15: General Inference

Lisa Yan

May 8, 2020

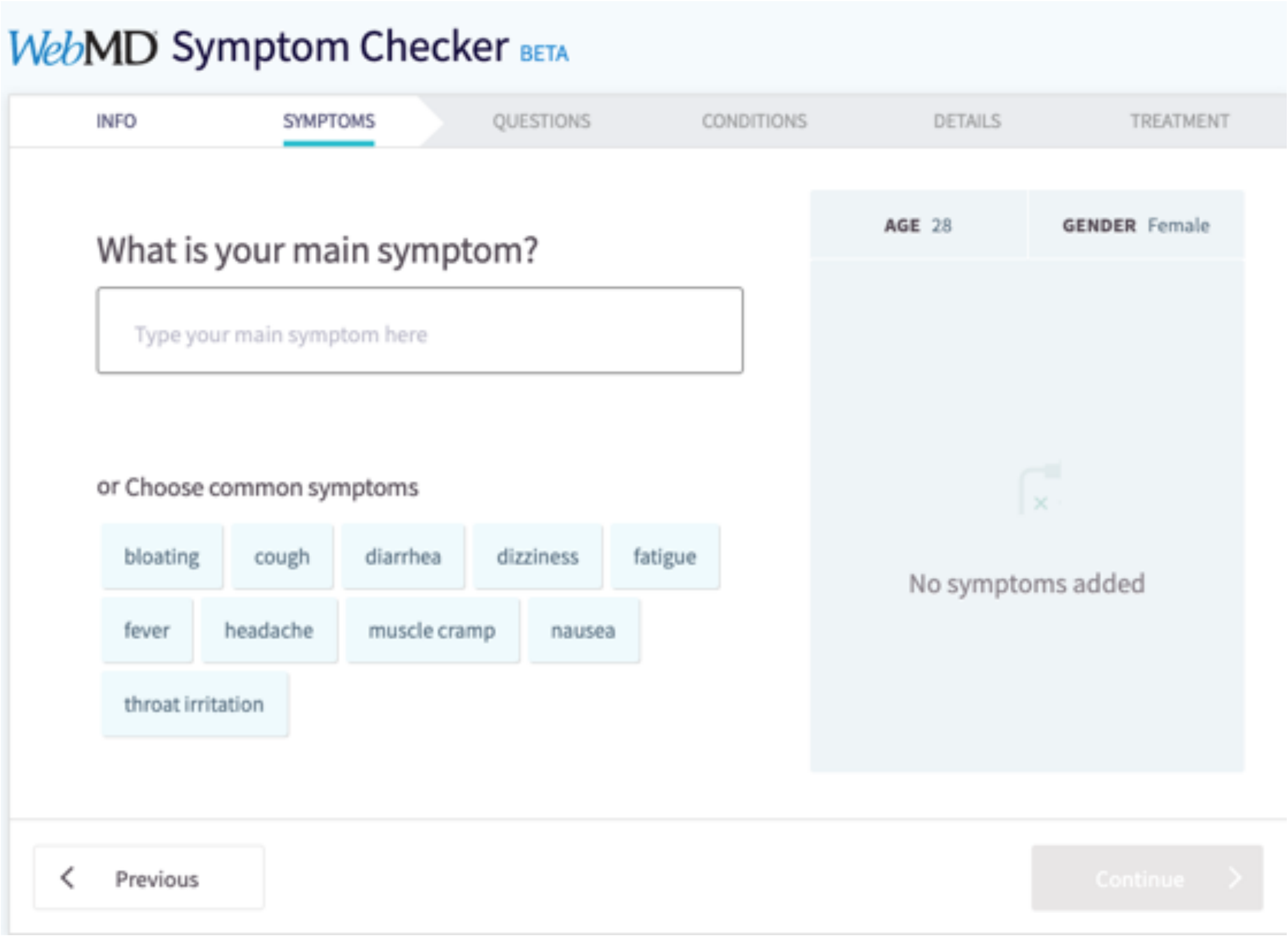
Quick slide reference

General Inference: Introduction

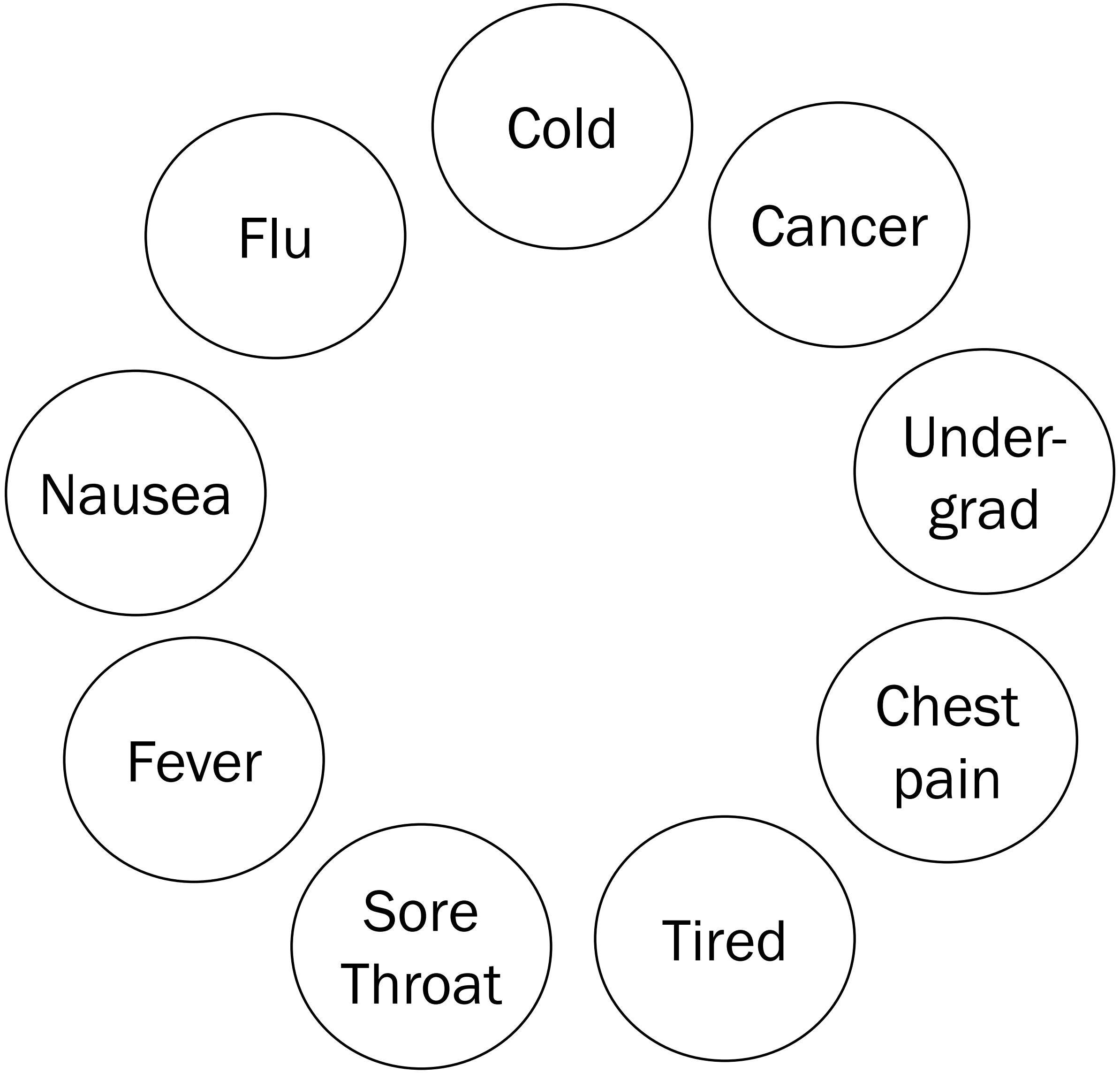
Inference

WebMD®

Inference


WebMD Symptom Checker BETA
INFO
SYMPTOMS
QUESTIONS
CONDITIONS
DETAILS
TREATMENT
What is your main symptom?
Type your main symptom here
or Choose common symptoms
bloating
cough
diarrhea
dizziness
fatigue
fever
headache
muscle cramp
nausea
throat irritation
AGE 28
GENDER Female
No symptoms added
Previous
Continue

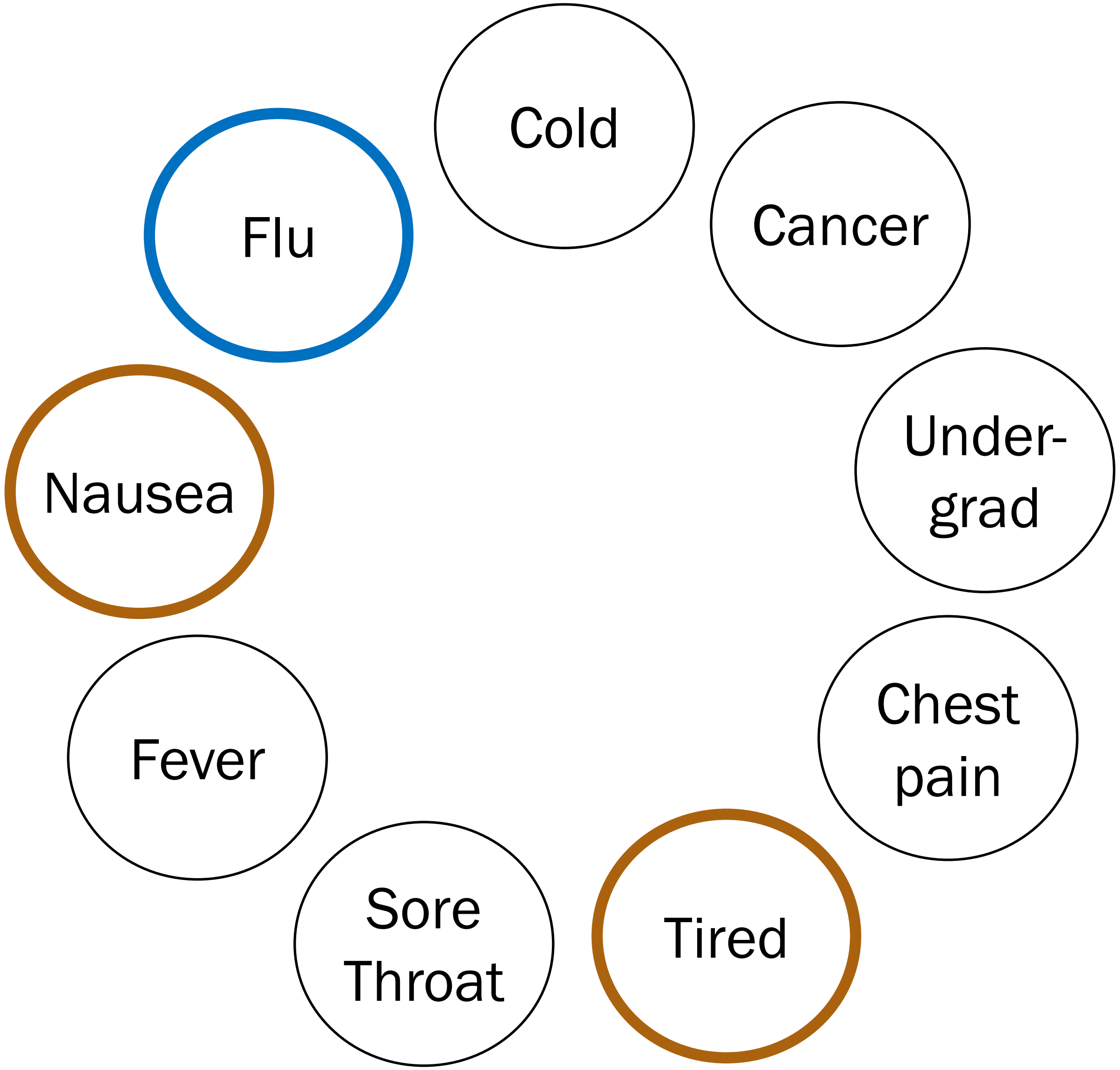
Inference



General inference question:

Given the values of some random variables, what is the conditional distribution of some other random variables?

Inference

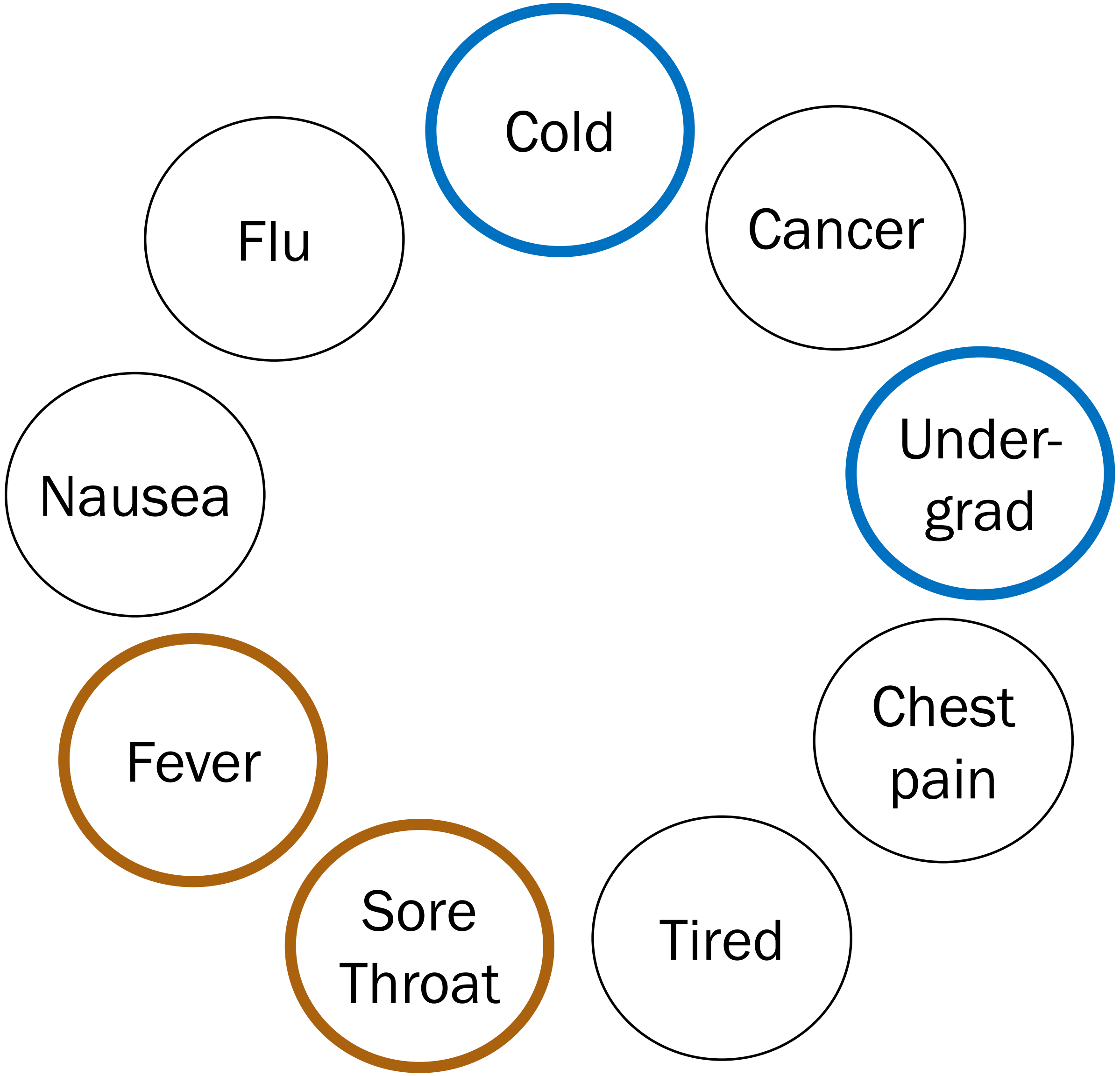


One inference question:

