02: Combinatorics

Lisa Yan
April 8, 2020

Quick slide reference

Today's discussion thread: https://us.edstem.org/courses/667/discussion/80935

Permutations II

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable)
- Choose k objects (combinations)
- Put objects in r buckets

Sort n distinct objects

Ayesha Tim Irina Joey Waddie

# of permutations =

Summary of Combinatorics

Counting tasks on n objects

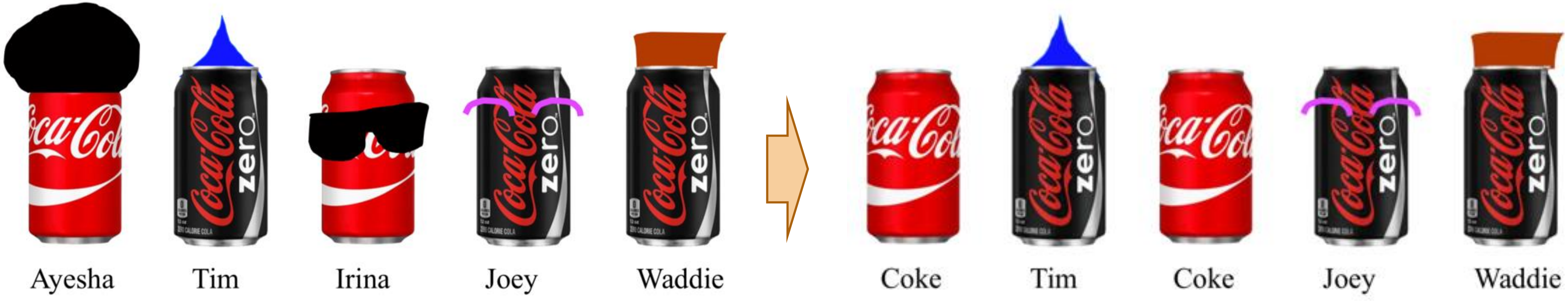
- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct
- Choose k objects (combinations)
- Put objects in r buckets

Sort semi-distinct objects

Order n distinct objects	$n!$

All distinct

Some indistinct



Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of <u>distinct</u> objects is a two-step process:

permutations of distinct objects $=$ permutations considering some objects are indistinct $\times$ Permutations of just the indistinct objects

Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

$$\frac{\text{permutations of distinct objects}}{\text{Permutations of just the indistinct objects}} = \text{permutations considering some objects are indistinct}$$

General approach to counting permutations

When there are n objects such that

n_1 are the same (indistinguishable or indistinct), and

n_2 are the same, and

…

n_r are the same,

The number of unique orderings (permutations) is

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}.$$

For each group of indistinct objects,
Divide by the overcounted permutations.

Sort semi-distinct objects

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many permutations?

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$
- Choose k objects (combinations)
- Put objects in r buckets

Strings

Order n semi-distinct objects	$\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many orderings of letters are possible for the following strings?

1. BOBA

2. MISSISSIPPI

Strings

Order n semi-distinct objects $\frac{n!}{n_1!n_2!\cdots n_r!}$

How many orderings of letters are possible for the following strings?

1. BOBA $= \frac{4!}{2!} = 12$

2. MISSISSIPPI $= \frac{11!}{1!4!4!2!} = 34{,}650$

Unique 6-digit passcodes with six smudges

Order n semi-distinct objects	$\frac{n!}{n_1!n_2!\cdots n_r!}$

How many unique 6-digit passcodes are possible if a phone password uses each of **six** distinct numbers?

$$\text{Total} = 6!$$
$$= 720 \text{ passcodes}$$

Unique 6-digit passcodes with five smudges

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many unique 6-digit passcodes are possible if a phone password uses each of five distinct numbers?

Steps:

1. Choose digit to repeat — 5 outcomes
2. Create passcode — (sort 6 digits: 4 distinct, 2 indistinct)

$$\text{Total} = 5 \times \frac{6!}{2!}$$

$$= 1{,}800 \text{ passcodes}$$

Combinations I

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\,n_2!\cdots n_r!}$
- Choose k objects (combinations)
  - Distinct
- Put objects in r buckets

