03: Intro to Probability

Lisa Yan
April 10, 2020

Quick slide reference

3	Defining Probability	03a_definitions
13	Axioms of Probability	03b_axioms
20	Equally likely outcomes	03c_elo
30	Corollaries	03d_corollaries
37	Exercises	LIVE

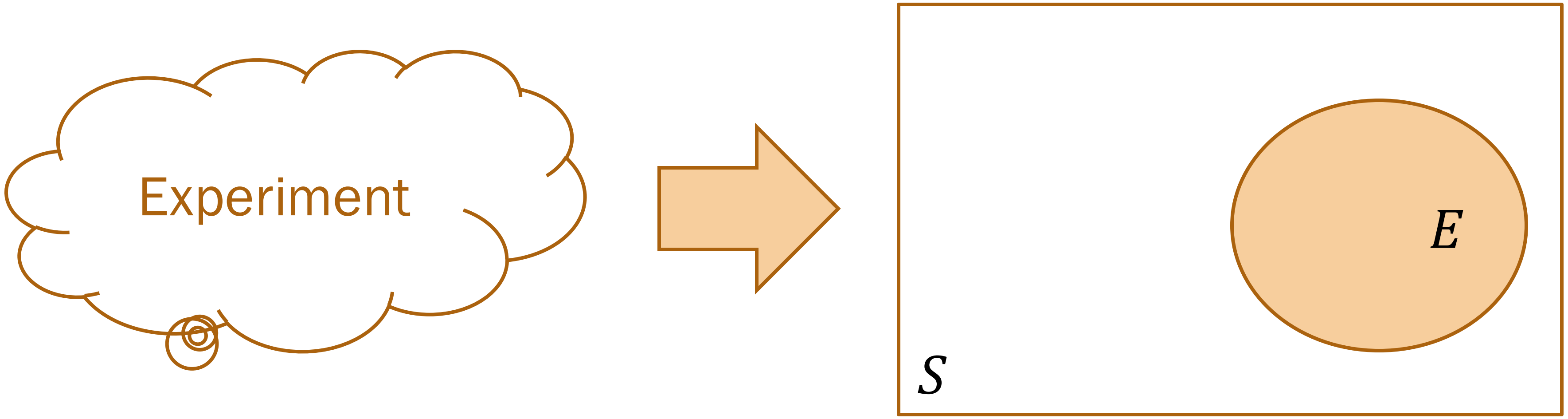
Today's discussion thread: https://us.edstem.org/courses/109/discussion/24492

Defining Probability

Gradescope quiz, blank slide deck, etc.
http://cs109.stanford.edu/

Key definitions

An experiment in probability:



Sample Space, S:	The set of all possible outcomes of an experiment
Event, E:	Some subset of S ($E \subseteq S$).

Key definitions

Sample Space, S

- Coin flip
  $S = \{\text{Heads}, \text{Tails}\}$
- Flipping two coins
  $S = \{(\text{H,H}), (\text{H,T}), (\text{T,H}), (\text{T,T})\}$
- Roll of 6-sided die
  $S = \{1, 2, 3, 4, 5, 6\}$
- # emails in a day
  $S = \{x \mid x \in \mathbb{Z}, x \geq 0\}$
- TikTok hours in a day
  $S = \{x \mid x \in \mathbb{R}, 0 \leq x \leq 24\}$

Event, E

- Flip lands heads
  $E = \{\text{Heads}\}$
- ≥ 1 head on 2 coin flips
  $E = \{(\text{H,H}), (\text{H,T}), (\text{T,H})\}$
- Roll is 3 or less:
  $E = \{1, 2, 3\}$
- Low email day (≤ 20 emails)
  $E = \{x \mid x \in \mathbb{Z}, 0 \leq x \leq 20\}$
- Wasted day (≥ 5 TT hours):
  $E = \{x \mid x \in \mathbb{R}, 5 \leq x \leq 24\}$

What is a probability?

A number between 0 and 1
to which we ascribe meaning.*

*our belief that an event E occurs.

What is a probability?

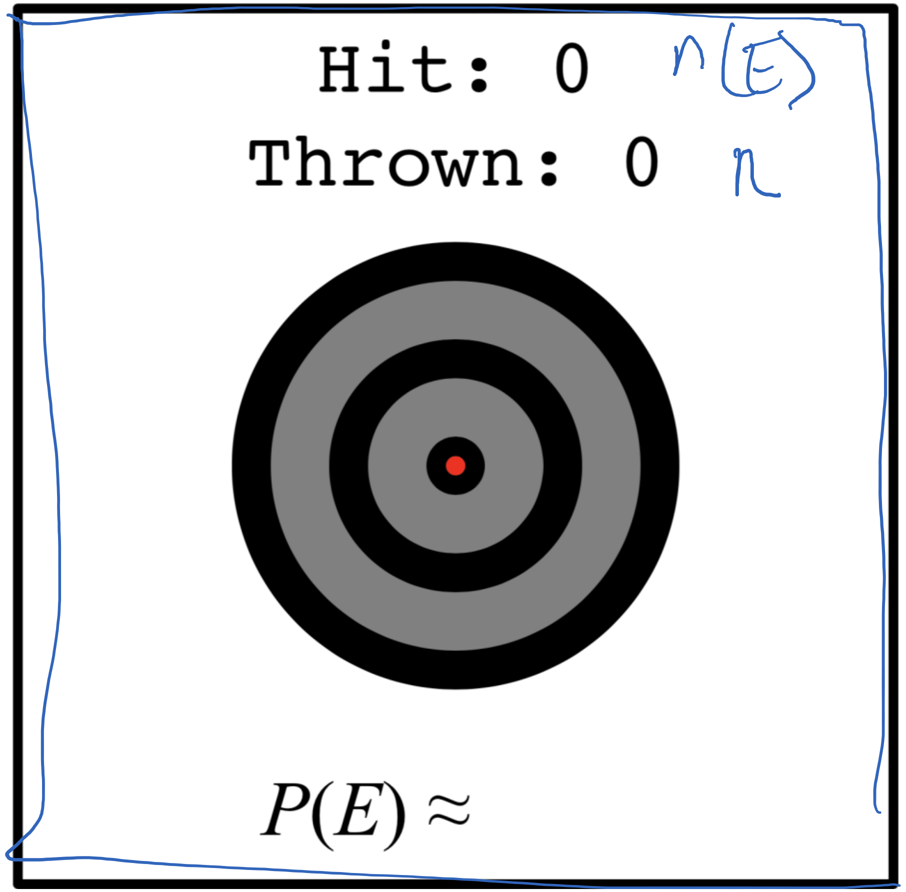
frequentist

$$\underline{P}(\underset{\uparrow}{E}) = \lim_{\underline{n}\to\infty} \frac{n(E)}{\underline{n}}$$

n = # of total trials

$n(E)$ = # trials where E occurs

Let E = the set of outcomes where you hit the target.