$$P(F = 1|N = 1, T = 1)$$

$$= \frac{P(F = 1, N = 1, T = 1)}{P(N = 1, T = 1)}$$

Inference

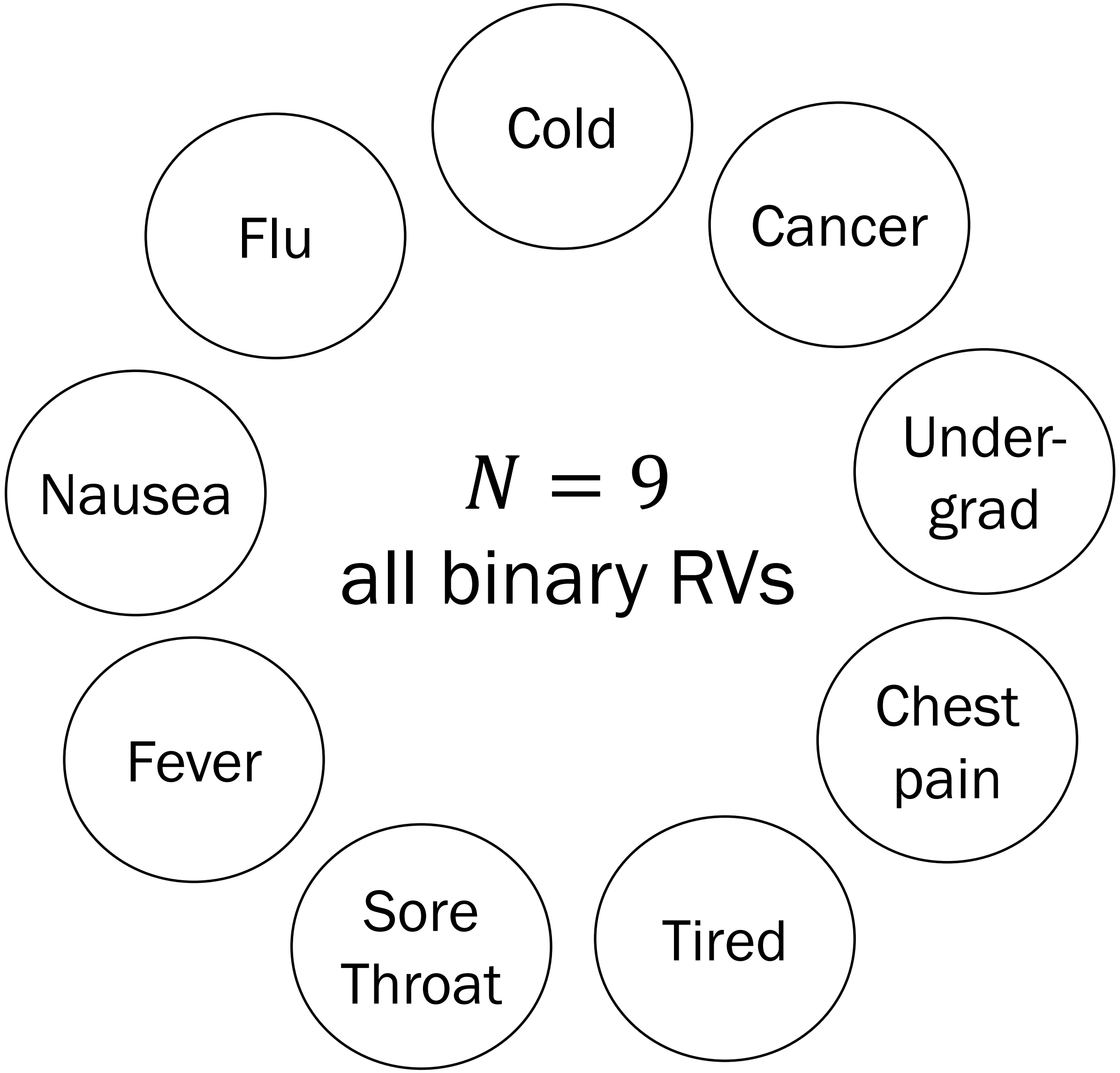


Another inference question:

$$P(C_o = 1, U = 1 | S = 0, F_e = 0)$$

$$= \frac{P(C_o = 1, U = 1, S = 0, F_e = 0)}{P(S = 0, F_e = 0)}$$

Inference



If we knew the **joint distribution**, we can answer all probabilistic inference questions.

What is the size of the joint probability table?

A. 2^{N-1} entries
B. N^2 entries
C. 2^N entries
D. None/other/don't know

Inference

Cold

Flu

Cancer

Nausea

$N = 9$

all binary RVs

Under-grad

Fever

Chest pain

Sore Throat

Tired

If we knew the **joint distribution**, we can answer all probabilistic inference questions.

What is the size of the joint probability table?

A. 2^{N-1} entries $2^N - 1$

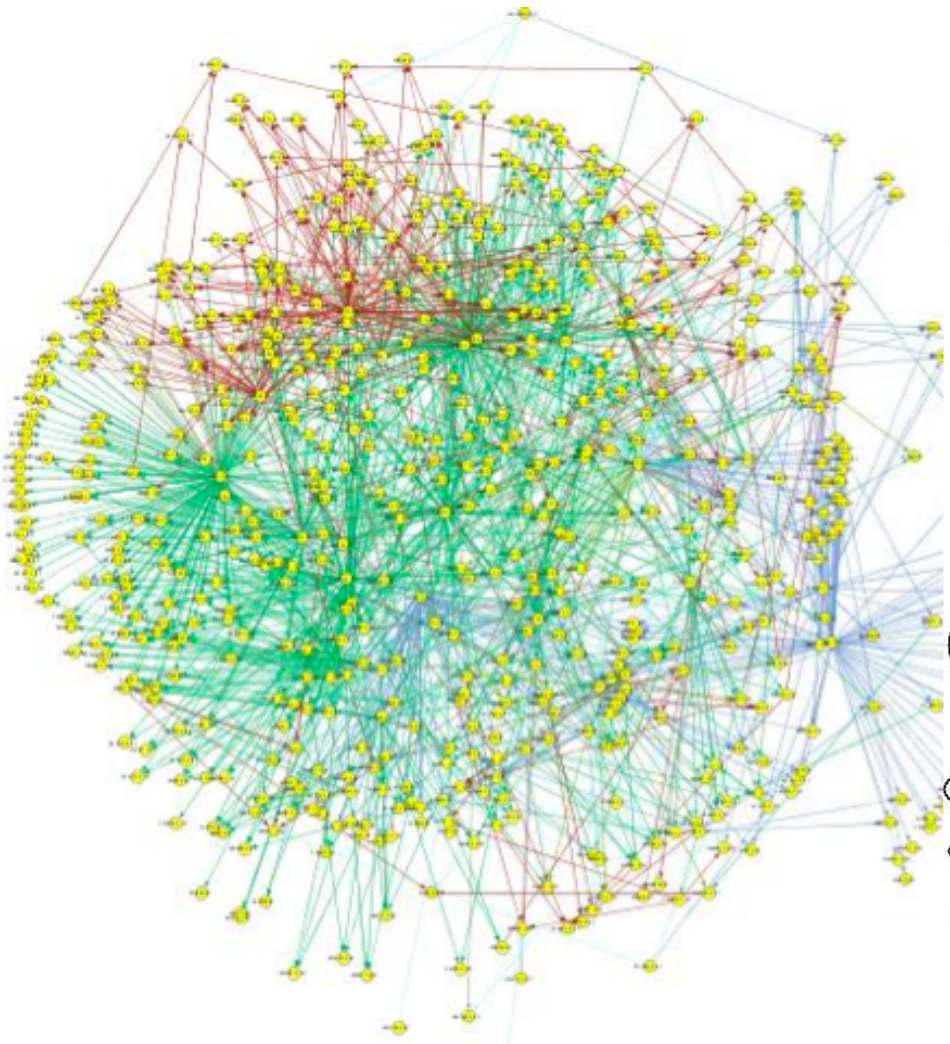
B. N^2 entries $P(X_1 = \_, X_2 = \_, \ldots, X_9 = \_)$

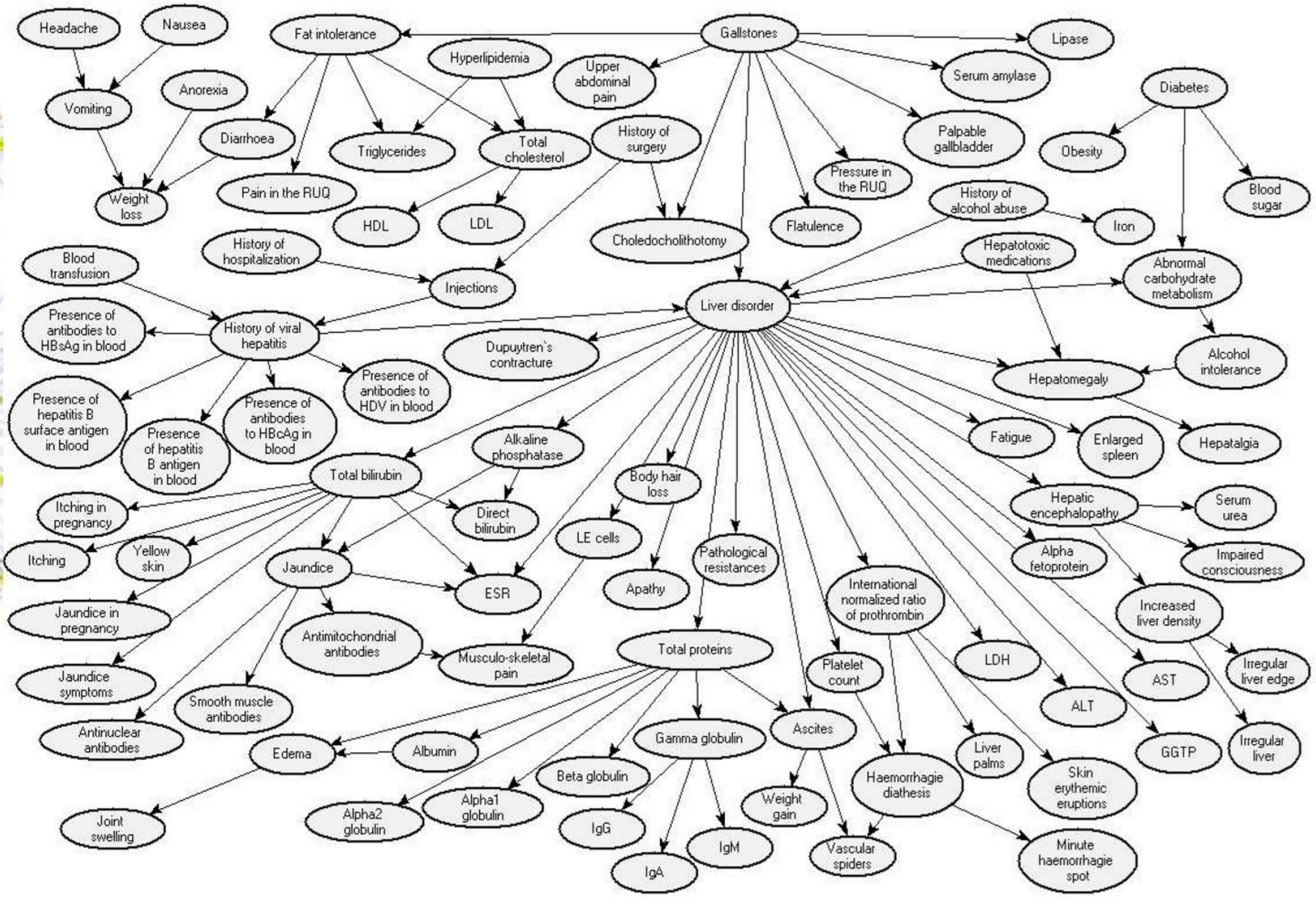
C. 2^N entries

D. None/other/don't know

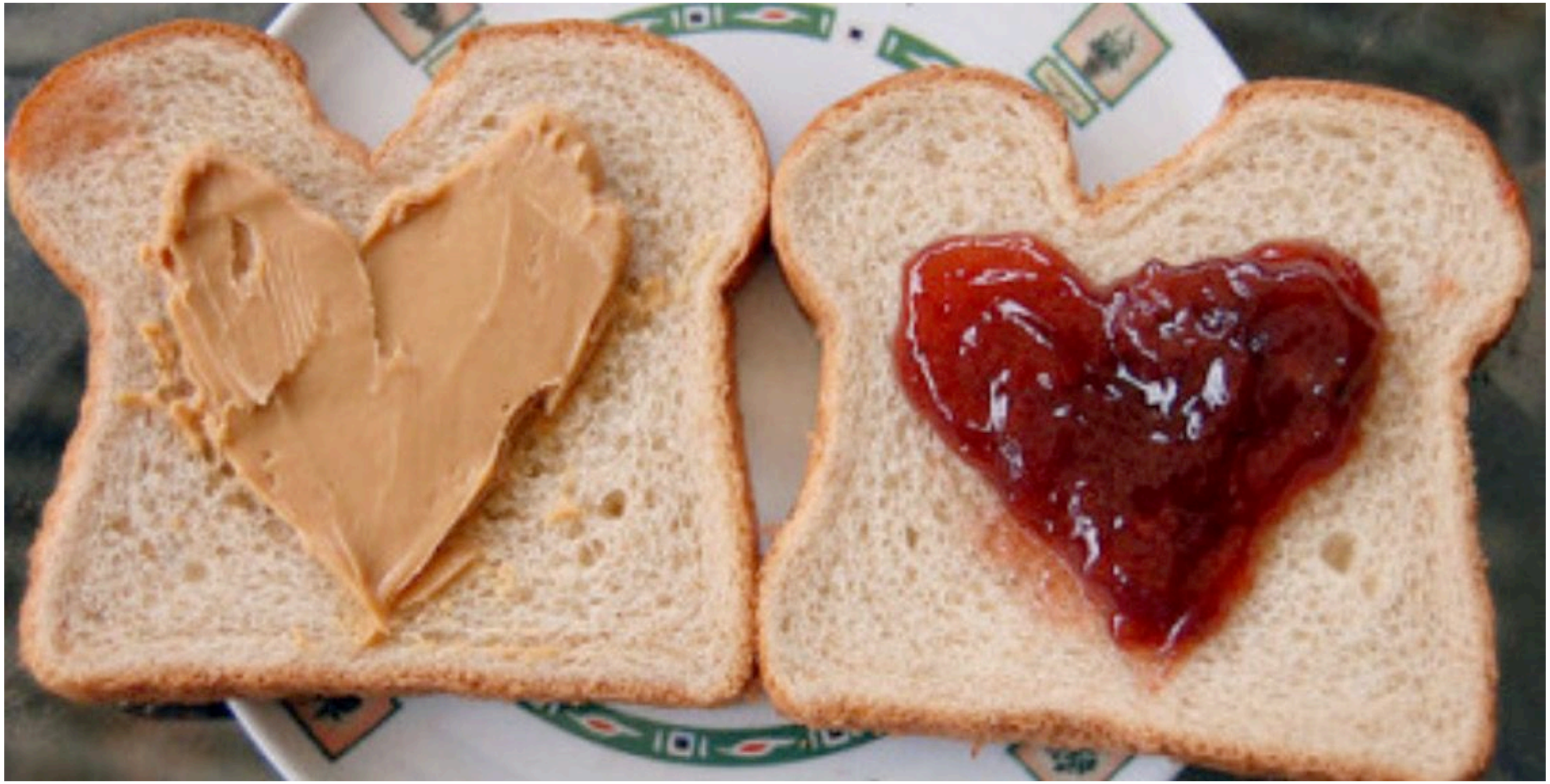
Naively specifying a joint distribution is often intractable.