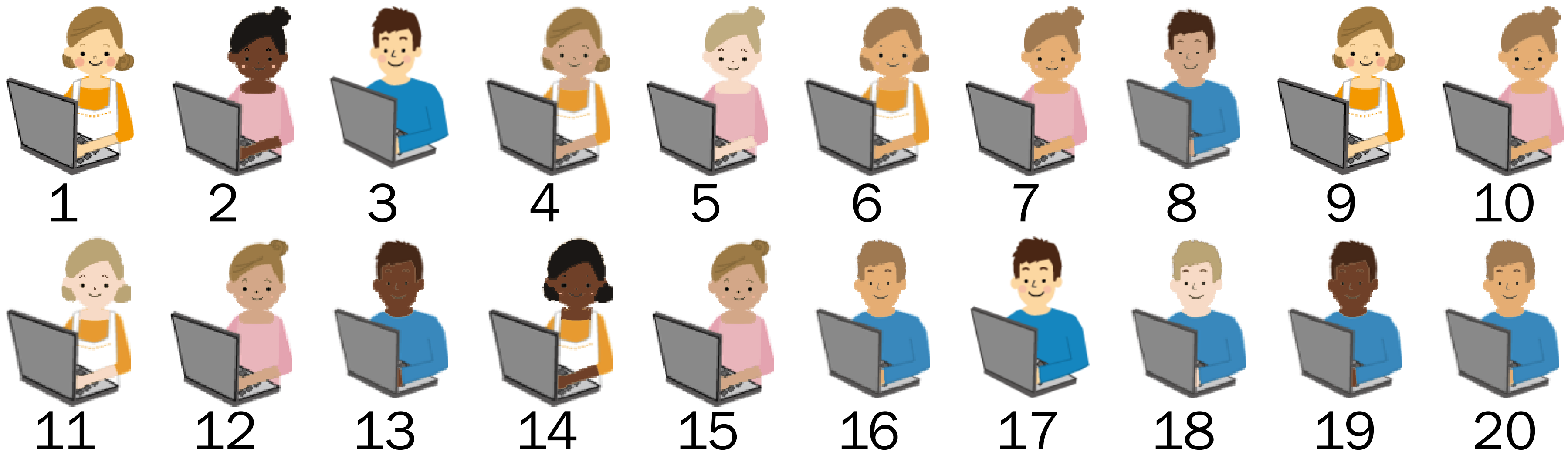
Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

Consider the following generative process...

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

$n!$ ways

Combinations with cake

There are $n = 20$ people.

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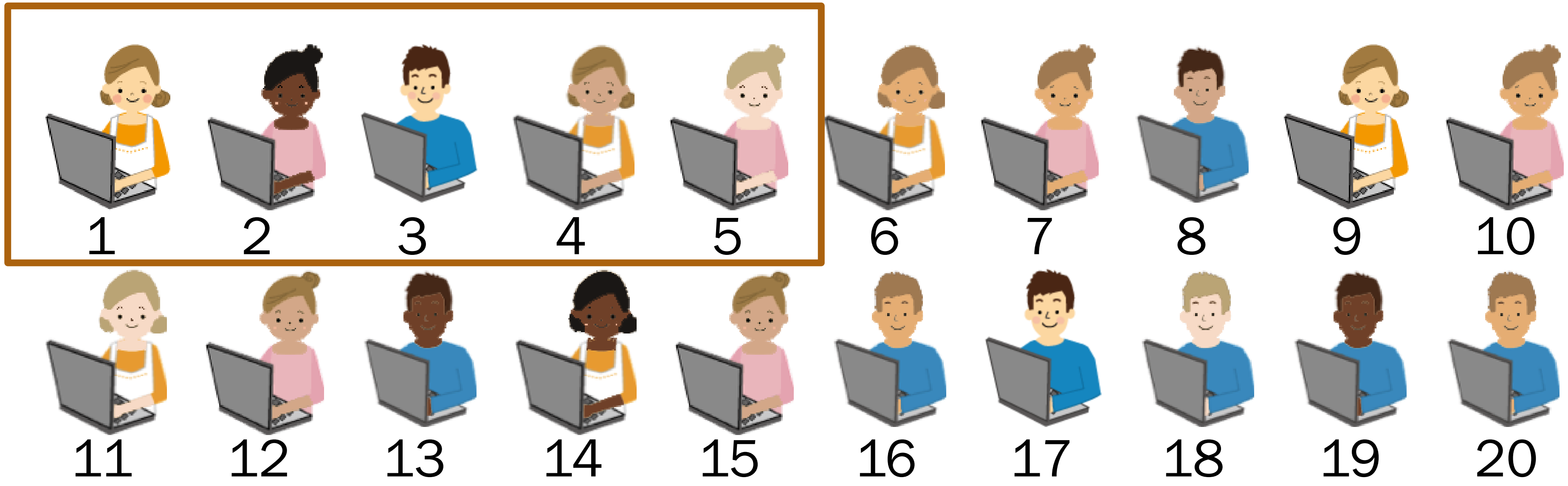

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

Combinations with cake

There are $n = 20$ people.
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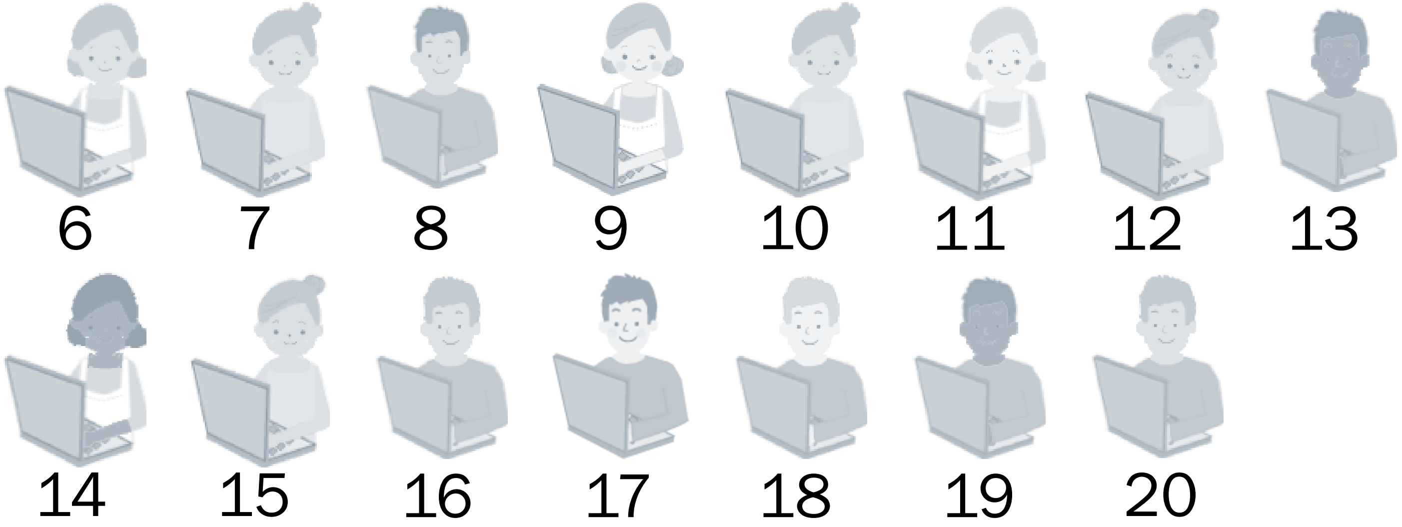