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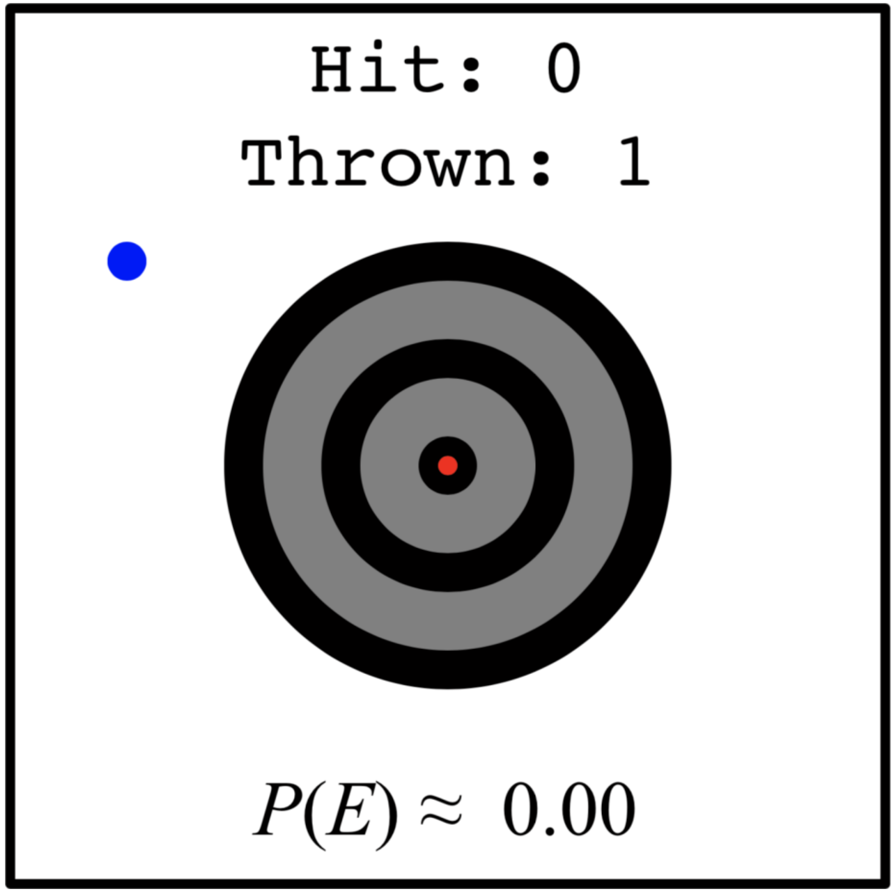

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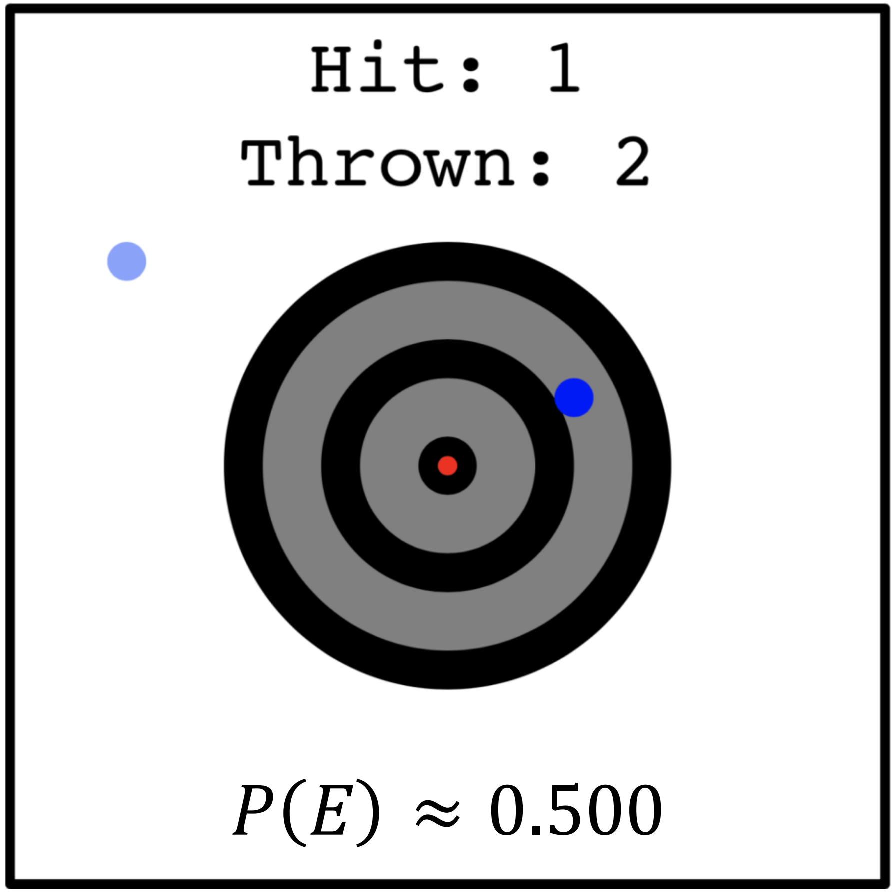

$P(E) \approx 0.500$

What is a probability?

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

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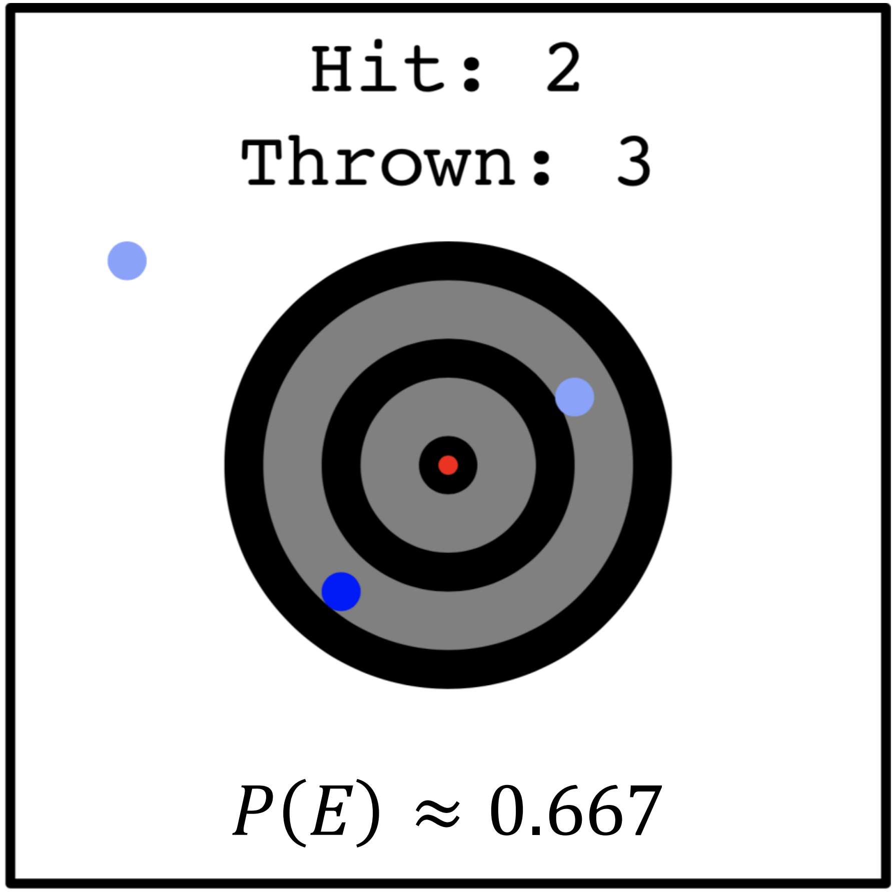
$n(E)$ = # trials where E occurs

Let E = the set of outcomes where you hit the target.