N can be large…





Conditionally Independent RVs



~~Conditional Probability~~
Conditional Distributions

~~Independence~~
Independent RVs

Conditionally Independent RVs

Recall that two events A and B are conditionally independent given E if:

$$P(AB|E) = P(A|E)P(B|E)$$

n discrete random variables $X_1, X_2, \ldots, X_n$ are called **conditionally independent given Y** if:

for all $x_1, x_2, \ldots, x_n, y$:

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n | Y = y) = \prod_{i=1}^{n} P(X_i = x_i | Y = y)$$

This implies the following (cool to remember for later):

$$\log P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n | Y = y) = \sum_{i=1}^{n} \log P(X_i = x_i | Y = y)$$

Lec. 12: Independence of multiple random variables

Errata

Recall independence of n events $E_1, E_2, \dots, E_n$:

for $r = 1, \dots, n$:
  for every subset $E_1, E_2, \dots, E_r$:
    $P(E_1, E_2, \dots, E_r) = P(E_1)P(E_2)\cdots P(E_r)$

We have independence of n discrete random variables $X_1, X_2, \dots, X_n$ if
  for all $x_1, x_2, \dots, x_n$:

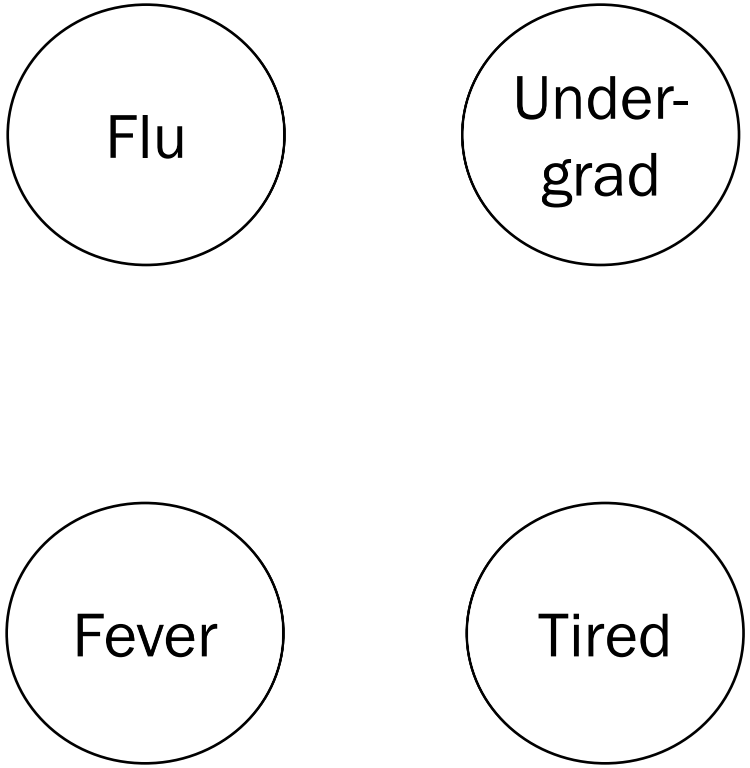
$$P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = \prod_{i=1}^{n} P(X_i = x_i)$$

Errata (edited May 3): **Removed the independent RV requirement for all subsets of size $r = 1, \dots, n$.** Do you see why this requirement is unnecessary?
(Hint: independence of RVs implies independence of all events)

Bayesian Networks

Bayes Nets

A simpler WebMD



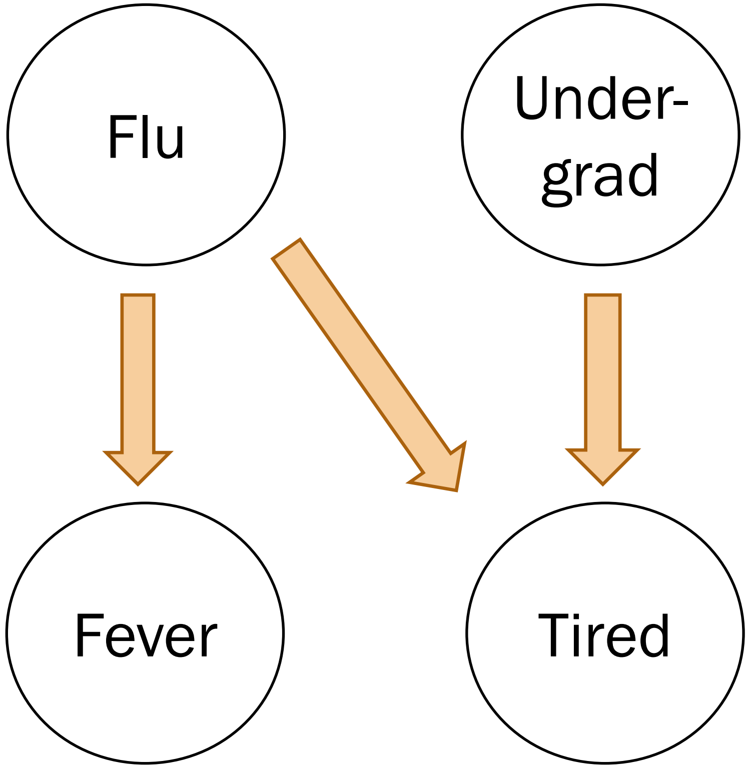
Great! Just specify $2^4 = 16$ joint probabilities…?

$$P(F_{lu} = a, F_{ev} = b, U = c, T = d)$$

What would a Stanford flu expert do?

Describe the joint distribution using causality!!!

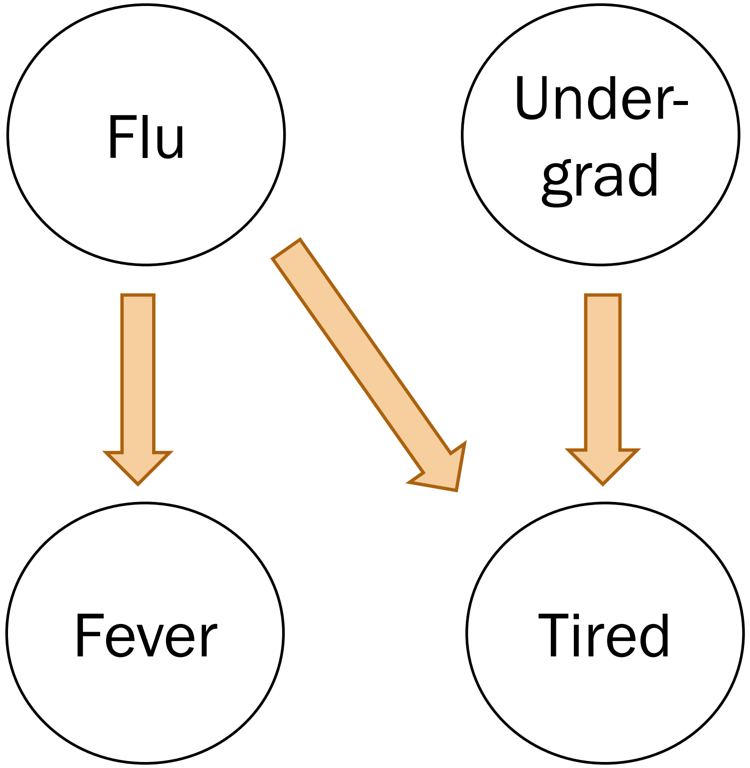
Constructing a Bayesian Network



What would a Stanford flu expert do?

1. Describe the joint distribution using causality.
2. <u>**Assume conditional independence.**</u>

Constructing a Bayesian Network



In a Bayesian Network,

Each random variable is conditionally independent of its non-descendants, given its parents.

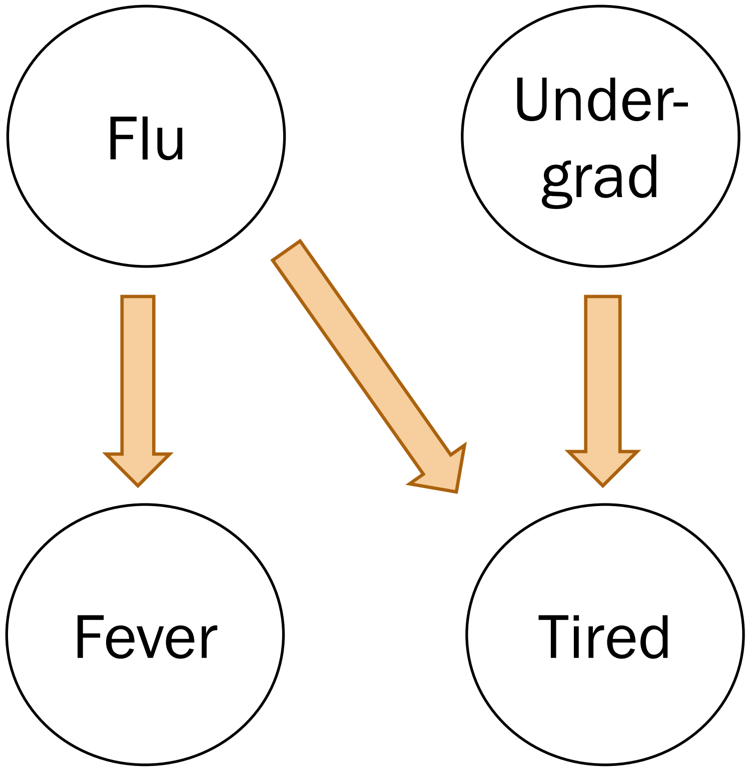
- Node: random variable
- Directed edge: conditional dependency

Examples:

- $P(F_{ev} = 1 | T \neq 0, F_{lu} = 1) = P(F_{ev} = 1 | F_{lu} = 1)$
- $P(F_{lu} = 1, U = 0) = P(F_{lu} = 1)P(U = 0)$

Constructing a Bayesian Network

$P(F_{lu} = 1) = 0.1$ $P(U = 1) = 0.8$



$P(F_{ev} = 1 | F_{lu} = 1) = 0.9$
$P(F_{ev} = 1 | F_{lu} = 0) = 0.05$

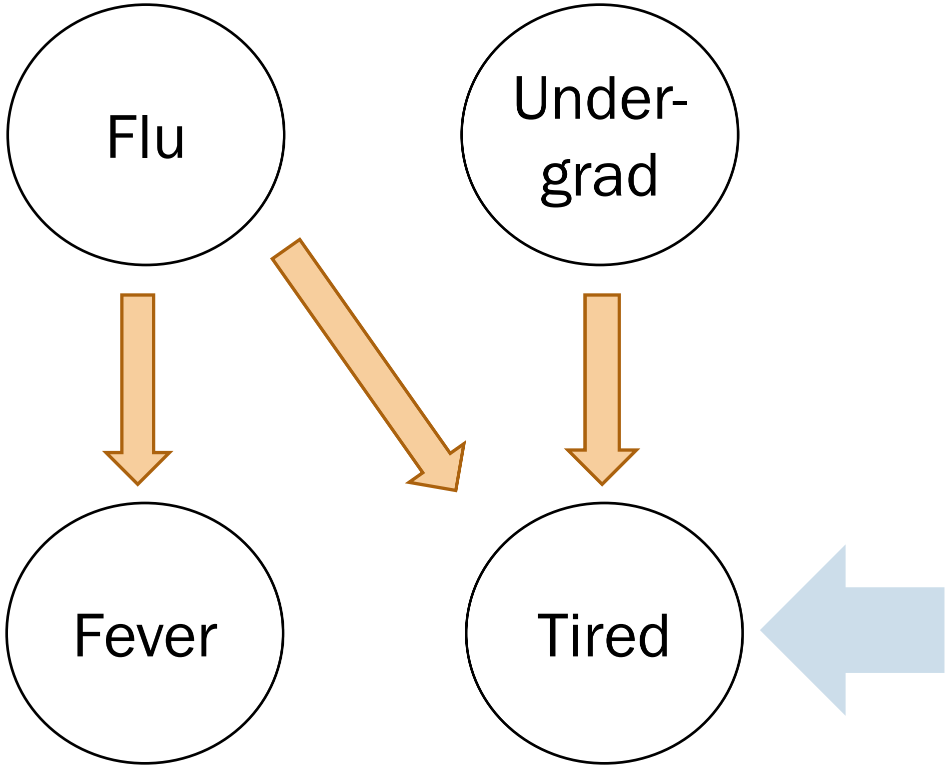
What would a Stanford flu expert do?

1. Describe the joint distribution using causality.
2. Assume conditional independence.
3. Provide $P(\text{values}|\text{parents})$ for each random variable

What conditional probabilities should our expert specify?

Constructing a Bayesian Network

$P(F_{lu} = 1) = 0.1$ $\quad$ $P(U = 1) = 0.8$



$P(F_{ev} = 1|F_{lu} = 1) = 0.9$
$P(F_{ev} = 1|F_{lu} = 0) = 0.05$

What would a Stanford flu expert do?