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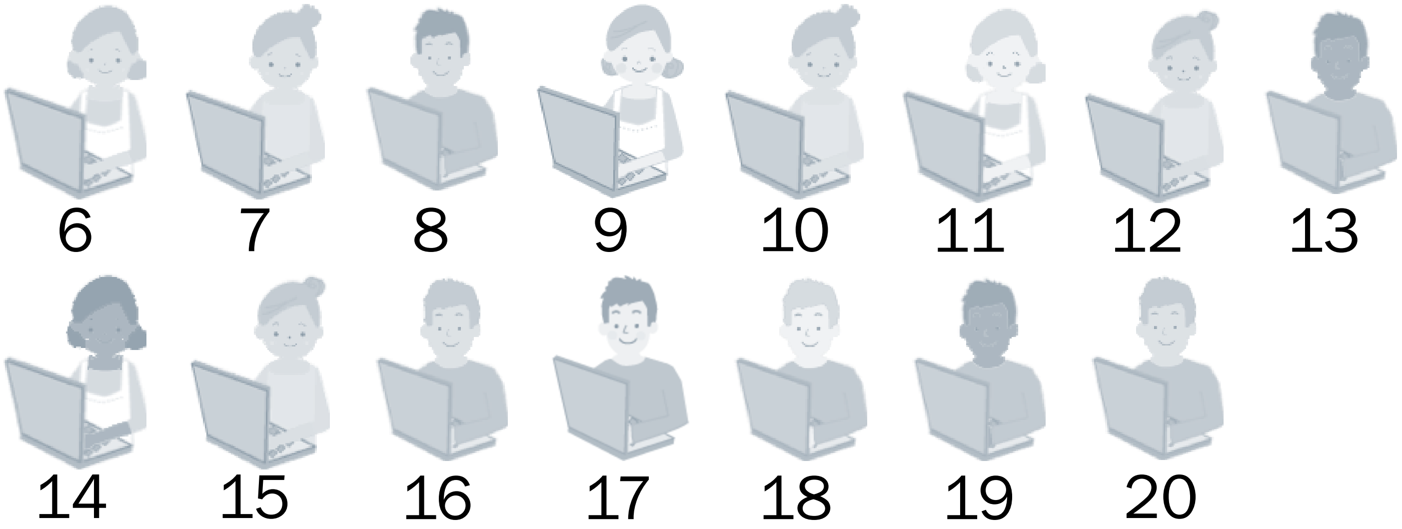

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

Combinations with cake

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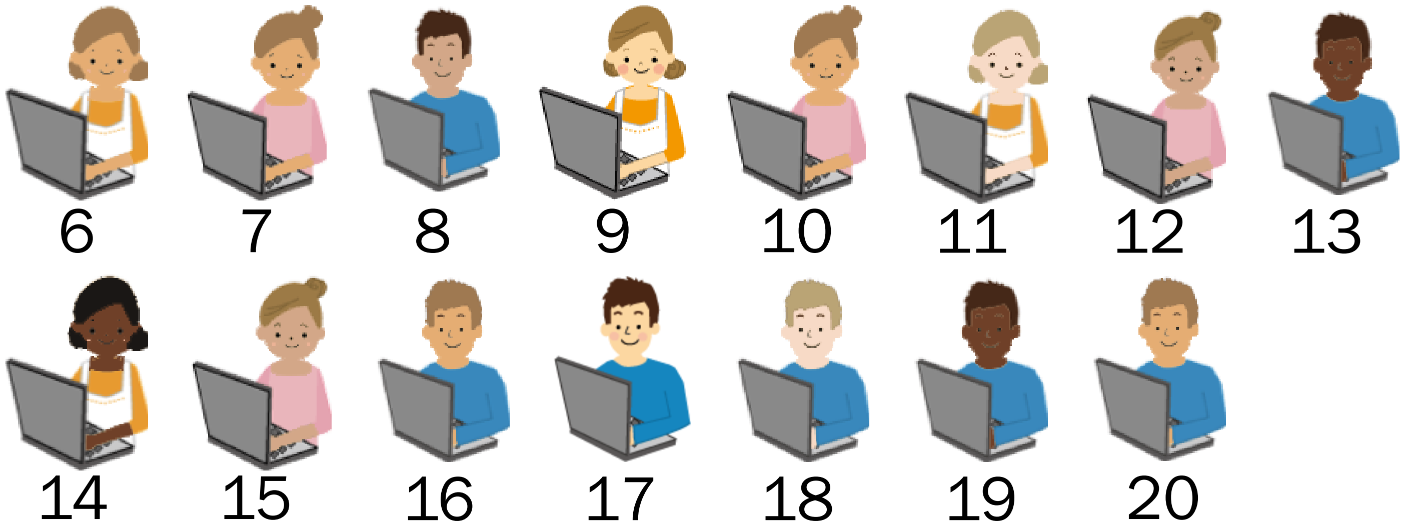