What is a probability?

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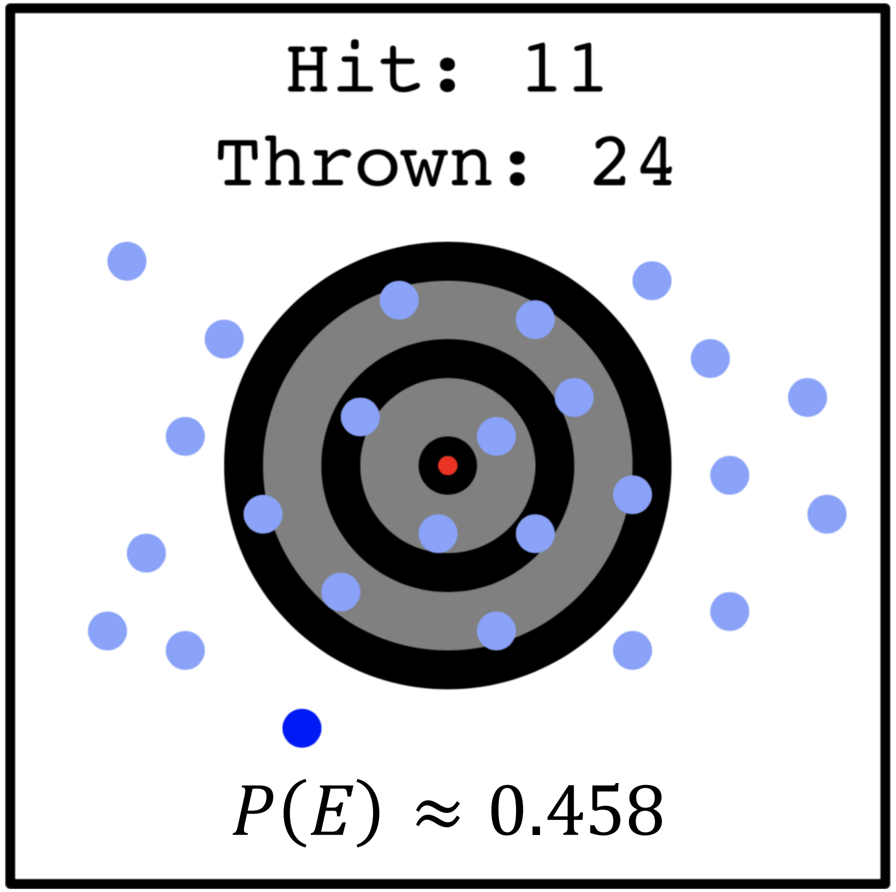
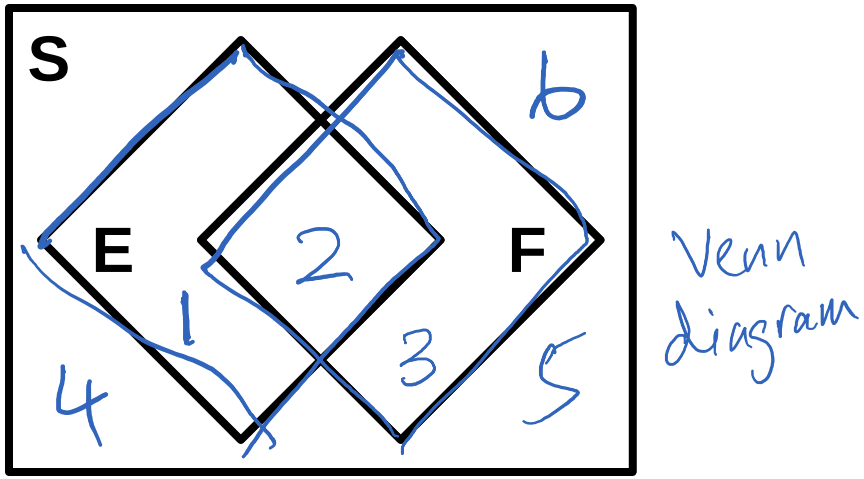


Not just yet...

03b_axioms

Axioms of Probability

Quick review of sets



E and F are events in S.

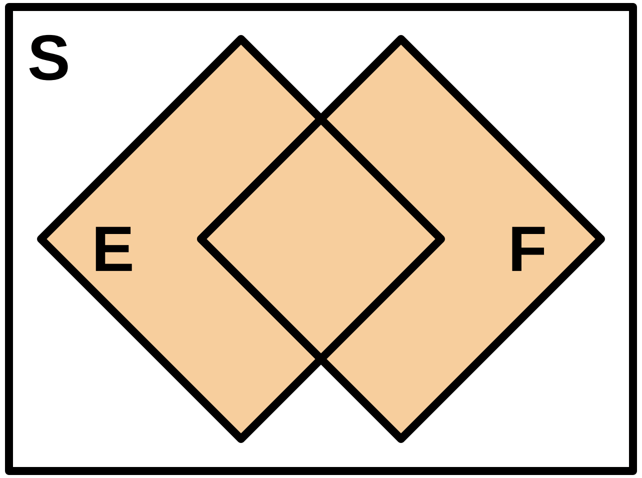
Experiment:

Die roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

Quick review of sets



E and F are events in S.

Experiment:

Die roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

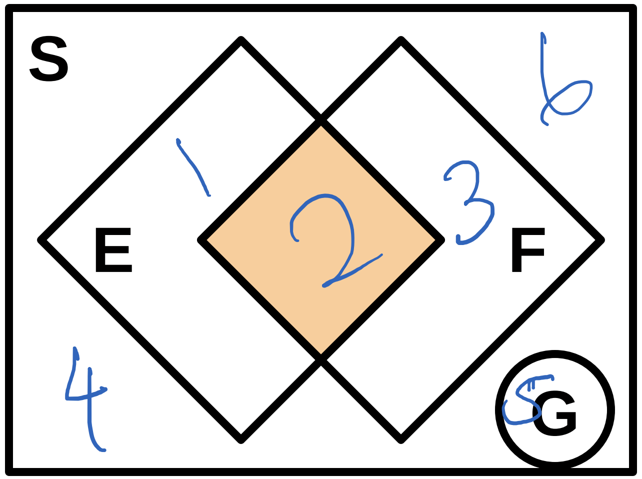
<u>def</u> Union of events, $E \cup F$

↑ \cup

The event containing all outcomes in E <u>or</u> F.

$$E \cup F = \{1,2,3\}$$

Quick review of sets



E and F are events in S.

Experiment:

Die roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

def Intersection of events, $E \cap F$

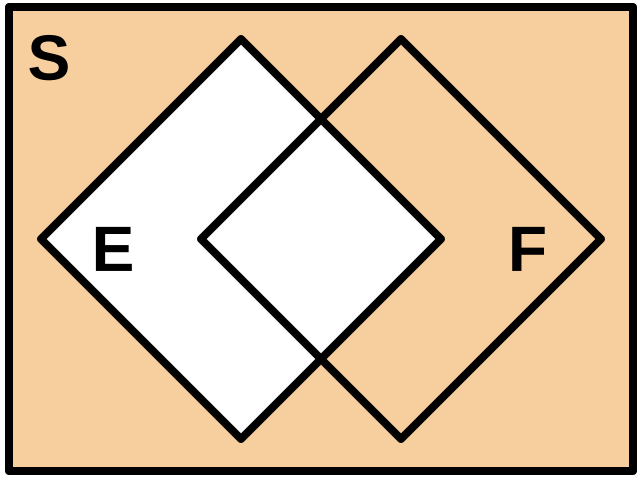
The event containing all outcomes in E **and** F.

$$\underset{\text{cap}}{E \cap F} = EF = \{2\}$$

def Mutually exclusive events F and G means that $F \cap G = \emptyset$

$G = \{5\}$

Quick review of sets



E and F are events in S.