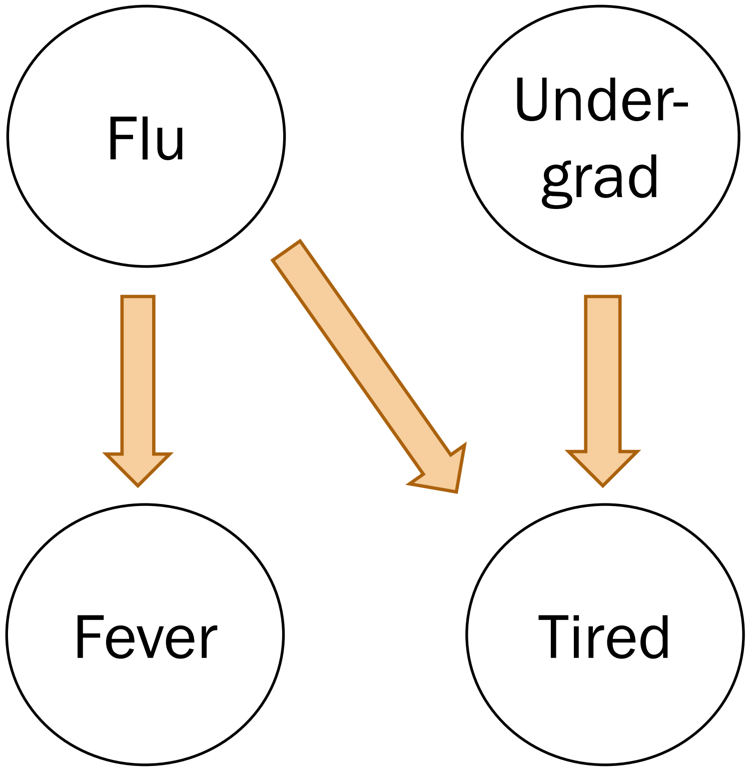
1. Describe the joint distribution using causality.
2. Assume conditional independence.
3. Provide P(values|parents) for each random variable

What conditional probabilities should our expert specify?

$P(T = 1|F_{lu} = 0, U = 0)$
$P(T = 1|F_{lu} = 0, U = 1)$
$P(T = 1|F_{lu} = 1, U = 0)$
$P(T = 1|F_{lu} = 1, U = 1)$

Using a Bayes Net

$P(F_{lu} = 1) = 0.1$ $P(U = 1) = 0.8$



$P(F_{ev} = 1|F_{lu} = 1) = 0.9$
$P(F_{ev} = 1|F_{lu} = 0) = 0.05$

$P(T = 1|F_{lu} = 0, U = 0) = 0.1$
$P(T = 1|F_{lu} = 0, U = 1) = 0.8$
$P(T = 1|F_{lu} = 1, U = 0) = 0.9$
$P(T = 1|F_{lu} = 1, U = 1) = 1.0$

What would a CS109 student do?

1. Populate a Bayesian network by asking a Stanford flu expert
   or
   by using reasonable assumptions

2. **Answer inference questions**

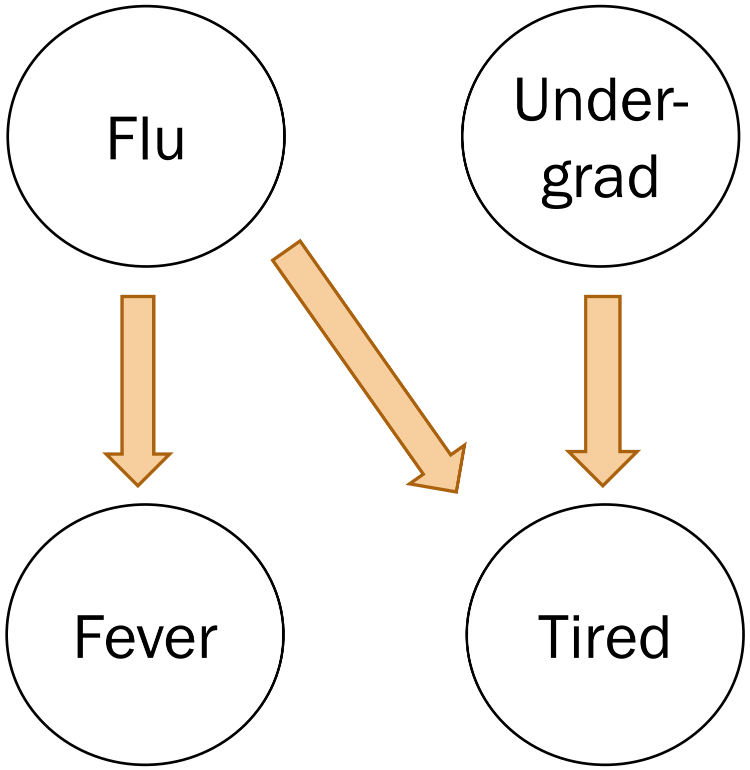
Our focus today

Inference (I): Math

Bayes Nets: Conditional independence

Review

$P(F_{lu} = 1) = 0.1$ $\quad$ $P(U = 1) = 0.8$



$P(F_{ev} = 1 | F_{lu} = 1) = 0.9$
$P(F_{ev} = 1 | F_{lu} = 0) = 0.05$

$P(T = 1 | F_{lu} = 0, U = 0) = 0.1$
$P(T = 1 | F_{lu} = 0, U = 1) = 0.8$
$P(T = 1 | F_{lu} = 1, U = 0) = 0.9$
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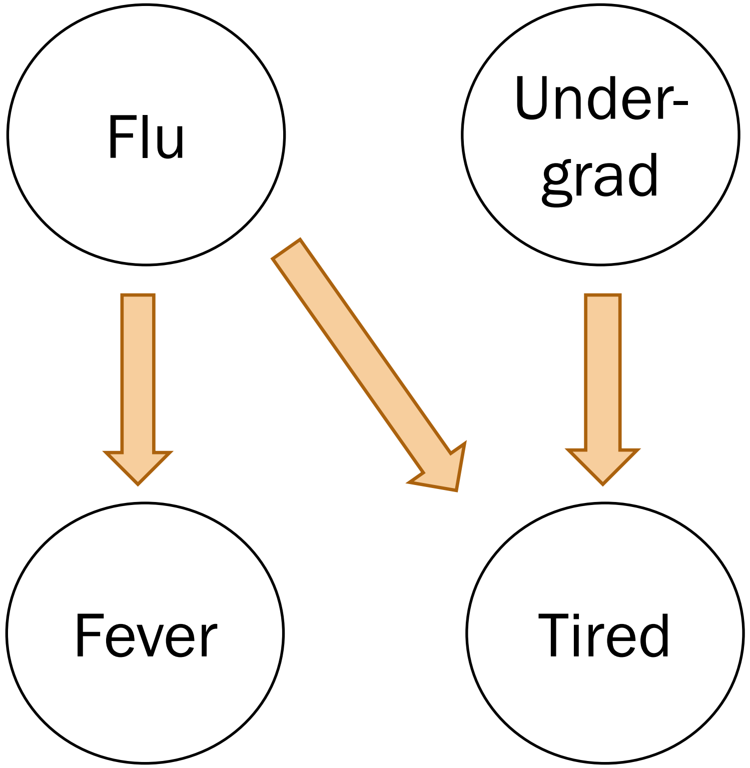
In a Bayesian Network,
Each random variable is conditionally independent of its non-descendants, given its parents.

- Node: random variable
- Directed edge: conditional dependency

Inference via math

$P(F_{lu} = 1) = 0.1$ $\quad$ $P(U = 1) = 0.8$



$P(F_{ev} = 1 | F_{lu} = 1) = 0.9$
$P(F_{ev} = 1 | F_{lu} = 0) = 0.05$

$P(T = 1 | F_{lu} = 0, U = 0) = 0.1$
$P(T = 1 | F_{lu} = 0, U = 1) = 0.8$
$P(T = 1 | F_{lu} = 1, U = 0) = 0.9$
$P(T = 1 | F_{lu} = 1, U = 1) = 1.0$

1. $P(F_{lu} = 0, U = 1, F_{ev} = 0, T = 1)$?

   Compute joint probabilities using chain rule.

$= P(F_{lu}=0) P(U=1 | F_{lu}=0) P(F_{ev}=0 | F_{lu}=0, U=1)$
$\cdot P(T=1 | F_{lu}=0, U=1, F_{ev}=0)$

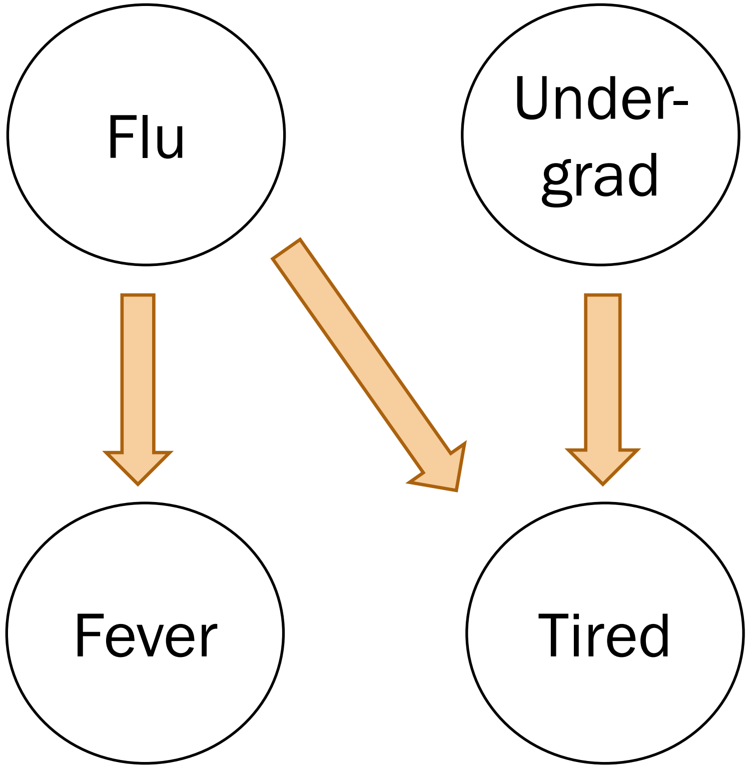
$= P(F_{lu}=0) P(U=1) P(F_{ev}=0 | F_{lu}=0)$
$\cdot P(T=1 | F_{lu}=0, U=1)$

$= 0.9 (0.8) 0.95 (0.8)$

$= 0.5472$

Inference via math

$P(F_{lu} = 1) = 0.1$ $\quad$ $P(U = 1) = 0.8$



$P(F_{ev} = 1 | F_{lu} = 1) = 0.9$
$P(F_{ev} = 1 | F_{lu} = 0) = 0.05$

$P(T = 1 | F_{lu} = 0, U = 0) = 0.1$
$P(T = 1 | F_{lu} = 0, U = 1) = 0.8$
$P(T = 1 | F_{lu} = 1, U = 0) = 0.9$
$P(T = 1 | F_{lu} = 1, U = 1) = 1.0$

2. $P(F_{lu} = 1 | F_{ev} = 0, U = 0, T = 1)$?

   1. Compute joint probabilities

      $P(F_{lu} = 1, F_{ev} = 0, U = 0, T = 1)$

      $P(F_{lu} = 0, \underbrace{F_{ev} = 0, U = 0, T = 1}_{\text{given}})$

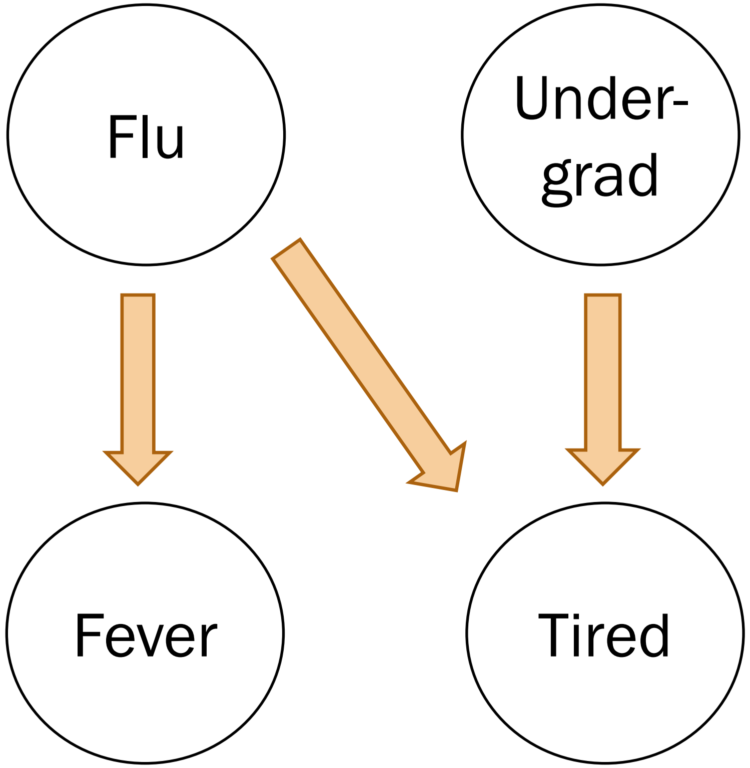
   2. Definition of conditional probability

$$\frac{P(F_{lu} = 1, F_{ev} = 0, U = 0, T = 1)}{\sum_x P(F_{lu} = x, F_{ev} = 0, U = 0, T = 1)}$$

$$= 0.095$$

Inference via math

$P(F_{lu} = 1) = 0.1$ $\qquad$ $P(U = 1) = 0.8$



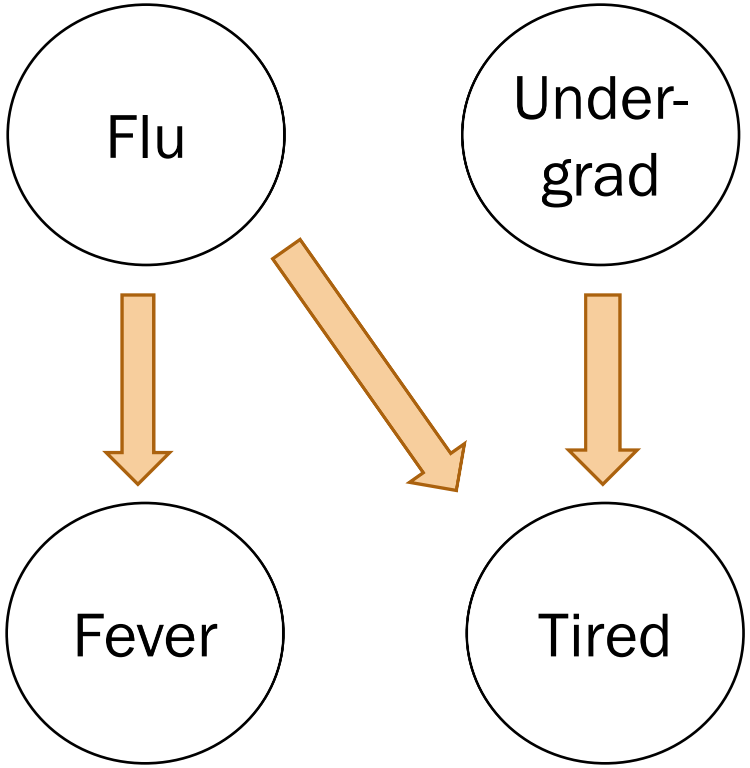
$P(F_{ev} = 1|F_{lu} = 1) = 0.9$
$P(F_{ev} = 1|F_{lu} = 0) = 0.05$

$P(T = 1|F_{lu} = 0, U = 0) = 0.1$
$P(T = 1|F_{lu} = 0, U = 1) = 0.8$
$P(T = 1|F_{lu} = 1, U = 0) = 0.9$
$P(T = 1|F_{lu} = 1, U = 1) = 1.0$

3. $P(F_{lu} = 1|U = 1, T = 1)$?