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

   $(n-k)!$ different permutations lead to the same mingle

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!}$$

1. Order n distinct objects
2. Take first k as chosen
3. Overcounted: any ordering of chosen group is same choice
4. Overcounted: any ordering of unchosen group is same choice

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!} = \binom{n}{k}$$

Binomial coefficient

Note: $\binom{n}{n-k} = \binom{n}{k}$

Probability textbooks

Choose k of
n distinct objects $\binom{n}{k}$

How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

Combinations II

Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\,n_2!\cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups
- **Put objects in r buckets**

General approach to combinations

The number of ways to choose r groups of n distinct objects such that

For all $i = 1, \dots, r$, group i has size n_i, and

$\sum_{i=1}^{r} n_i = n$ (all objects are assigned), is

$$\frac{n!}{n_1!\, n_2! \cdots n_r!} = \binom{n}{n_1, n_2, \cdots, n_r}$$

Multinomial coefficient

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	6
B	4
C	3

A. $\binom{13}{6,4,3} = 60{,}060$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

C. $6 \cdot 1001 \cdot 10 = 60{,}060$

D. A and B

E. All of the above

Datacenters

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How many different divisions are possible?

A. $\binom{13}{6,4,3} = 60{,}060$

Strategy: Combinations into 3 groups

Group 1 (datacenter A): $n_1 = 6$

Group 2 (datacenter B): $n_2 = 4$

Group 3 (datacenter C): $n_3 = 3$

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

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Group 3 (datacenter C): $n_3 = 3$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

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Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

Your approach will determine if you use binomial/multinomial coefficients or factorials.

02: Combinatorics (live)

Lisa Yan
April 8, 2020

Reminders: Lecture with zoom

- Turn on your camera if you are able, mute your mic in the big room
- Virtual backgrounds are encouraged (classroom-appropriate)

Breakout Rooms for meeting your classmates

- Just like sitting next to someone new

We will use Ed instead of Zoom chat (for now)

Today's discussion thread: https://us.edstem.org/courses/667/discussion/80935

Summary of Combinatorics

Review

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group (really 2): $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\, n_2! \cdots n_r!}$
- Choose k objects (combinations)
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
  - Indistinct?
- Put objects in r buckets

Think

Slide 42 is a question to think over by yourself (~1min).

Post any clarifications here!

https://us.edstem.org/courses/667/discussion/80935

A trick question

How many ways are there to group 6 indistinct (indistinguishable) objects into 3 groups, where groups A, B, and C have sizes 1, 2, and 3, respectively?

A. $\binom{6}{1,2,3}$

B. $\frac{6!}{1!2!3!}$

C. 0

D. 1

E. Both A and B

F. Something else

(by yourself)

Ask: https://us.edstem.org/courses/667/discussion/80935

A trick question

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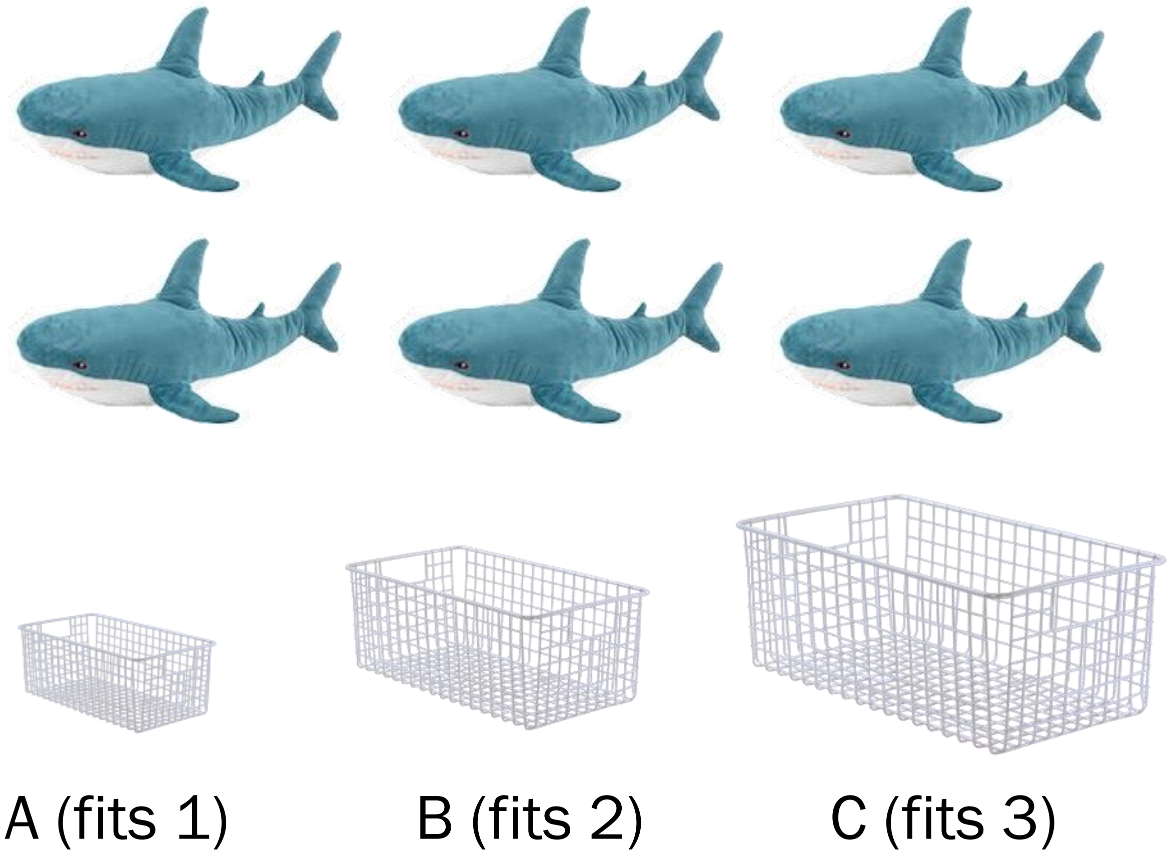