Experiment:

Die roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

def Complement of event E, E^C

The event containing all outcomes in that are ***not*** in E.

$$E^C = \{3, 4, 5, 6\}$$

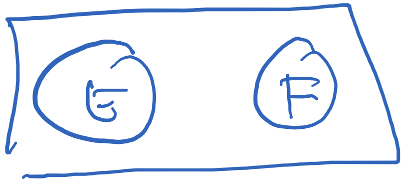
3 Axioms of Probability

Definition of probability: $P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$

Axiom 1: $0 \leq P(E) \leq 1$

Axiom 2: $P(S) = 1$

Axiom 3: If E and F are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$



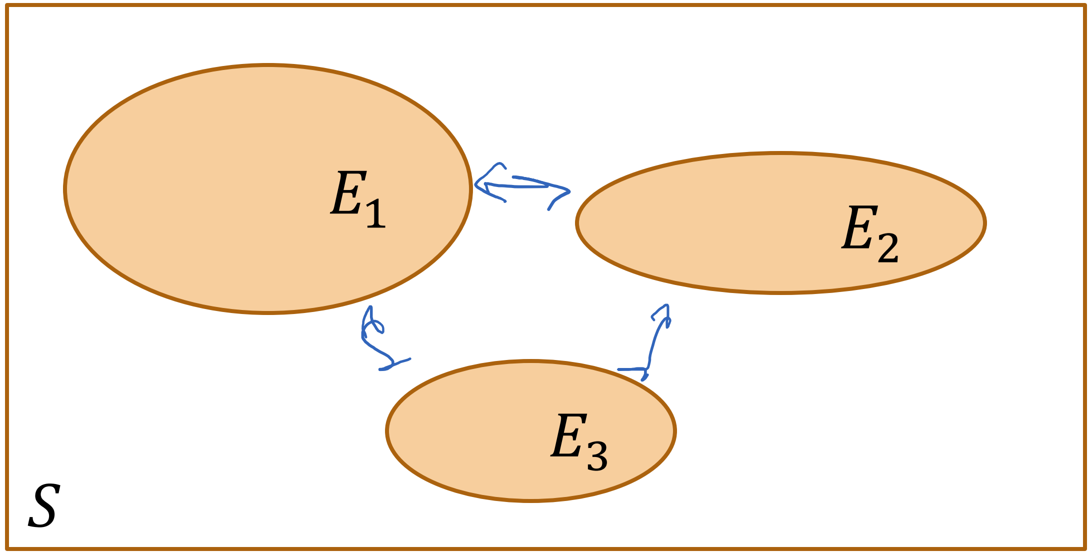
Axiom 3 is the (analytically) useful Axiom

Axiom 3: If E and F are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$

More generally, for any sequence of mutually exclusive events $E_1, E_2, \ldots$:

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$$



(like the Sum Rule of Counting, but for probabilities)

Equally Likely Outcomes

Equally Likely Outcomes

Some sample spaces have **equally likely outcomes**.

- Fair Coin flip: S = {Head, Tails}
- Flipping two fair coins: S = {(H, H), (H, T), (T, H), (T, T)}
- Roll of fair 6-sided die: S = {1, 2, 3, 4, 5, 6}

$P(\text{Heads}) = 1/2$

$P(\{(H,H)\}) = 1/4$

$P(\{3\}) = 1/6$

If we have equally likely outcomes, then P(Each outcome) $= \dfrac{1}{|S|}$

Therefore $$P(E) = \frac{\#\text{ outcomes in E}}{\#\text{ outcomes in S}} = \frac{|E|}{|S|}$$ (by Axiom 3)

E: 3 or lower

$E = \{1, 2, 3\}$

$E_1 = \{1\}, E_2 = \{2\},$

$E_3 = \{3\}, P(E_i) = \frac{1}{|S|}$

$$P(E) = P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$$

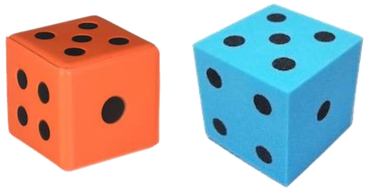
$$= 3 \cdot \frac{1}{|S|}$$

$$= |E| \cdot \frac{1}{|S|} = \frac{|E|}{|S|}$$

Roll two dice

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Roll two 6-sided fair dice. What is P(sum = 7)?



$$
\begin{aligned}
S = \{ & (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (\underline{1, 6}),\\
& (2, 1), (2, 2), (2, 3), (2, 4), (\underline{2, 5}), (2, 6),\\
& (3, 1), (3, 2), (3, 3), (\underline{3, 4}), (3, 5), (3, 6),\\
& (4, 1), (4, 2), (\underline{4, 3}), (4, 4), (4, 5), (4, 6),\\
& (5, 1), (\underline{5, 2}), (5, 3), (5, 4), (5, 5), (5, 6),\\
& (\underline{6, 1}), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}
\end{aligned}
$$

$|S| = 36$

$E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$ $|E| = 6$

Sum is 7

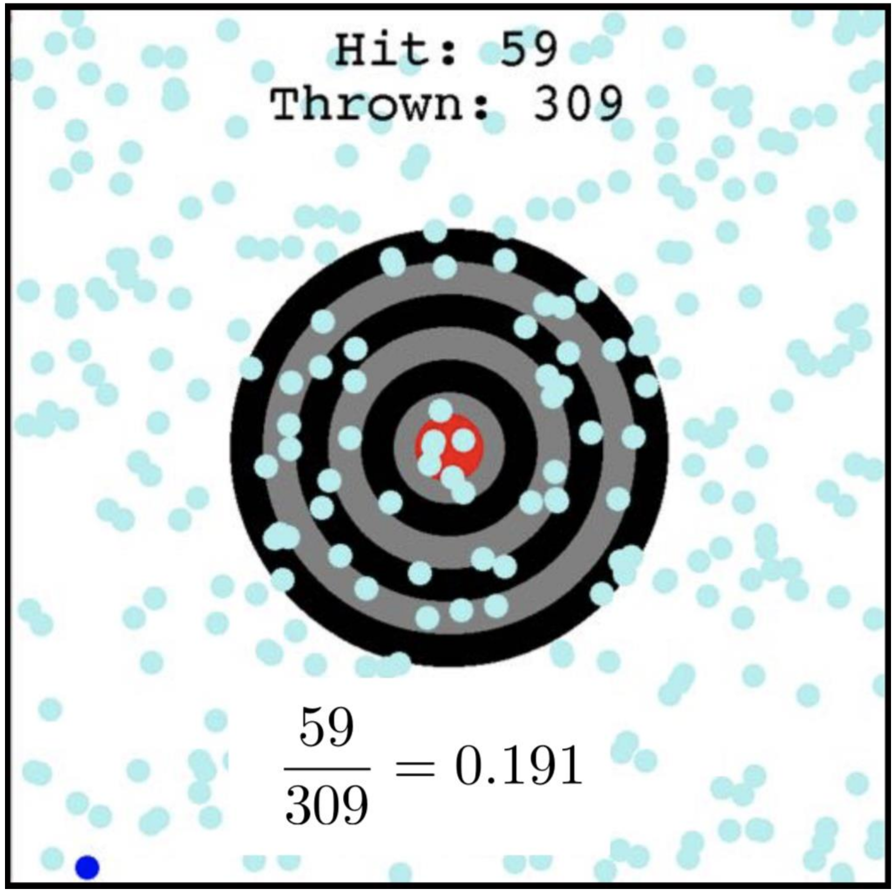
$P(E) = \frac{6}{36} = \boxed{\frac{1}{6}}$

Target revisited

Target revisited

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Let E = the set of outcomes where you hit the target.