Inference via math

$P(F_{lu} = 1) = 0.1$ $\quad$ $P(U = 1) = 0.8$



$P(F_{ev} = 1 | F_{lu} = 1) = 0.9$
$P(F_{ev} = 1 | F_{lu} = 0) = 0.05$

$P(T = 1 | F_{lu} = 0, U = 0) = 0.1$
$P(T = 1 | F_{lu} = 0, U = 1) = 0.8$
$P(T = 1 | F_{lu} = 1, U = 0) = 0.9$
$P(T = 1 | F_{lu} = 1, U = 1) = 1.0$

3. $P(F_{lu} = 1 | U = 1, T = 1)$?

   1. Compute joint probabilities

   $$P(F_{lu} = 1, U = 1, F_{ev} = 1, T = 1)$$
   $$\ldots$$
   $$P(F_{lu} = 0, U = 1, F_{ev} = 0, T = 1)?$$

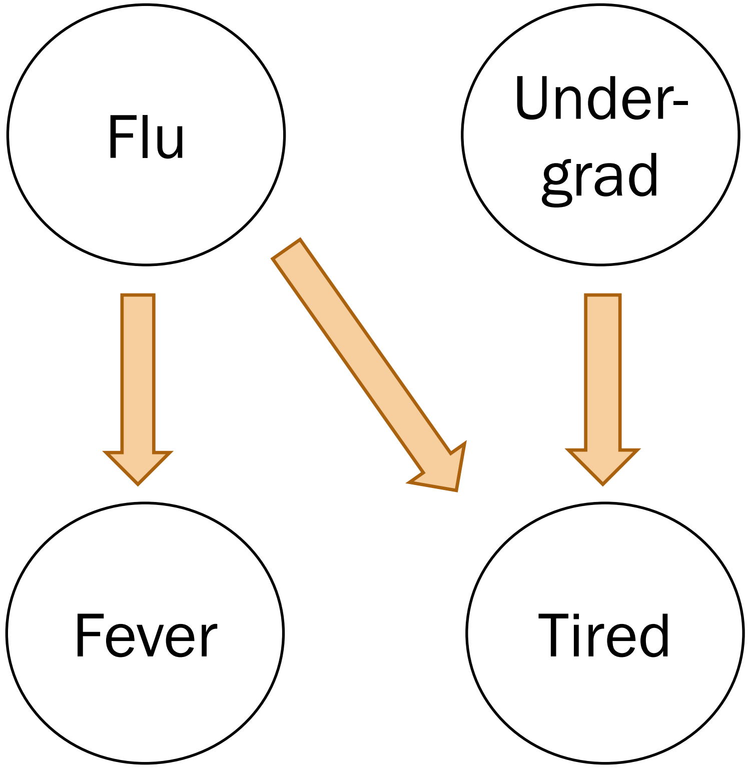
   2. Definition of conditional probability

   $$\frac{\sum_y P(F_{lu} = 1, U = 1, F_{ev} = y, T = 1)}{\sum_x \sum_y P(F_{lu} = x, U = 1, F_{ev} = y, T = 1)}$$

$$= 0.122$$

Inference via math

$P(F_{lu} = 1) = 0.1$ $\quad$ $P(U = 1) = 0.8$



$P(F_{ev} = 1 | F_{lu} = 1) = 0.9$
$P(F_{ev} = 1 | F_{lu} = 0) = 0.05$

$P(T = 1 | F_{lu} = 0, U = 0) = 0.1$
$P(T = 1 | F_{lu} = 0, U = 1) = 0.8$
$P(T = 1 | F_{lu} = 1, U = 0) = 0.9$
$P(T = 1 | F_{lu} = 1, U = 1) = 1.0$

Solving inference questions precisely is possible, but sometimes tedious.

Can we use sampling to do approximate inference?

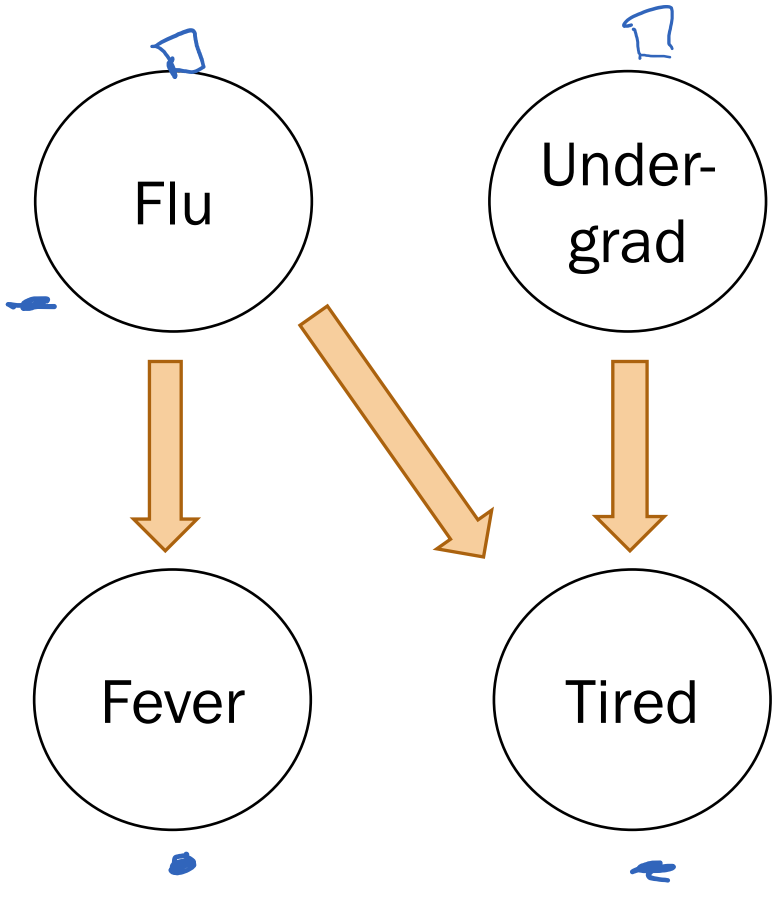
Yes.

(live) 15: General Inference

Lisa Yan

May 8, 2020

Constructing a Bayesian Network



In a Bayesian Network,

Each random variable is conditionally independent of its non-descendants, given its parents.

- Node: random variable
- Directed edge: conditional dependency

causality

Examples:

- $P(F_{ev} = 1 | T \cancel{=} 0, F_{lu} = 1) = P(F_{ev} = 1 | F_{lu} = 1)$
- $P(F_{lu} = 1, U = 0) = P(F_{lu} = 1)P(U = 0)$

Breakout Rooms

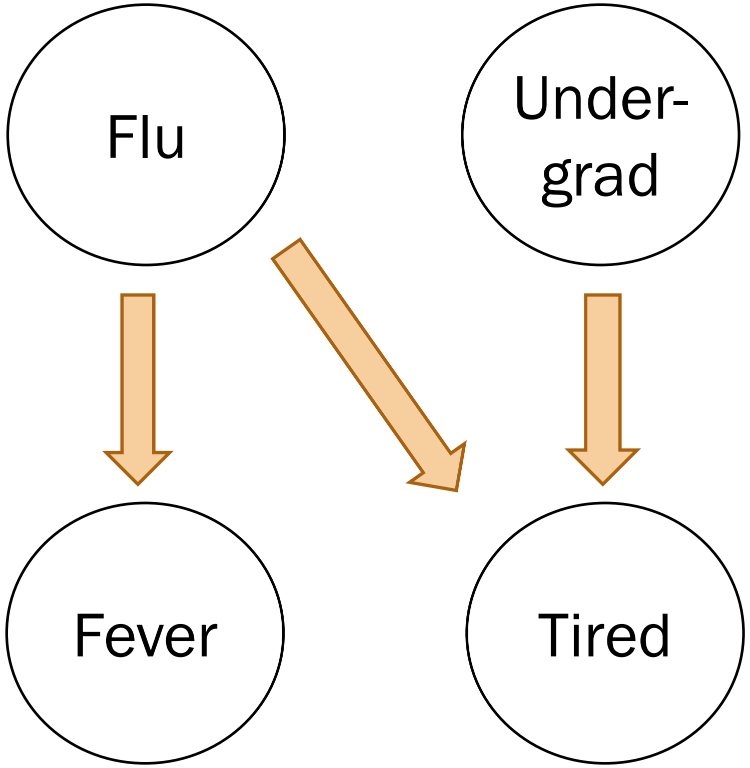
Check out the question on the next slide (Slide 31). Post any clarifications here!

https://us.edstem.org/courses/109/discussion/59206

Breakout rooms: 4 min. Introduce yourself!

Inference via math

$P(F_{lu} = 1) = 0.1$ $\quad$ $P(U = 1) = 0.8$



$$P(F_{lu} = 1 | F_{ev} = 1, U = 1, T = 1)?$$

$0.9 + 0.05 = 1$?

$1 = \underbrace{P(Fev=1 | Flu=1)}_{0.9} + \underbrace{P(Fev=0 | Flu=1)}_{0.1}$

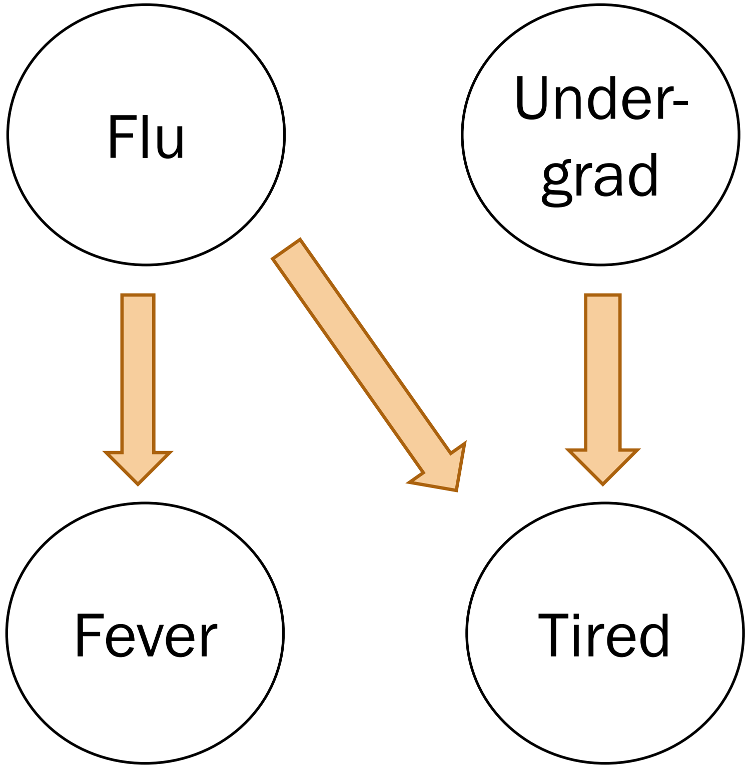
$1 = \underbrace{P(Fev=1 | Flu=0)}_{0.05} + \underbrace{P(Fev=0 | Flu=0)}_{0.95}$

$P(F_{ev} = 1 | F_{lu} = 1) = 0.9$
$P(F_{ev} = 1 | F_{lu} = 0) = 0.05$

$P(T = 1 | F_{lu} = 0, U = 0) = 0.1$
$P(T = 1 | F_{lu} = 0, U = 1) = 0.8$
$P(T = 1 | F_{lu} = 1, U = 0) = 0.9$
$P(T = 1 | F_{lu} = 1, U = 1) = 1.0$

Inference via math

$P(F_{lu} = 1) = 0.1$ $\quad$ $P(U = 1) = 0.8$



$$P(F_{lu} = 1 | F_{ev} = 1, U = 1, T = 1)?$$

$$= \frac{\star\ P(Flu=1, Fev=1, U=1, T=1)}{\square\ P(Fev=1, U=1, T=1)}$$

$$= \frac{\star\ (0.1)(0.8)(0.9)(1.0)}{\square\ (0.1)(0.8)(0.9)1.0 + (0.9)(0.8)0.05(0.8)}$$

$$\approx 0.714$$

$P(F_{ev} = 1 | F_{lu} = 1) = 0.9$
$P(F_{ev} = 1 | F_{lu} = 0) = 0.05$

$P(T = 1 | F_{lu} = 0, U = 0) = 0.1$
$P(T = 1 | F_{lu} = 0, U = 1) = 0.8$
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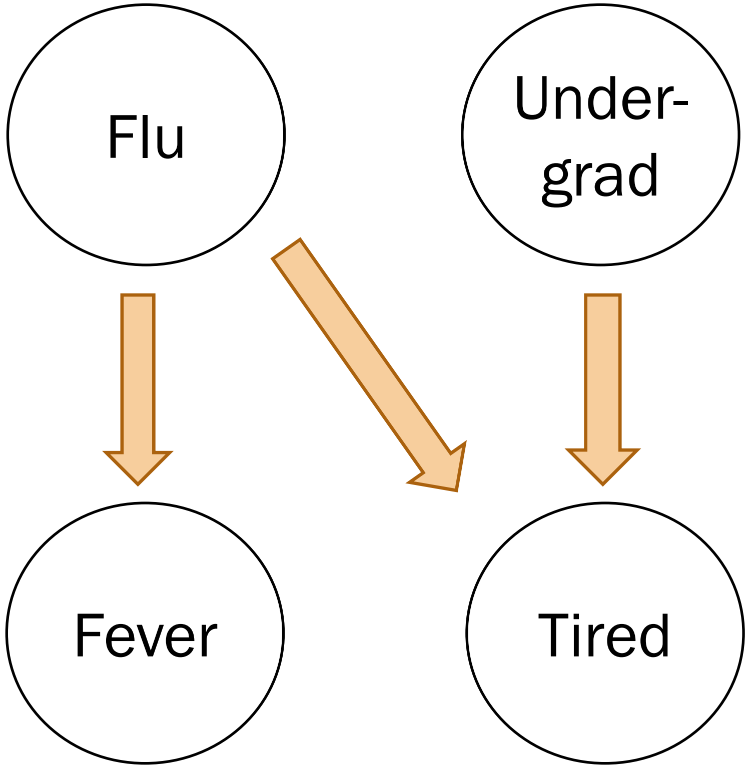
Inference via math