A. $\binom{6}{1,2,3}$
B. $\frac{6!}{1!2!3!}$
C. 0
D. 1
E. Both A and B
F. Something else

Probability textbooks

Review

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books? $\binom{6}{3} = \frac{6!}{3!\,3!} = 20$ ways

2. What if we do not want to read both the 9th and 10th edition of Ross?

A. $\binom{6}{3} - \binom{6}{2} = 5$ ways

B. $\frac{6!}{3!3!2!} = 10$

C. $2 \cdot \binom{4}{2} + \binom{4}{3} = 16$

D. $\binom{6}{3} - \binom{4}{1} = 16$

E. Both C and D

F. Something else

Ask: https://us.edstem.org/courses/667/discussion/80935

Breakout Rooms

Introduce yourself!

Then check out the question on the previous slide. Post any clarifications here!

https://us.edstem.org/courses/667/discussion/80935

Breakout Room time: 4 minutes

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

2. What if we do not want to read both the 9th and 10th edition of Ross?

Strategy 1: Sum Rule

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

2. What if we do not want to read both the 9th and 10th edition of Ross?

Strategy 2: "Forbidden method" (unofficial name)

Forbidden method: It is sometimes easier to exclude invalid cases than to include cases.

Interlude for Fun/announcements

Announcements

PS#1

Out: today
Due: Wednesday 7/1, 1pm
Covers: through ~Monday 6/29
Gradescope submission: posted soon

Python help

When: (asynchronous recording)

Notes: to be posted online

Optional Section

Update: 1st live section during **Week 2**

Week 1: Python tutorial to be posted

Staff help

Ed discussion: find study buddies!
Office hours: start tmrw, on QS/Zoom
http://cs109.stanford.edu/staff.html

The website is a great resource

You can access every *other* course resource from the front page!

Lecture Notes/Slides:

- You are responsible for material in both
- Lecture notes generally a subset of lecture slides

Ross Textbook

- Optional, good for a second perspective
- 8th edition online 1-hr checkout system in Administrivia handout
- Table of Contents comparison for 8th, 9th, 10th editions (TL;DR: they're the same) https://cs109.stanford.edu/restricted/ross_editions_toc.pdf

Interesting probability news (from spring)