Screen size = 800 × 800

Radius of target: 200

The dart is equally likely to land anywhere on the screen. What is $P(E)$, the probability of hitting the target?

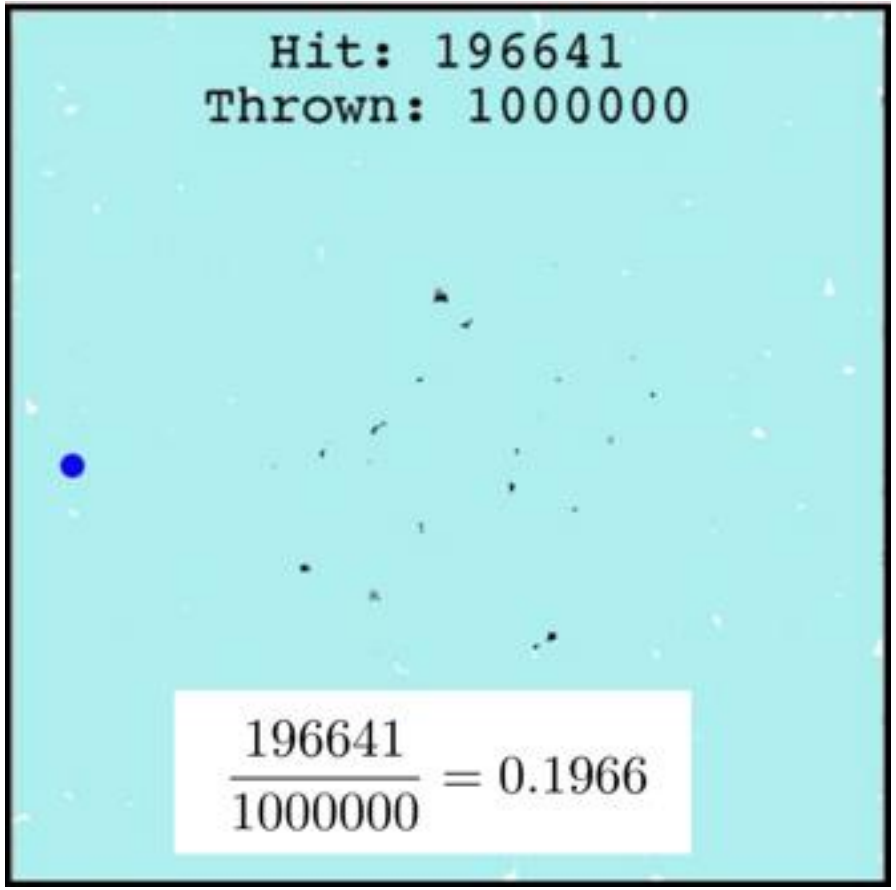
$|S| = 800^2$ $|E| \approx \pi \cdot 200^2$

$$P(E) = \frac{|E|}{|S|} \approx \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$

Target revisited

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Let E = the set of outcomes where you hit the target.



Hit: 196641
Thrown: 1000000

$$\frac{196641}{1000000} = 0.1966$$

Screen size = 800 × 800

Radius of target: 200

The dart is equally likely to land anywhere on the screen. What is $P(E)$, the probability of hitting the target?

$$|S| = 800^2 \qquad |E| \approx \pi \cdot 200^2$$

$$P(E) = \frac{|E|}{|S|} \approx \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$

Not equally likely outcomes

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Play the lottery.

What is $P(\text{win})$?

$S = \{\text{Lose}, \text{Win}\}$

$E = \{\text{Win}\}$

$$P(E) = \frac{|E|}{|S|} = \frac{1}{2} = 50\%?$$

41,416,355 tickets sold

1 winning

$P(\text{wins}) = 1/41416355$

The hard part: defining outcomes consistently across sample space and events

Cats and sharks

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 sharks in a bag. 3 drawn.

What is P(1 cat and 2 sharks drawn)?

Make indistinct items distinct to get equally likely outcomes.

A. $\frac{3}{7}$

B. $\frac{1}{4} \cdot \frac{2}{3}$

C. $\frac{4}{7} + 2 \cdot \frac{3}{6}$

D. $\frac{12}{35}$

E. Zero/other

Note: Do indistinct objects give you an equally likely sample space?

CCC
CCS
CSS
SSS

(No)

Cats and sharks (ordered solution)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 sharks in a bag. 3 drawn.

What is P(1 cat and 2 sharks drawn)?

Make indistinct items distinct to get equally likely outcomes.

Define

- S = Pick 3 distinct items
- E = 1 distinct cat, 2 distinct sharks

$\underline{7} \cdot \underline{6} \cdot \underline{5}$ $|S|=210$

pick C first: $\underset{c}{4} \cdot \underset{s}{3} \cdot \underset{s}{2}$

pick C second $\underset{s}{3} \cdot \underset{c}{4} \cdot \underset{s}{2}$

pick C third $\underset{s}{3} \cdot \underset{s}{2} \cdot \underset{c}{4}$

$+$

$|E| = 72$

$P(E) = \boxed{\frac{12}{35}} = \frac{72}{210}$

Cats and sharks (unordered solution)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 sharks in a bag. 3 drawn.

What is P(1 cat and 2 sharks drawn)?

Make indistinct items distinct to get equally likely outcomes.

Define

- S = Pick 3 distinct items
- E = 1 distinct cat, 2 distinct sharks

$|S| = \binom{7}{3} = \frac{7!}{3!4!} = 35$

$|E| = \binom{4}{1}\binom{3}{2} = 4 \cdot 3 = 12$ $P(E) = \boxed{\frac{12}{35}}$

03d_corollaries

Corollaries of Probability

Axioms of Probability

Review

Definition of probability: $P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$