$P(F_{lu} = 1) = 0.1$ $P(U = 1) = 0.8$



$$P(F_{lu} = 1 | F_{ev} = 1, U = 1, T = 1)?$$

numerator $P(Flu=1, Fev=1, U=1, T=1)$

$= P(Flu=1)\, P(U=1 \mid \cancel{Flu=1})$

$\cdot\, P(Fev=1 \mid Flu=1, \cancel{U=1})$

$\cdot\, P(T=1 \mid Flu=1, U=1, \cancel{Fev=1})$

$0.1\ (0.9)\ (0.8)\ 1$

$P(F_{ev} = 1 | F_{lu} = 1) = 0.9$
$P(F_{ev} = 1 | F_{lu} = 0) = 0.05$

$P(T = 1 | F_{lu} = 0, U = 0) = 0.1$
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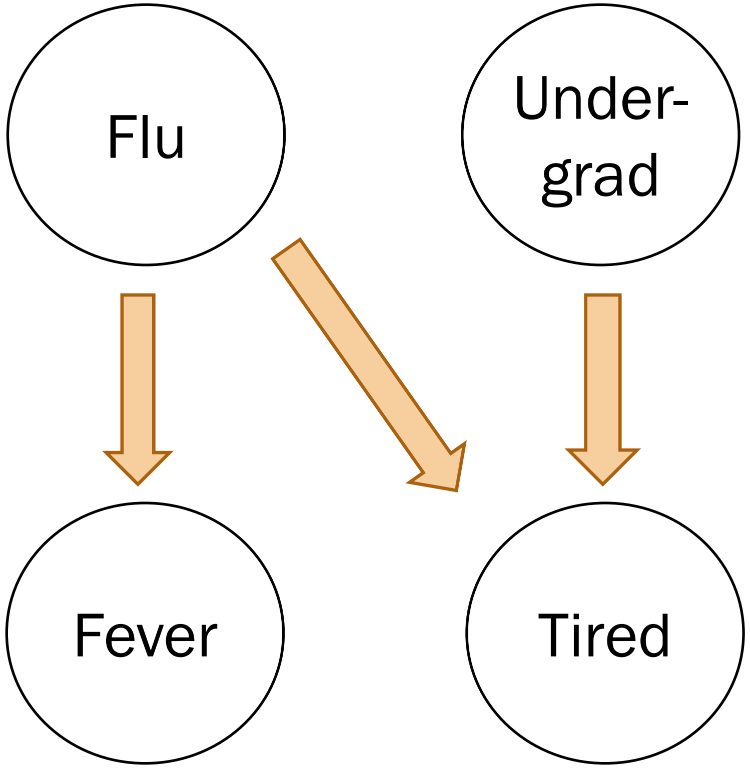
Inference via math

$P(F_{lu} = 1) = 0.1$ $\quad$ $P(U = 1) = 0.8$



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$$P(T = 1 | F_{lu} = 1, U = 0) = 0.9$$
$$P(T = 1 | F_{lu} = 1, U = 1) = 1.0$$

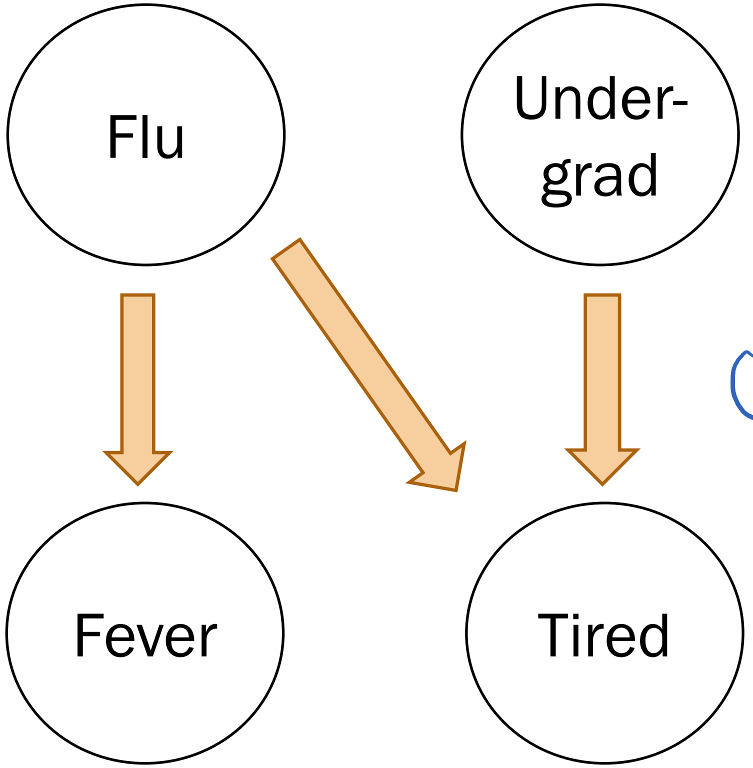
Solving inference questions precisely is possible, but sometimes tedious.

Can we use sampling to do approximate inference?

Yes.

Inference via math

$P(F_{lu} = 1) = 0.1$ $\quad$ $P(U = 1) = 0.8$



(3) What is $P(F_{lu} = 1 | U = 1, T = 1)$?

$= 0.122$

(from pre-lecture video)

$P(F_{ev} = 1 | F_{lu} = 1) = 0.9$
$P(F_{ev} = 1 | F_{lu} = 0) = 0.05$

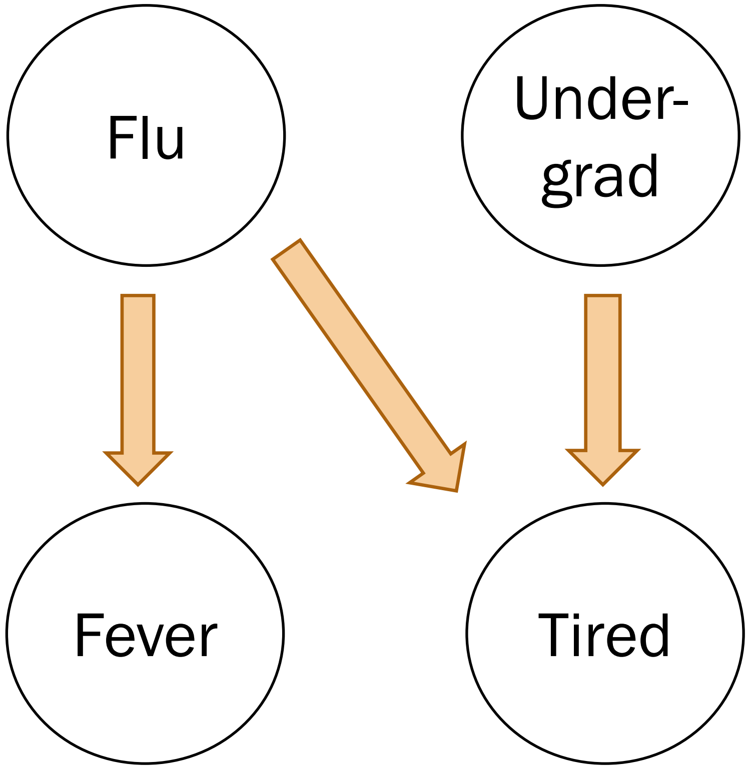
$P(T = 1 | F_{lu} = 0, U = 0) = 0.1$
$P(T = 1 | F_{lu} = 0, U = 1) = 0.8$
$P(T = 1 | F_{lu} = 1, U = 0) = 0.9$
$P(T = 1 | F_{lu} = 1, U = 1) = 1.0$

Rejection sampling algorithm

Step 0:

Have a fully specified Bayesian Network

$P(F_{lu} = 1) = 0.1$ $\quad$ $P(U = 1) = 0.8$



$P(F_{ev} = 1 | F_{lu} = 1) = 0.9$
$P(F_{ev} = 1 | F_{lu} = 0) = 0.05$

$P(T = 1 | F_{lu} = 0, U = 0) = 0.1$
$P(T = 1 | F_{lu} = 0, U = 1) = 0.8$
$P(T = 1 | F_{lu} = 1, U = 0) = 0.9$
$P(T = 1 | F_{lu} = 1, U = 1) = 1.0$

Rejection sampling algorithm

Inference question: What is $P(F_{lu} = 1 | U = 1, T = 1)$?

```
def rejection_sampling(event, observation):
    samples = sample_a_ton()
    samples_observation = …
        # number of samples with (U = 1, T = 1)
    samples_event =
        # number of samples with (F_lu = 1, U = 1, T
    return len(samples_event)/len(samples_observ
```

[flu, und, fev, tir]

```
Sampling...
[0, 1, 0, 1]
[0, 1, 0, 1]
[0, 1, 0, 1]
[0, 0, 0, 0]
[0, 1, 0, 1]
[0, 1, 1, 1]
[0, 1, 0, 0]
[1, 1, 1, 1]
[0, 0, 1, 1]
...
[0, 1, 0, 1]
Finished sampling
```

Rejection sampling algorithm

Inference question: What is $P(F_{lu} = 1 | U = 1, T = 1)$?

```
def rejection_sampling(event, observation):
    samples = sample_a_ton()
    samples_observation = …
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            # number of samples with (F_lu = 1, U = 1, T = 1)
    return len(samples_event)/len(samples_observation)
```

$$\text{Approximate Probability} = \frac{\#\text{ samples with } (F_{lu} = 1, U = 1, T = 1)}{\#\text{ samples with } (U = 1, T = 1)}$$

Rejection sampling algorithm

Inference question: What is $P(F_{lu} = 1|U = 1, T = 1)$?

Approximate Probability = $\dfrac{\text{\# samples with } (F_{lu} = 1, U = 1, T = 1)}{\text{\# samples with } (U = 1, T = 1)}$

Why would this definition of approximate probability make sense?

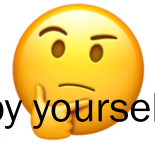
Think

Slide 41 has a question to go over by yourself.

Post any clarifications here!

https://us.edstem.org/courses/109/discussion/59206

Think by yourself: 2 min



Why would this approximate probability make sense?

Inference question: What is $P(F_{lu} = 1 | U = 1, T = 1)$?

Approximate Probability = $\dfrac{\text{\# samples with } (F_{lu} = 1, U = 1, T = 1)}{\text{\# samples with } (U = 1, T = 1)}$

Recall our definition of probability as a frequency:

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

n = # of total trials
$n(E)$ = # trials where E occurs

Why would this approximate probability make sense?

Inference question: What is $P(F_{lu} = 1 | U = 1, T = 1)$?

$$= \frac{P(Flu=1, U=1, T=1)}{P(U=1, T=1)} = \frac{\frac{n(Flu=1, U=1, T=1)}{n \text{ of samples}}}{\frac{n(U=1, T=1)}{n \text{ of samples}}}$$

Approximate Probability = $\dfrac{\text{\# samples with } (F_{lu} = 1, U = 1, T = 1)}{\text{\# samples with } (U = 1, T = 1)}$

Recall our definition of probability as a frequency: $P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$

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Rejection sampling algorithm

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[flu, und, fev, tir]

```
Sampling...
[0, 1, 0, 1]
[0, 1, 0, 1]
[0, 1, 0, 1]
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[0, 1, 0, 1]
[0, 1, 1, 1]
[0, 1, 0, 0]
[1, 1, 1, 1]
[0, 0, 1, 1]
...
[0, 1, 0, 1]
Finished sampling
```

Rejection sampling algorithm

```
N_SAMPLES = 100000
# Method: Sample a ton
# -------------------
# create N_SAMPLES with likelihood proportional
# to the joint distribution
def sample_a_ton():
    samples = []
    for i in range(N_SAMPLES):
        sample = make_sample() # a particle
        samples.append(sample)
    return samples
```

How do we make a sample $(F_{lu} = a, U = b, F_{ev} = c, T = d)$ according to the joint probability?

Create a sample using the Bayesian Network!!

Rejection sampling algorithm

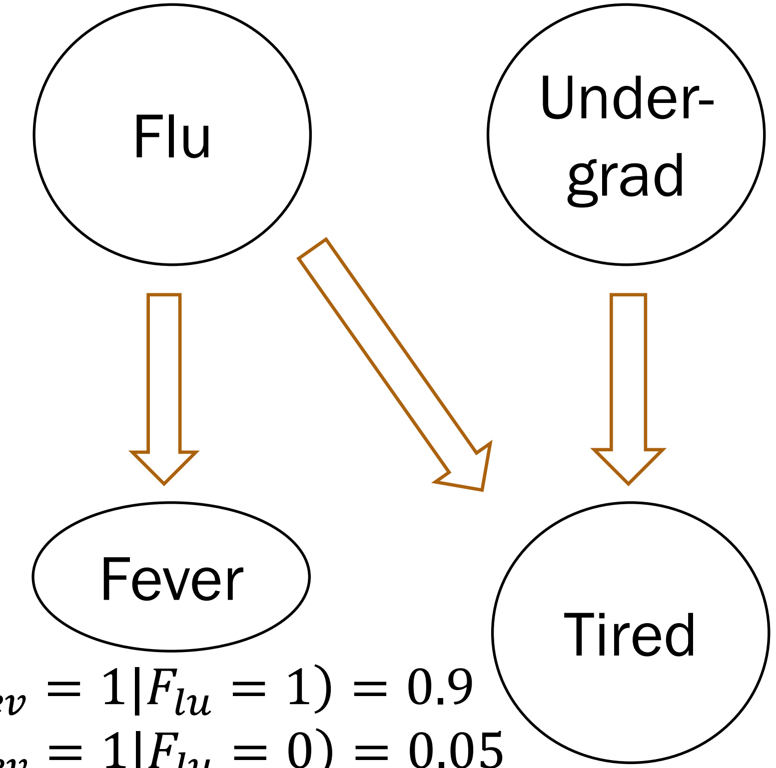
```python
# Method: Make Sample
# --------------------
# create a single sample from the joint distribution
# based on the medical "WebMD" Bayesian Network
def make_sample():
    # prior on causal factors
    flu = bernoulli(0.1)
    und = bernoulli(0.8)

    # choose fever based on flu
    if flu == 1: fev = bernoulli(0.9)
    else:        fev = bernoulli(0.05)

    # choose tired based on (undergrad and flu)
    #
    # TODO: fill in
    #
    #

    # a sample from the joint has an
    # assignment to *all* random variables
    return [flu, und, fev, tir]
```