APR. 6, 2020

It Was A Roller-Coaster Season For Michigan State, Our Men's Bracket Champion

By Josh Planos

Filed under College Basketball

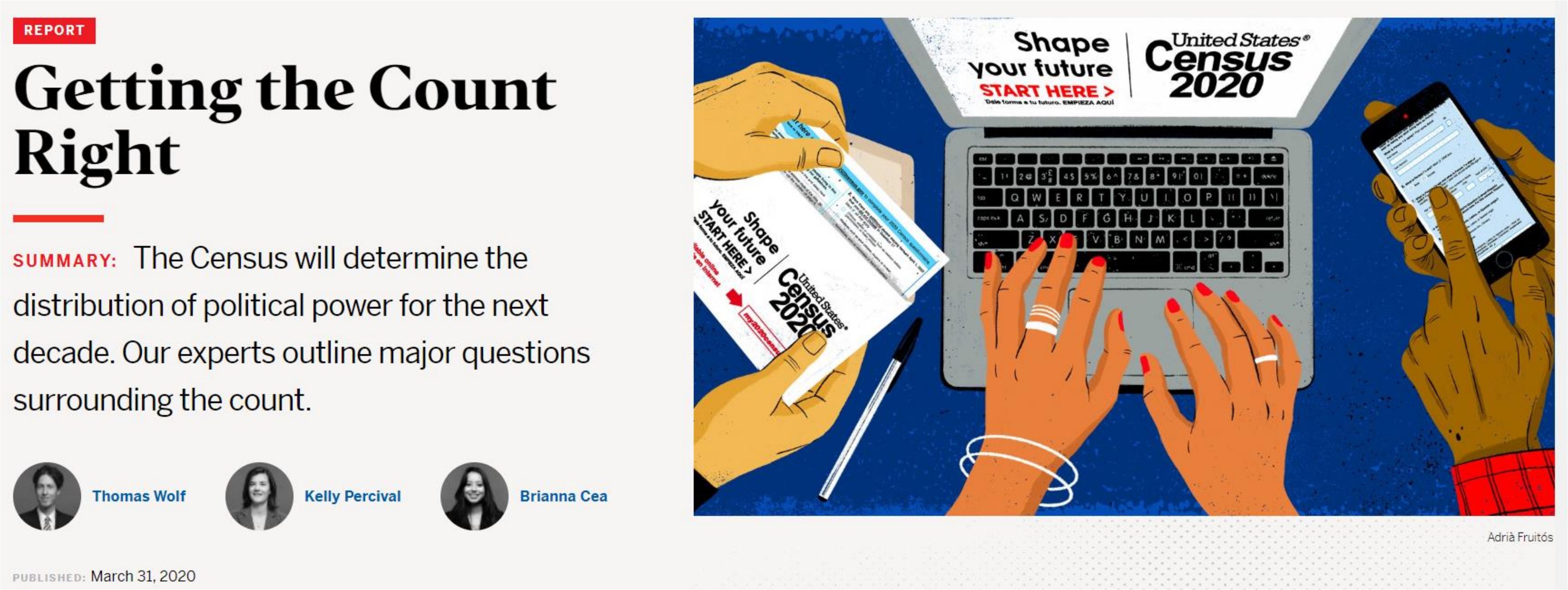
*"Though **there are no actual games to be played**, FiveThirtyEight is still taking a shot at a little March Madness. We built an NCAA Tournament bracket, using ESPN's Bracketology, and we're simulating the results of each game by using a **simple "100-sided dice roll" against our forecast probabilities**…"*

https://fivethirtyeight.com/features/it-was-a-roller-coaster-season-for-michigan-state-our-mens-bracket-champion/

Ethics in probability

https://2020census.gov/en.html

RESPOND



REPORT

Getting the Count Right

SUMMARY: The Census will determine the distribution of political power for the next decade. Our experts outline major questions surrounding the count.

Thomas Wolf Kelly Percival Brianna Cea

PUBLISHED: March 31, 2020

"Historically, the census has not counted all demographic groups equally well."

"If the Census undercounts communities of color, they risk going underrepresented — or unrepresented entirely — in bodies ranging from state legislatures to local school boards."

https://www.brennancenter.org/our-work/research-reports/getting-count-right

Buckets and Dividers

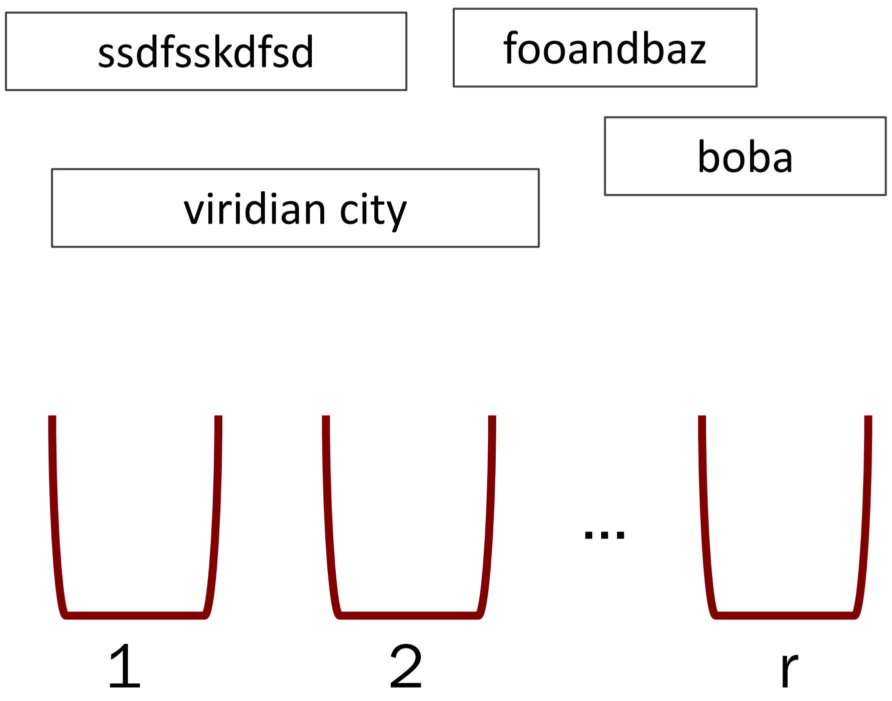
Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\, n_2! \cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct
  - Indistinct

~~Balls and urns~~ Hash tables and distinct strings

How many ways are there to hash n distinct strings to r buckets?