Axiom 1: $0 \leq P(E) \leq 1$

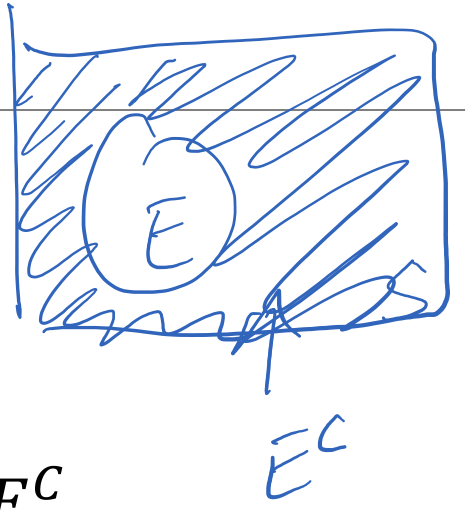
Axiom 2: $P(S) = 1$

Axiom 3: If E and F are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$

3 Corollaries of Axioms of Probability

Corollary 1: $P(E^C) = 1 - P(E)$

Proof of Corollary 1



Corollary 1: $P(E^C) = 1 - P(E)$

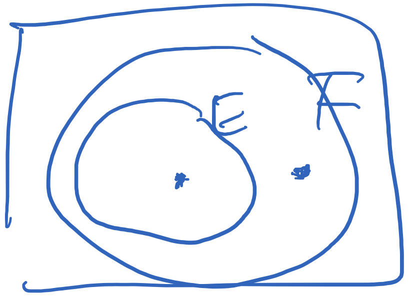
Proof:

E, E^C are mutually exclusive	Definition of E^C
$P(E \cup E^C) = P(E) + P(E^C)$	Axiom 3
$S = E \cup E^C$	Everything must either be in E or E^C, by definition
$1 = P(S) = P(E) + P(E^C)$	Axiom 2
$P(E^C) = 1 - P(E)$	Rearrange

3 Corollaries of Axioms of Probability

Corollary 1: $P(E^C) = 1 - P(E)$

Corollary 2: If $E \subseteq F$, then $P(E) \leq P(F)$



Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$

(Inclusion-Exclusion Principle for Probability)

Selecting Programmers

- P(student programs in Java) = 0.28 $= P(E)$
- P(student programs in Python) = 0.07 $= P(F)$
- P(student programs in Java and Python) = 0.05. $= P(E \cap F) = P(EF)$

What is P(student does not program in (Java or Python))?

1. Define events & state goal
2. Identify <u>known</u> probabilities
3. Solve

E: Java

F: Python

$$P((\underset{\substack{\uparrow \\ \text{Java}}}{E} \cup \underset{\substack{\uparrow \\ \text{Python}}}{F})^c) \leftarrow$$

Corollary 1: $P((E \cup F)^c) = 1 - P(\underline{E \cup F})$

Corollary 3: $P(\underline{E \cup F}) = P(E) + P(F) - P(EF)$

$= 0.28 + 0.07 - 0.05$

$= 0.3$

$$P((E \cup F)^c) = \boxed{0.7}$$

Inclusion-Exclusion Principle (Corollary 3)

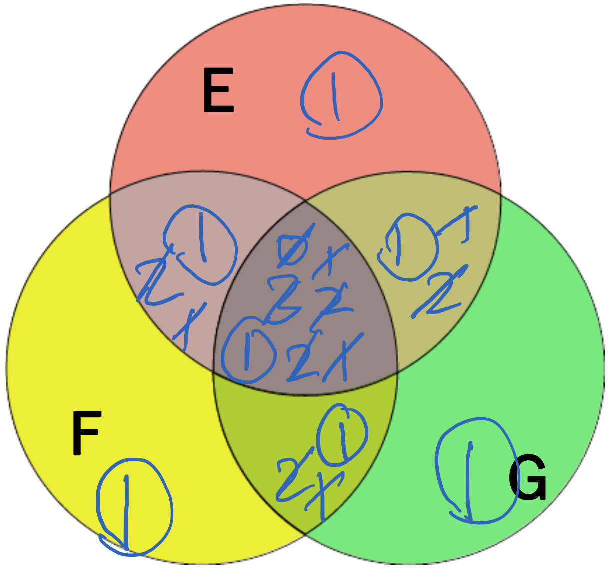
Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$

(Inclusion-Exclusion Principle for Probability)

General form:

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{(r+1)} \sum_{i_1 < \cdots < i_r} P\left(\bigcap_{j=1}^{r} E_{i_j}\right)$$

\# of sets in intersection



$P(E \cup F \cup G) =$

r = 1: $P(E) + P(F) + P(G)$

r = 2: $- P(E \cap F) - P(E \cap G) - P(F \cap G)$

r = 3: $+ P(E \cap F \cap G)$

(live)
03: Intro to Probability

Oishi Banerjee and Cooper Raterink
Adapted from Lisa Yan
June 26, 2020

Reminders: Lecture with zoom

- Turn on your camera if you are able, mute your mic in the big room
- Virtual backgrounds are encouraged (classroom-appropriate)

Breakout Rooms for meeting your classmates

- ~~Just like sitting next to someone new~~ Our best approximation to sitting next to someone new

We will use Ed instead of Zoom chat

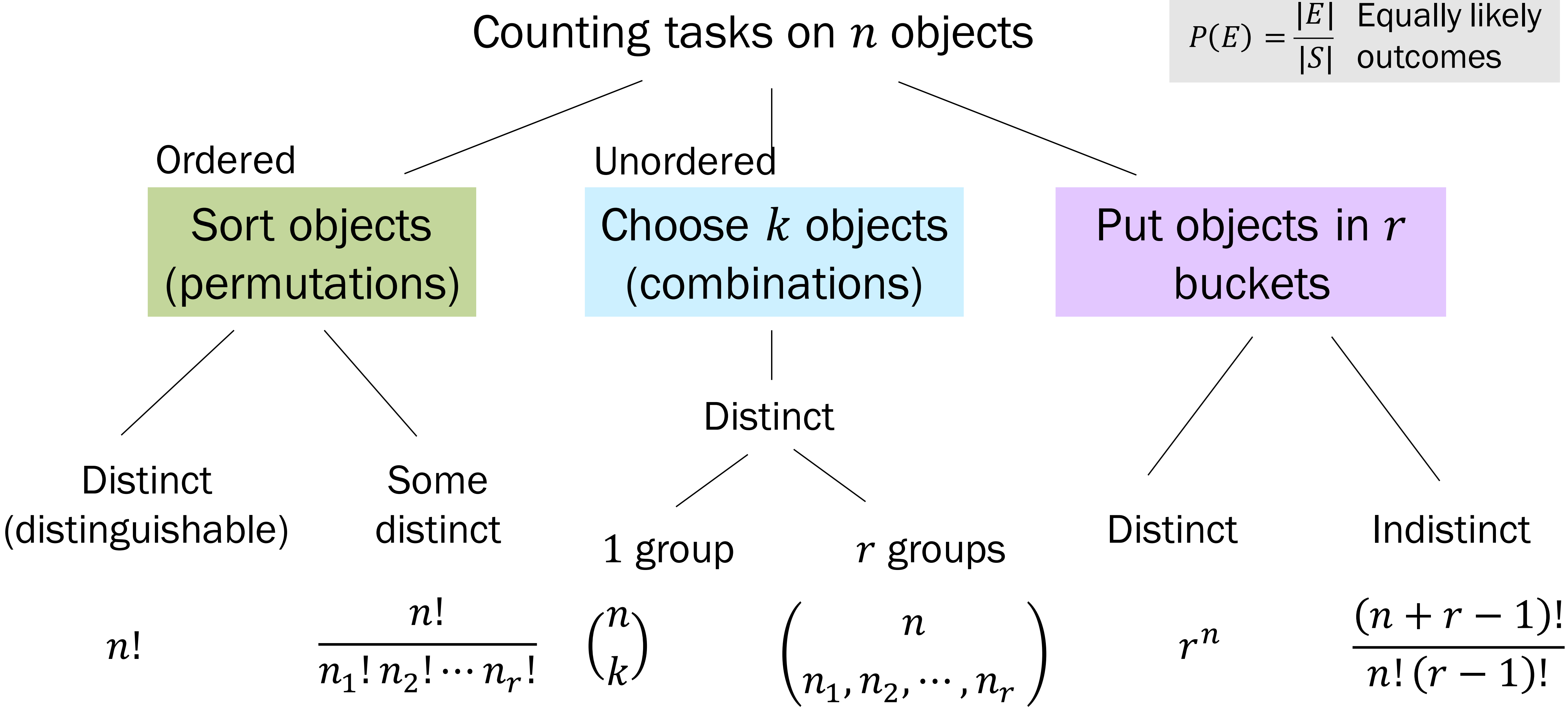
- Lots of activity and questions, thank you all!