$P(F_{lu} = 1) = 0.1$ $\quad$ $P(U = 1) = 0.8$



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    #
    # TODO: fill in
    #
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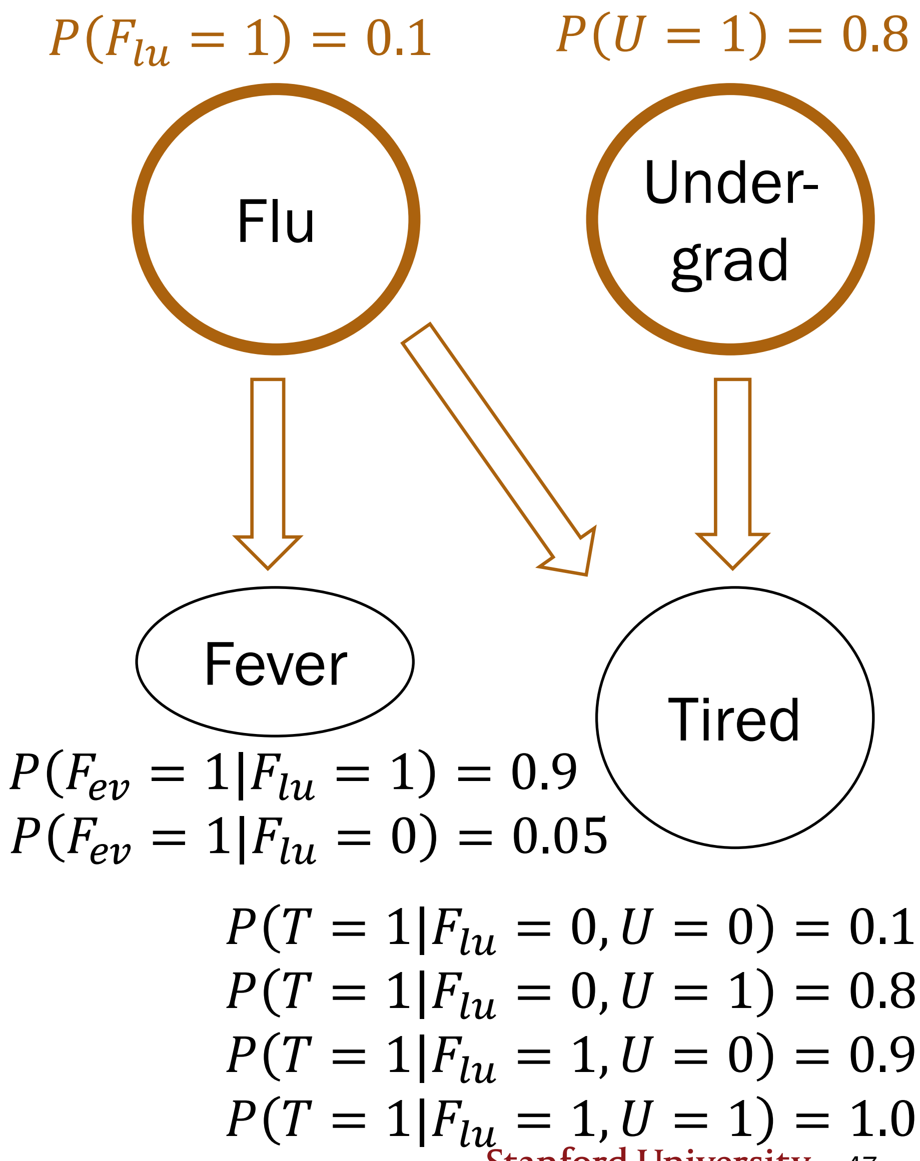
    # assignment to *all* random variables
    return [flu, und, fev, tir]
```



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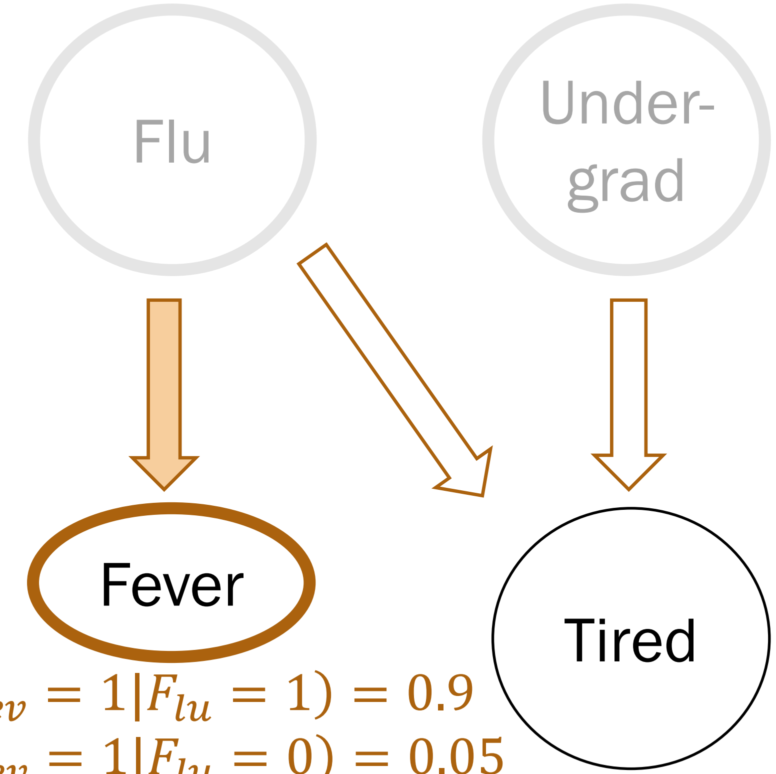

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$$\sum_{i=90}^{99} i$$

Interlude for jokes/announcements

Announcements

<u>Quiz #1</u>

Solutions: after class
Grades: after class
Regrades: by next Friday

<u>Mid-quarter feedback form</u>

link

Open until: next Friday

Giant sheet of all equations covered so far?

<u>Problem Set 4</u>

Out: later today
Due: Monday 5/18 10am
Covers: Up to and including today

Announcements: CS109 contest

Do something cool and creative with probability

Replaces one "passing" work requirement

Week 7

Optional Proposal: Sat. 5/23 11:59pm

Due: Week 10 Monday 6/8, 11:59pm

Winner, 2 finalists $

https://web.stanford.edu/class/cs109/psets/cs109_contest.pdf

Interesting probability news

http://www.intuitor.com/statistics/TwentyQs.html

CS109 Current Events Spreadsheet

Rejection sampling algorithm

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    # TODO: fill in
    #
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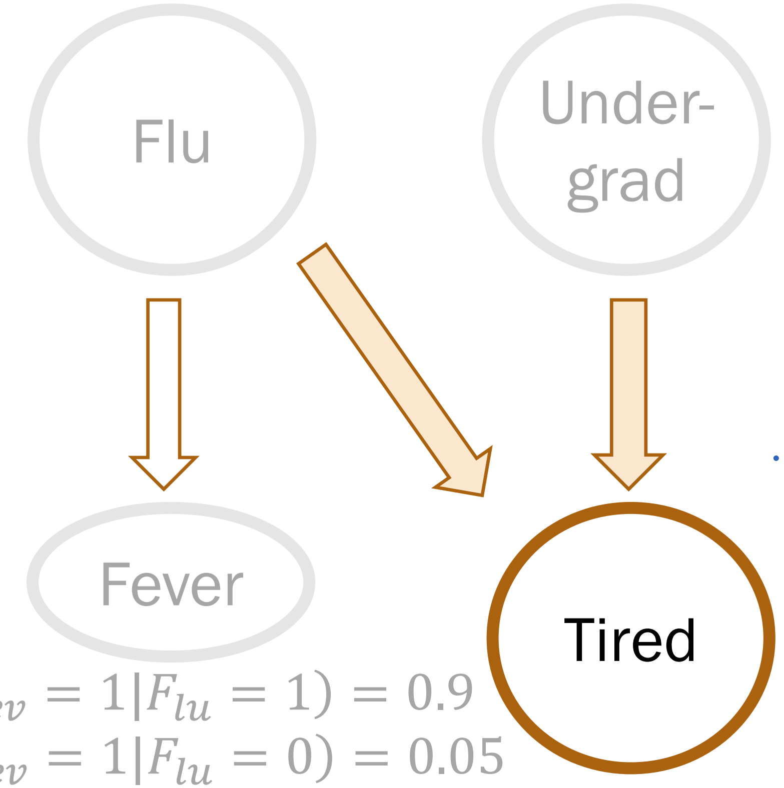
$P(F_{lu} = 1) = 0.1$ $P(U = 1) = 0.8$



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Rejection sampling algorithm

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    else:        fev = bernoulli(0.05)

    # choose tired based on (undergrad and flu)
    if   flu == 0 and und == 0: tir = bernoulli(0.1)
    elif flu == 0 and und == 1: tir = bernoulli(0.8)
    elif flu == 1 and und == 0: tir = bernoulli(0.9)
    else:                       tir = bernoulli(1.0)

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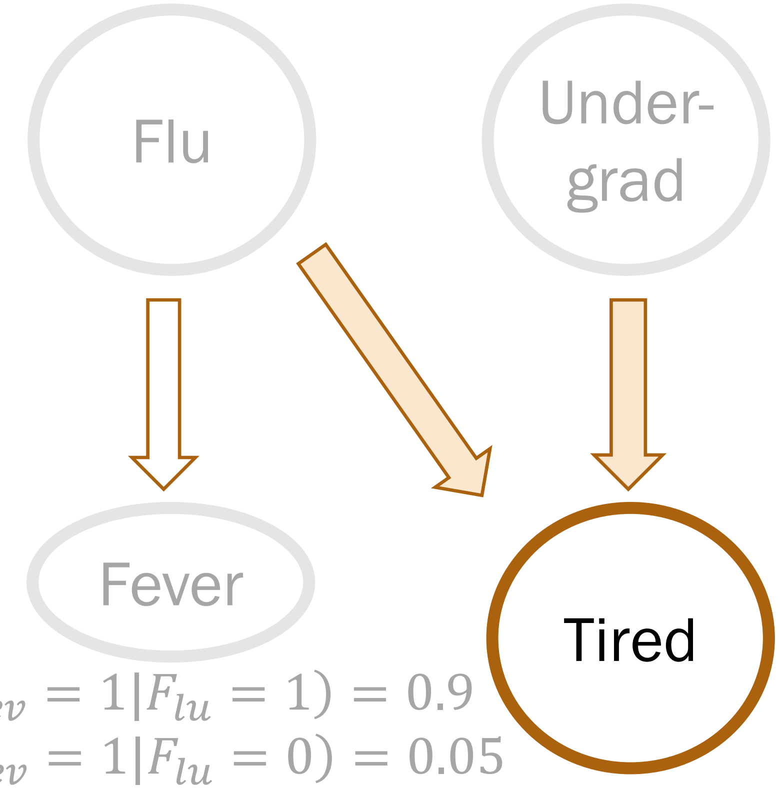
    # assignment to *all* random variables
    return [flu, und, fev, tir]
```

$P(F_{lu} = 1) = 0.1$ $P(U = 1) = 0.8$



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```

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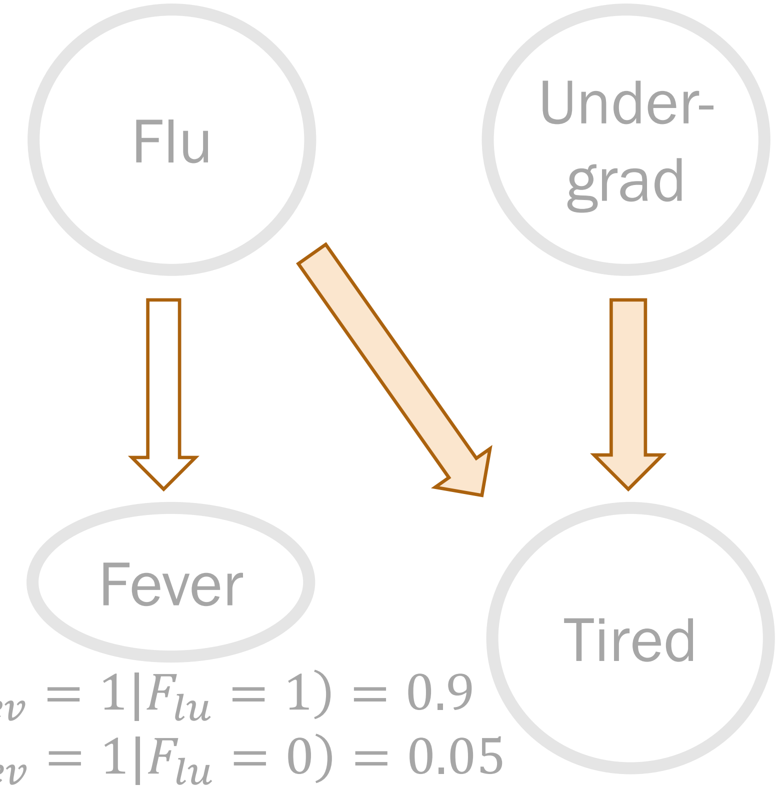

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Rejection sampling algorithm

Inference question: What is $P(F_{lu} = 1 | U = 1, T = 1)$?

```
def rejection_sampling(event, observation):
  samples = sample_a_ton()
  samples_observation = …
        # number of samples with (U = 1, T = 1)
  samples_event =
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  return len(samples_event)/len(samples_obser
```

[flu, und, fev, tir]

```
Sampling...
[0, 1, 0, 1]
[0, 1, 0, 1]
[0, 1, 0, 1]
[0, 0, 0, 0]
[0, 1, 0, 1]
[0, 1, 1, 1]
[0, 1, 0, 0]
[1, 1, 1, 1]
[0, 0, 1, 1]
...
[0, 1, 0, 1]
Finished sampling
```

Rejection sampling algorithm

Inference question: What is $P(F_{lu} = 1 | U = 1, T = 1)$?

```
def rejection_sampling(event, observation):
  samples = sample_a_ton()
  samples_observation = …
        # number of samples with (U = 1, T = 1)
  samples_event =
        # number of samples with (F_lu = 1, U = 1, T = 1)
  return len(samples_event)/len(samples_observation)
```

Rejection sampling algorithm

Inference question: What is $P(F_{lu} = 1 | U = 1, T = 1)$?

```
def rejection_sampling(event, observation):
  samples = sample_a_ton()
  samples_observation =
        reject_inconsistent(samples, observation)
  samples_event =
        # number of samples with (F_lu = 1, U = 1, T = 1)
  return len(samples_event)/len(samples_observation)
```

Keep only samples that are consistent with the observation $(U = 1, T = 1)$.

Rejection sampling algorithm

Inference question: What is $P(F_{lu} = 1 | U = 1, T = 1)$?

```python
def rejection_sampling(event, observation):
    samples = sample_a_ton()
    samples_observation =
            reject_inconsistent(samples, observation)
    samples_
    
    return
```

```python
# Method: Reject Inconsistent
# -------------------
# Rejects all samples that do not align with the outcome.
# Returns a list of consistent samples.
def reject_inconsistent(samples, outcome):
    consistent_samples = []
    for sample in samples:    100,000
        if check_consistent(sample, outcome):
            consistent_samples.append(sample)
    return consistent_samples
```

$(U = 1, T = 1)$

Rejection sampling algorithm

Inference question: What is $P(F_{lu} = 1 | U = 1, T = 1)$?

```
def rejection_sampling(event, observation):
    samples = sample_a_ton()
    samples_observation =
            reject_inconsistent(samples, observation)
    samples_event =
            reject_inconsistent(samples_observation, event)
    return len(samples_event)/len(samples_observation)
```

$(U=1, T=1)$

$(F_{lu}=1, U=1, T=1)$

Conditional event = samples with $(F_{lu} = 1, U = 1, T = 1)$.

Rejection sampling algorithm

Inference question: What is $P(F_{lu} = 1 | U = 1, T = 1)$?

```
def rejection_sampling(event, observation):
    samples = sample_a_ton()
    samples_observation =
            reject_inconsistent(samples, observation)
    samples_event =
            reject_inconsistent(samples_observation, event)
    return l                                         ation)
```

Conditi $= 1)$.

```
def reject_inconsistent(samples, outcome):
```

$(F_{lu} = x, U = 1, F_{ev} = y, T = 1)$ $(F_{lu} = 1)$

```
    return consistent_samples
```

Rejection sampling algorithm

Inference question: What is $P(F_{lu} = 1 | U = 1, T = 1)$?

```
def rejection_sampling(event, observation):
    samples = sample_a_ton()
    samples_observation =
            reject_inconsistent(samples, observation)
    samples_event =
            reject_inconsistent(samples_observation, event)
    return len(samples_event)/len(samples_observation)
```

Approximate Probability = $\dfrac{\#\text{ samples with }(F_{lu} = 1, U = 1, T = 1)}{\#\text{ samples with }(U = 1, T = 1)}$

To the code!

Rejection sampling

If you can sample enough from the joint distribution, you can answer any probability inference question.