<u>Steps</u>:

1. Bucket 1st string
2. Bucket 2nd string

...

n. Bucket n^{th} string

r^n outcomes

Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
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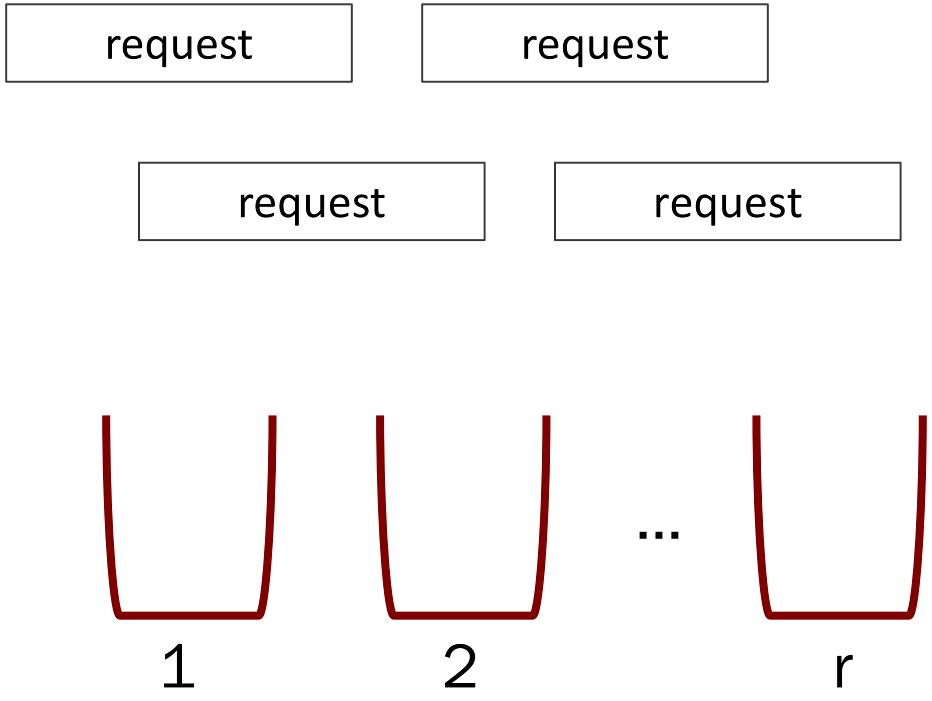
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct: r^n
  - Indistinct

Servers and indistinct requests

How many ways are there to distribute n indistinct web requests to r servers?



Goal

Server 1 has x_1 requests,

Server 2 has x_2 requests,

…

Server r has x_r requests (the rest)

Simple example: $n = 3$ requests and $r = 2$ servers

Bicycle helmet sales

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

Consider the following generative process...

Bicycle helmet sales: 1 possible assignment outcome

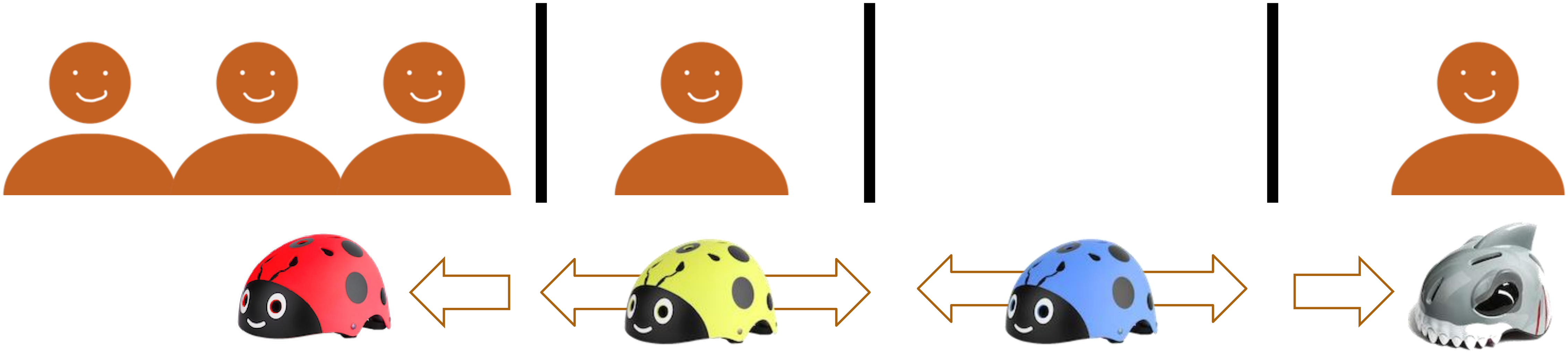
How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

$n = 5$ indistinct objects $\qquad$ $r = 4$ distinct buckets

Bicycle helmet sales: 1 possible assignment outcome

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

$n = 5$ indistinct objects $\qquad$ $r = 4$ distinct buckets



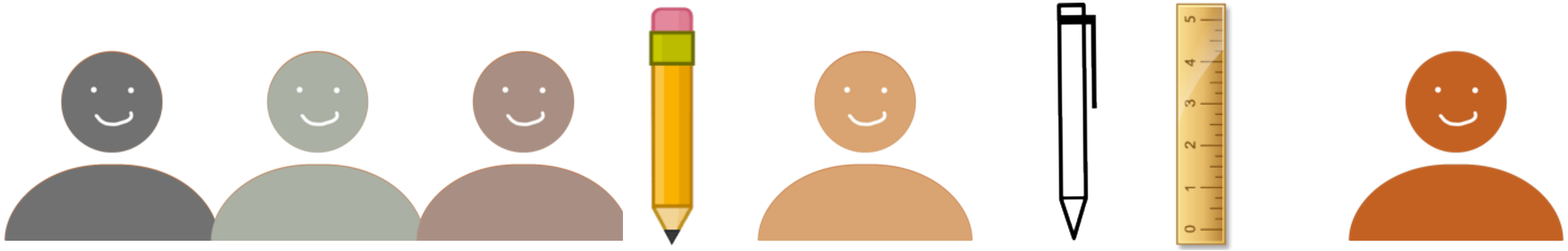
Goal Order n indistinct objects and $r - 1$ indistinct dividers.