Today's discussion thread: https://us.edstem.org/courses/667/discussion/82037

Summary so far

Review

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes



Counting tasks on n objects

Ordered

Sort objects (permutations)

Unordered

Choose k objects (combinations)

Put objects in r buckets

Distinct (distinguishable)

$n!$

Some distinct

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

Distinct

1 group

$\binom{n}{k}$

r groups

$$\binom{n}{n_1, n_2, \cdots, n_r}$$

Distinct

r^n

Indistinct

$$\frac{(n+r-1)!}{n!\,(r-1)!}$$

Indistinguishable? Distinguishable? Probability?

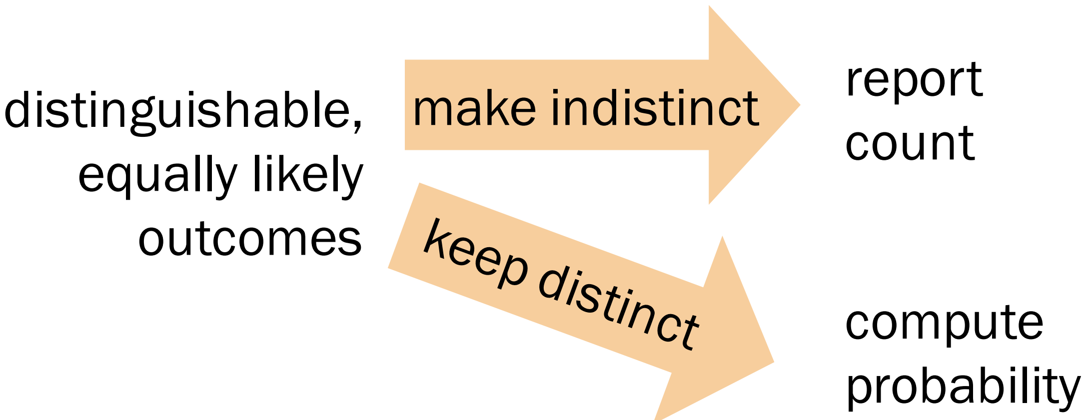
Review

We choose 3 books from a set of 4 distinct (distinguishable) and 2 indistinct (indistinguishable) books.

Let event E = our choice does not include both indistinct books.

1. What is $|E|$?

2. What is $P(E)$?



distinguishable, equally likely outcomes

make indistinct

report count

keep distinct

compute probability

Think, then Breakout Rooms

Then check out the question on the next slide (Slide 44). Post any clarifications here!

https://us.edstem.org/courses/667/discussion/82037

Think by yourself: 2 min

Breakout rooms: 5 min. Introduce yourself!

Poker Straights and Computer Chips

1. Consider 5-card poker hands.
   - "straight" is 5 consecutive rank cards of any suit

   What is P(Poker straight)?

- What is an example of an outcome?
- Is each outcome equally likely?
- Should objects be ordered or unordered?

2. Consider the "official" definition of a Poker Straight:
   - "straight" is 5 consecutive rank cards of any suit
   - straight flush" is 5 consecutive rank cards of same suit

   What is P(Poker straight, but not straight flush)?

3. Computer chips: n chips are manufactured, 1 of which is defective. k chips are randomly selected from n for testing.

   What is P(defective chip is in k selected chips?)

Any Poker Straight

1. Consider 5-card poker hands.
   - "straight" is 5 consecutive rank cards of any suit

   What is P(Poker straight)?

Define

- S (unordered)
- E (unordered, consistent with S)

Compute $P(\text{Poker straight}) =$

"Official" Poker Straight

Consider 5-card poker hands.

- "straight" is 5 consecutive rank cards of any suit
- "straight flush" is 5 consecutive rank cards of same suit

What is P(Poker straight, but not straight flush)?

Define

- S (unordered)
- E (unordered, consistent with S)

Compute $P(\text{Official Poker straight}) =$

Chip defect detection

n chips are manufactured, 1 of which is defective.
k chips are randomly selected from n for testing.

What is P(defective chip is in k selected chips?)

Define

- S (unordered)
- E (unordered,
  consistent with S)

Compute $P(E) =$

Chip defect detection, solution #2

n chips are manufactured, 1 of which is defective.
k chips are randomly selected from n for testing.

What is P(defective chip is in k selected chips?)

Redefine experiment

1. Choose k indistinct chips (1 way)
2. Throw a dart and make one defective

Define

- S (unordered)
- E (unordered, consistent with S)

Interlude for announcements

Section

Week 1’s section: pre-recorded Python review session
Week 2+: 12:30-1:30PT Thursdays, live on Zoom (will be recorded)

Interesting probability news

EPFL

NEWS

› News

Decoding Beethoven's music style using data science

"The study finds that **very few chords govern most of the music, a phenomenon that is also known in linguistics**, where very few words dominate language corpora.... It characterizes Beethoven's specific composition style for the String Quartets, through a distribution of all the chords he used, **how often they occur**, and how they commonly transition from one to the other."

https://actu.epfl.ch/news/decoding-beethoven-s-music-style-using-data-scienc/

3 Corollaries of Axioms of Probability

Review

Corollary 1: $P(E^C) = 1 - P(E)$

Corollary 2: If $E \subseteq F$, then $P(E) \leq P(F)$

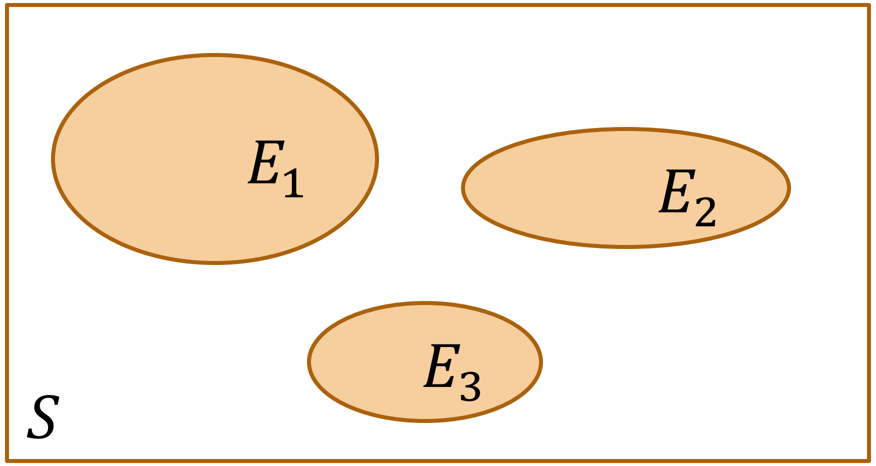
Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$