With enough samples, you can correctly compute:

- Probability estimates
- Conditional probability estimates
- Expectation estimates

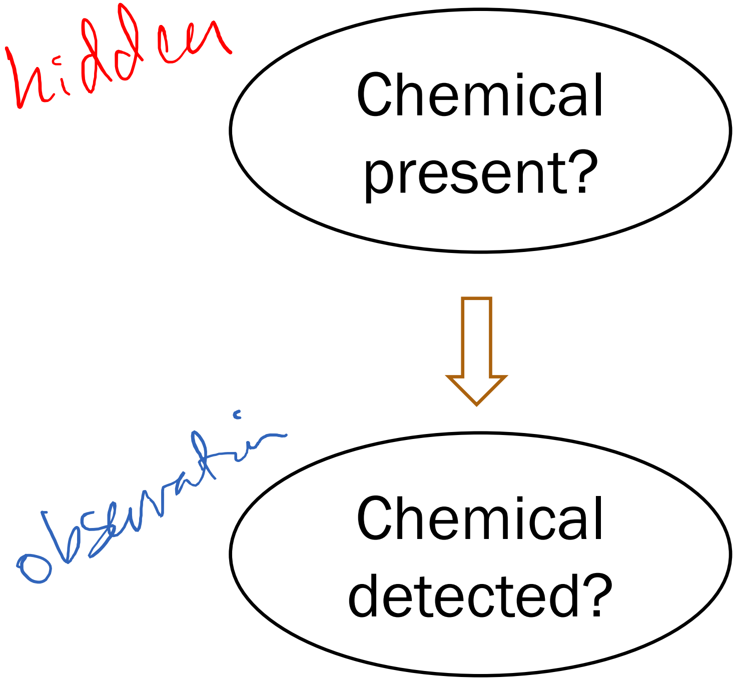
Because your samples are a representation of the joint distribution!

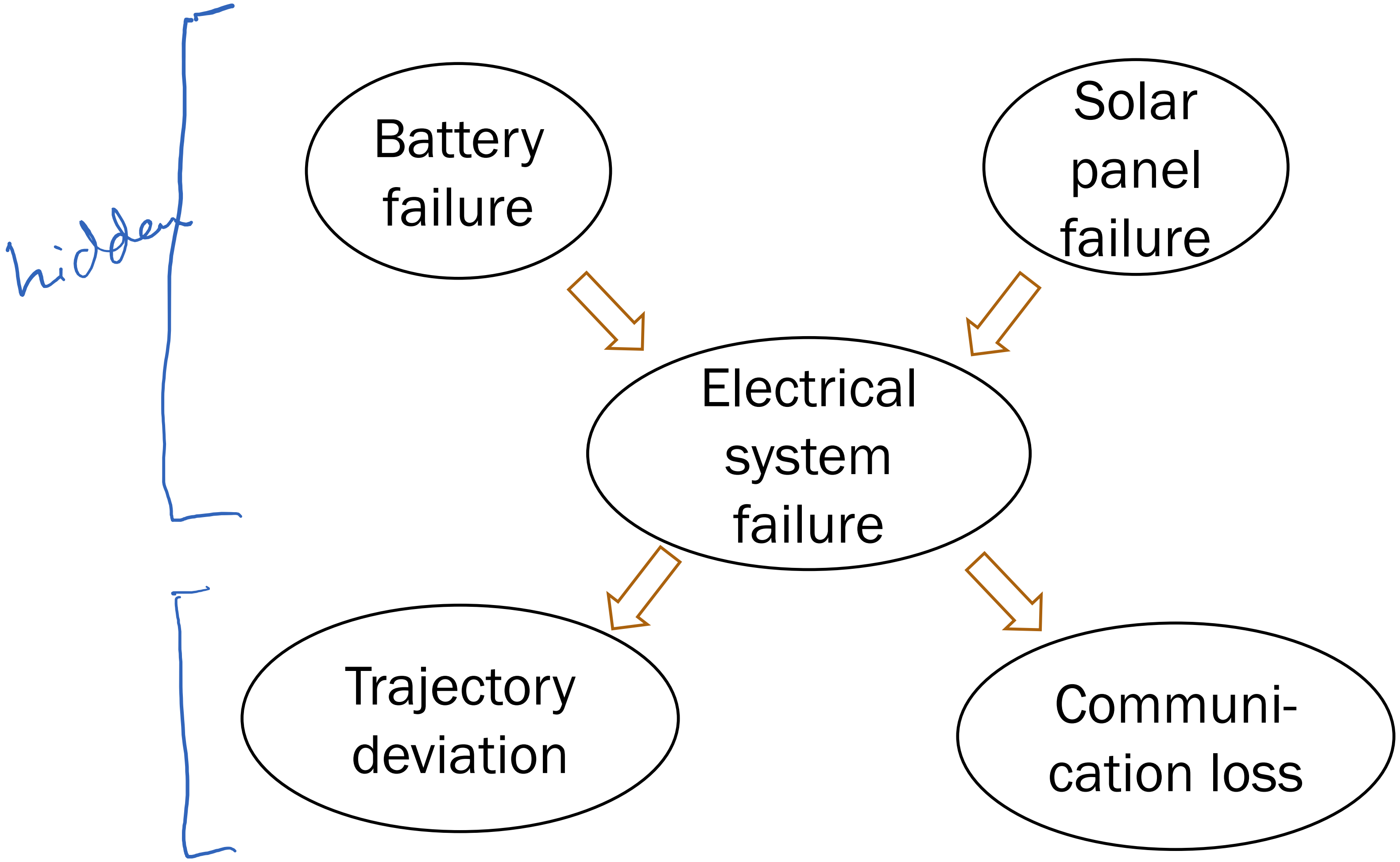
```
[flu, und, fev, tir]
```

```
Sampling...
[0, 1, 0, 1]
[0, 1, 0, 1]
[0, 1, 0, 1]
[0, 0, 0, 0]
[0, 1, 0, 1]
[0, 1, 1, 1]
[0, 1, 0, 0]
[1, 1, 1, 1]
[0, 0, 1, 1]
...
[0, 1, 0, 1]
Finished sampling
```

```
P(has flu | undergrad and is tired) = 0.122
```

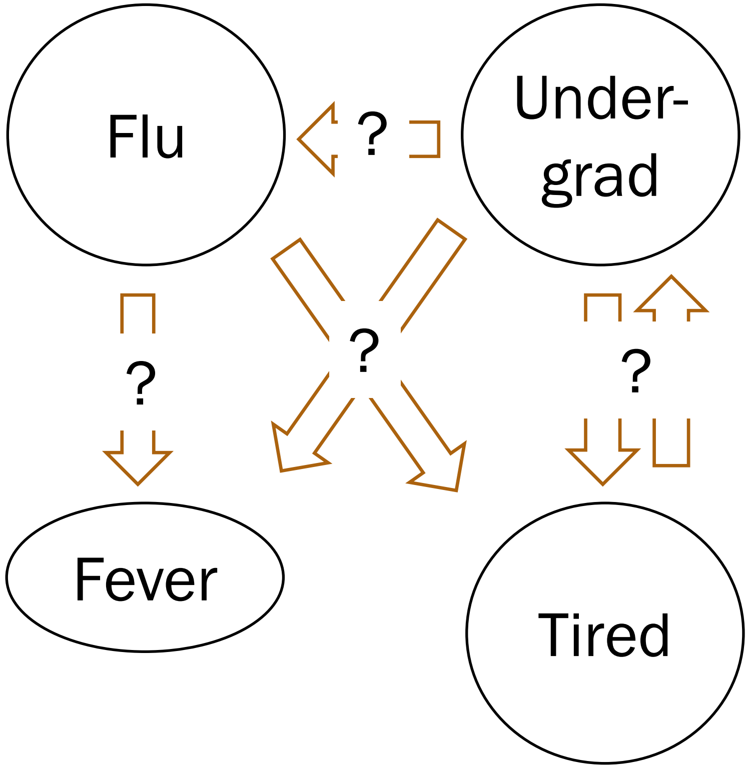
Other applications





Take CS238/AA228: Decision Making under Uncertainty!

Challenge with Bayesian Networks



What if we don't know the structure?

Take CS228: Probabilistic Graphical Models!

Disadvantages of rejection sampling

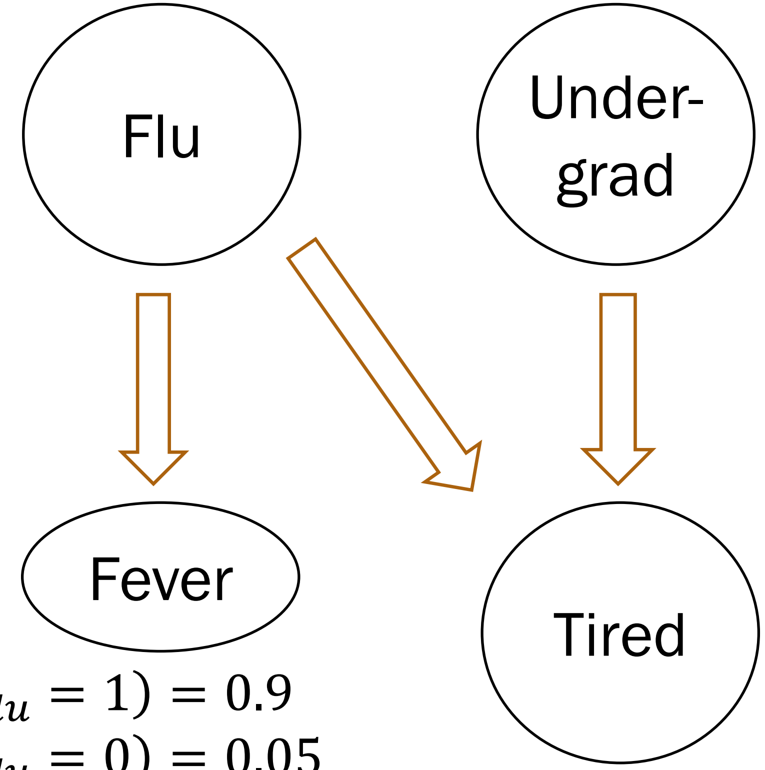
$$P(F_{lu} = 1 | F_{ev} = 1)?$$

What if we never encounter some samples?

```
[flu=0, und, fev=1, tir]
```

$P(F_{lu} = 1) = 0.1$

$P(U = 1) = 0.8$



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$P(T = 1 | F_{lu} = 1, U = 0) = 0.9$
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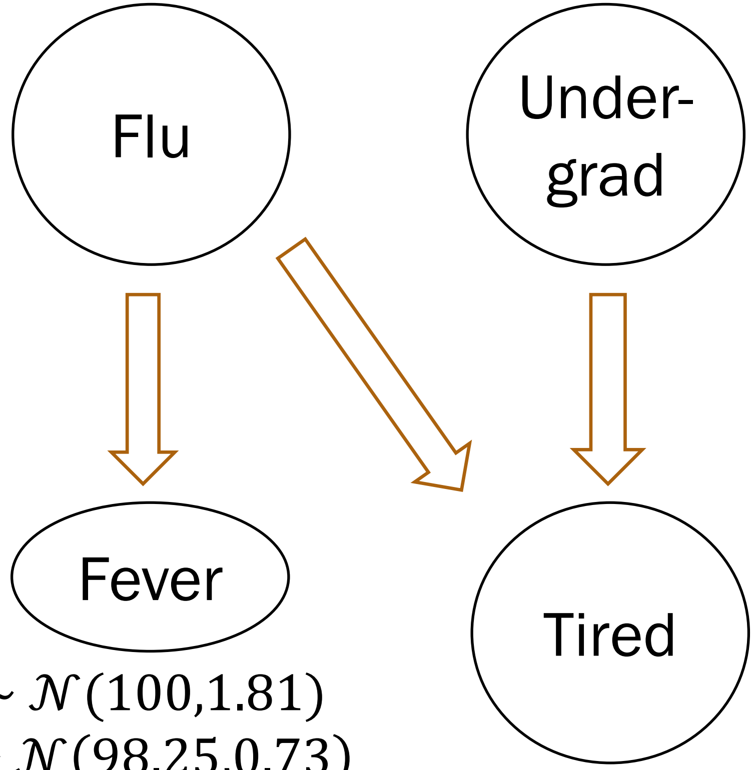
Disadvantages of rejection sampling

$$P(F_{lu} = 1 | F_{ev} = 99.4)?$$

What if we never encounter some samples?

What if random variables are continuous?

$P(F_{lu} = 1) = 0.1$ $\quad$ $P(U = 1) = 0.8$



$F_{ev} | F_{lu} = 1 \sim \mathcal{N}(100, 1.81)$

$F_{ev} | F_{lu} = 0 \sim \mathcal{N}(98.25, 0.73)$

$P(T = 1 | F_{lu} = 0, U = 0) = 0.1$

$P(T = 1 | F_{lu} = 0, U = 1) = 0.8$

$P(T = 1 | F_{lu} = 1, U = 0) = 0.9$

$P(T = 1 | F_{lu} = 1, U = 1) = 1.0$

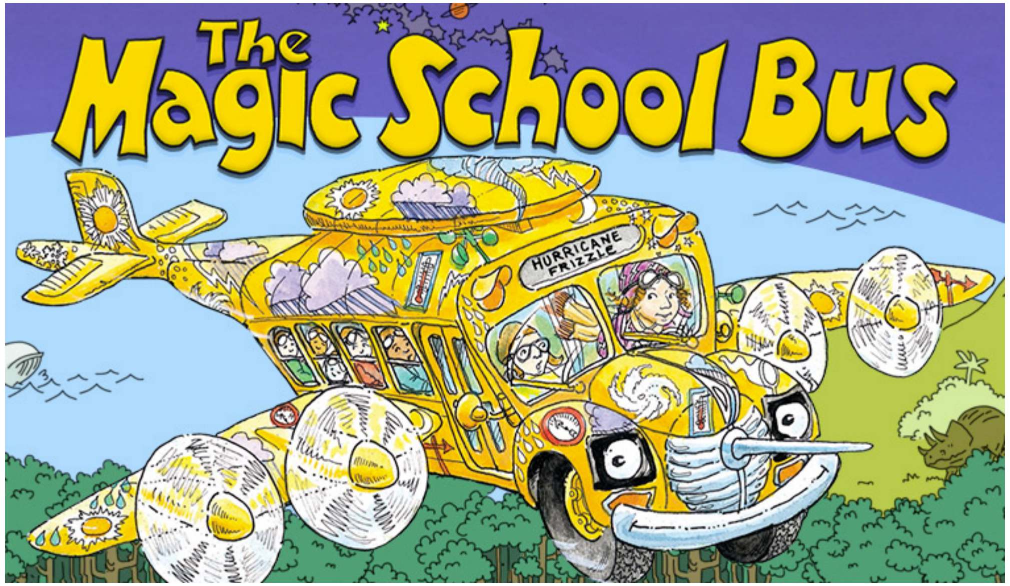
(no video)

Gibbs sampling (extra)

Gibbs Sampling (not covered)

Basic idea:

- Fix all observed events
- Incrementally sample a new value for each random variable
- Difficulty: More coding for computing different posterior probabilities



Learn in extra slides/extra notebook!

(or by taking CS228/CS238)