Bicycle helmet sales: A generative proof

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

Goal Order n indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct



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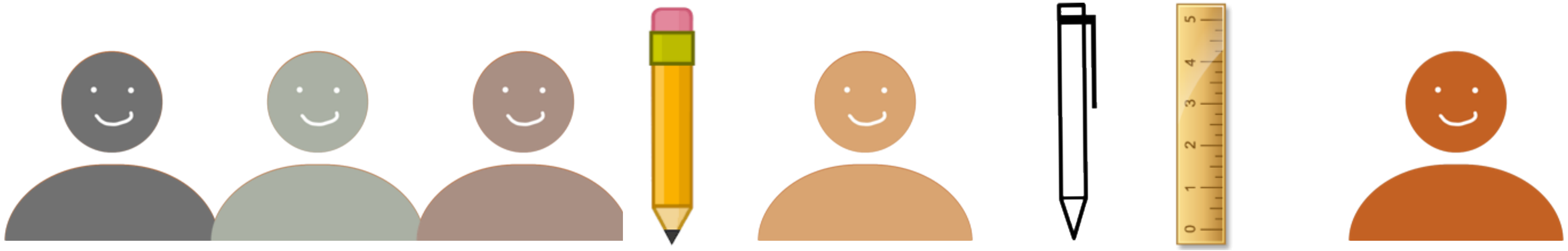

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

Bicycle helmet sales: A generative proof

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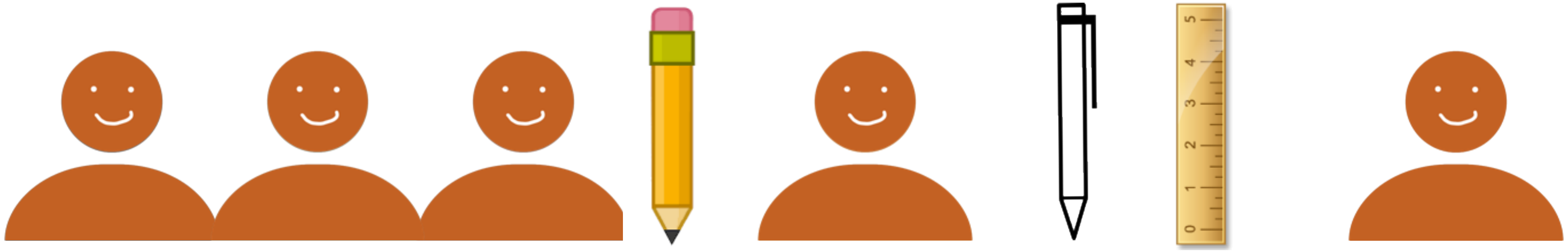

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

Bicycle helmet sales: A generative proof

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

Goal Order n indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

3. Make $r - 1$ dividers indistinct

$$\frac{1}{(r - 1)!}$$

Divider method

The number of ways to distribute n indistinct objects into r buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that

n are indistinct objects, and
$r - 1$ are indistinct dividers:

$$\text{Total} = (n + r - 1)! \times \frac{1}{n!} \times \frac{1}{(r-1)!}$$

$$= \binom{n + r - 1}{r - 1} \text{ outcomes}$$

Integer solutions to equations

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

How many integer solutions are there to the following equation:

$$x_1 + x_2 + \cdots + x_r = n,$$

where for all i, x_i is an integer such that $0 \le x_i \le n$?

Positive integer equations can be solved with the divider method.

Breakout Rooms

Hopefully you're all in the same rooms…

Then check out the three questions on the next slide. Post any clarifications here!

https://us.edstem.org/courses/667/discussion/80935

Breakout Room time: 4 minutes

We'll all come back as a big group to go over the answer.

Venture capitalists

Divider method (n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have $10 million to invest in 4 companies (in $1 million increments).

1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1? (and fully allocate)
3. What if you don't invest all your money?

Ask: https://us.edstem.org/courses/667/discussion/80935

Venture capitalists. #1

Divider method (n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

$$x_i \geq 0$$

Solve

Venture capitalists. #2

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

Solve

Venture capitalists. #3

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?
3. What if you don't invest all your money?

Set up

$$x_1 + x_2 + x_3 + x_4 \le 10$$

x_i: amount invested in company i

$$x_i \ge 0$$

Solve

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable)
  - Some distinct
- Choose k objects (combinations)
  - Distinct
    - 1 group
    - r groups
- Put objects in r buckets
  - Distinct
  - Indistinct

> - Determine if objects are distinct
> - Use Product Rule if several steps
> - Use Inclusion-Exclusion if different cases

See you next time...