- Law of Total Probability: $P(E) = P(EF) + P(EF^C) = P(E|F)P(F) + P(E|F^C)P(F^C)$
- Bayes Rule: $P(E|F) = \dfrac{P(F|E)P(E)}{P(F)} = \dfrac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^C)P(E^C)}$

This is a good time to commit these three to memory and start thinking about when each of them is useful.

Problem: Corrupted by their power, the judges running the popular game show America's Next Top Hot Dog Eater have been taking bribes from many of the contestants. During each of two episodes, a given contestant is either allowed to stay on the show or is kicked off. If the contestant has been bribing the judges, she will be allowed to stay with probability 1. If the contestant has not been bribing the judges, she will be allowed to stay with probability 1/3, independent of what happens in earlier episodes. Suppose that 1/4 of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds.

a. If you pick a random contestant, what is the probability that she is allowed to stay during the first episode?

b. If you pick a random contestant, what is the probability that she is allowed to stay during both episodes?

c. If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?

d. If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she was bribing the judge?

a. Let S_1 be the event she stayed during the first episode, and B the event she bribed the judges.

$$P(S_1) = P(S_1|B)P(B) + P(S_1|B^C)P(B^C) = 1 \cdot (1/4) + (1/3)(3/4) = 1/2$$

b. Let S_2 be the event she stayed during the second episode. Since staying in episodes are conditionally independent given whether she bribed the judges,

$$P(S_1 \cap S_2) = P(S_1 \cap S_2|B)P(B) + P(S_1 \cap S_2|B^C)P(B^C)$$

$$= P(S_1|B)P(S_2|B)P(B) + P(S_1|B^C)P(S_2|B^C)P(B^C) = 1^2(1/4) + (1/3)^2(3/4) = 1/3$$

c.

$$P(S_1 \cap S_2^C) = P(S_1|B)P(S_2^C|B)P(B) + P(S_1|B^C)P(S_2^C|B^C)P(B^C)$$

$$= 1 \cdot 0 \cdot (1/4) + (1/3) \cdot (2/3) \cdot (3/4) = 1/6$$

$$P(S_2^C|S_1) = \frac{P(S_1 \cap S_2^C)}{P(S_1)} = \frac{1/6}{1/2} = 1/3$$

d.

$$P(B|S_1) = \frac{P(S_1|B)P(B)}{P(S_1)} = \frac{(1)(1/4)}{1/2} = 1/2$$