(Inclusion-Exclusion Principle for Probability)

Takeaway: Mutually exclusive events

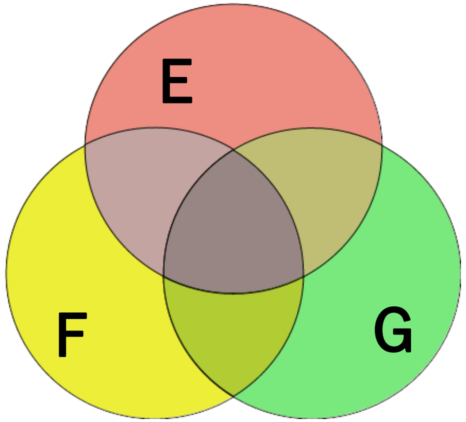
Review

Axiom 3,
Mutually exclusive events



$$P\left(\bigcup_{i=1}^{\infty} E_i\right) =$$

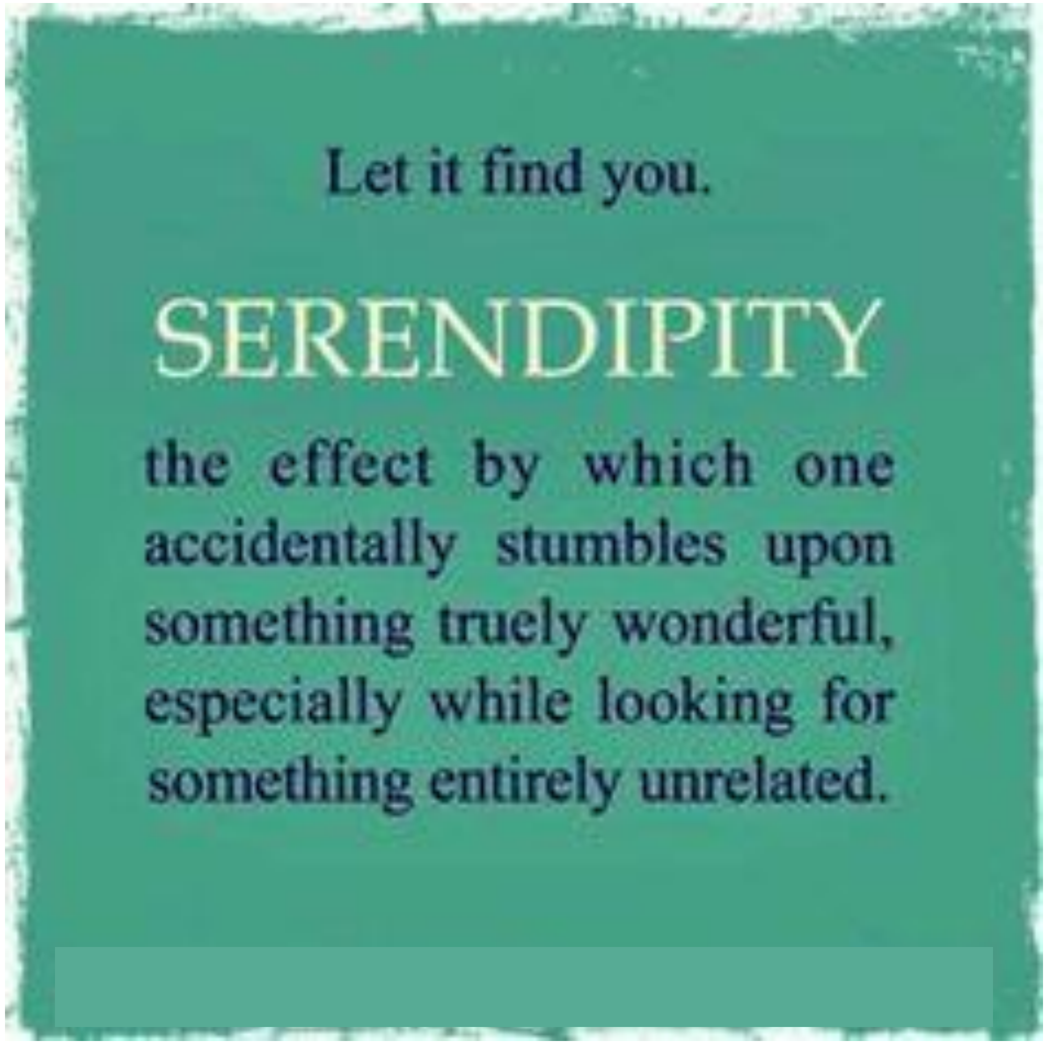
Inclusion-Exclusion Principle



$$P\left(\bigcup_{i=1}^{\infty} E_i\right) =$$

The challenge of probability is in defining events.
Some event probabilities are easier to compute than others.

Serendipity



Let it find you.

SERENDIPITY

the effect by which one accidentally stumbles upon something truely wonderful, especially while looking for something entirely unrelated.

Serendipity

- The population of Stanford is $n = 17{,}000$ people.
- You are friends with $r =$ people.
- Walk into a room, see $k = 360$ random people.
- Assume you are equally likely to see each person at Stanford.

What is the probability that you see someone you know in the room?

Breakout Rooms

Check out the question on the next slide (Slide 57). Post any clarifications here!

https://us.edstem.org/courses/667/discussion/82037

Breakout rooms: 5 min. Introduce yourself if you haven’t yet!

Serendipity

- The population of Stanford is $n = 17{,}000$ people.
- You are friends with $r = 100$ people.
- Walk into a room, see $k = 360$ random people.
- Assume you are equally likely to see each person at Stanford.

What is the probability that you see someone you know in the room?

Define

- S (unordered)
- E: ≥ 1 friend in the room

What strategy should you use?

A. $P(\text{exactly } 1) + P(\text{exactly } 2)$ $P(\text{exactly } 3) + \cdots$

B. $1 - P(\text{see no friends})$

Serendipity

- The population of Stanford is $n = 17{,}000$ people.
- You are friends with $r = 100$ people.
- Walk into a room, see $k = 360$ random people.
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What is the probability that you see someone you know in the room?

Define

- S (unordered)
- E: ≥ 1 friend in the room

It is often much easier to compute $P(E^c)$.

The Birthday ~~Paradox~~ Problem

What is the probability that in a set of *n* people, at least one pair of them will share the same birthday?

For you to think about (and discuss in section!)

Card Flipping

In a 52 card deck, cards are flipped one at a time.
After the first ace (of any suit) appears, consider the next card.

Is P(next card = Ace Spades) < P(next card = 2 Clubs)?

(by yourself)

Card Flipping

In a 52 card deck, cards are flipped one at a time.
After the first ace (of any suit) appears, consider the next card.

Is P(next card = Ace Spades) < P(next card = 2 Clubs)?

Sample space | S | = 52!

Event E_{AS}, next card is Ace Spades

1. Take out Ace of Spades.
2. Shuffle leftover 51 cards.
3. Add Ace Spades after first ace.

$$|E_{AS}| = 51! \cdot 1$$

E_{2C}, next card is 2 Clubs

1. Take out 2 Clubs.
2. Shuffle leftover 51 cards.
3. Add 2 Clubs after first ace.

$$|E_{2C}| = 51! \cdot 1$$

$$P(E_{AS}) = P(E_{2C})$$