16: Continuous Joint Distributions

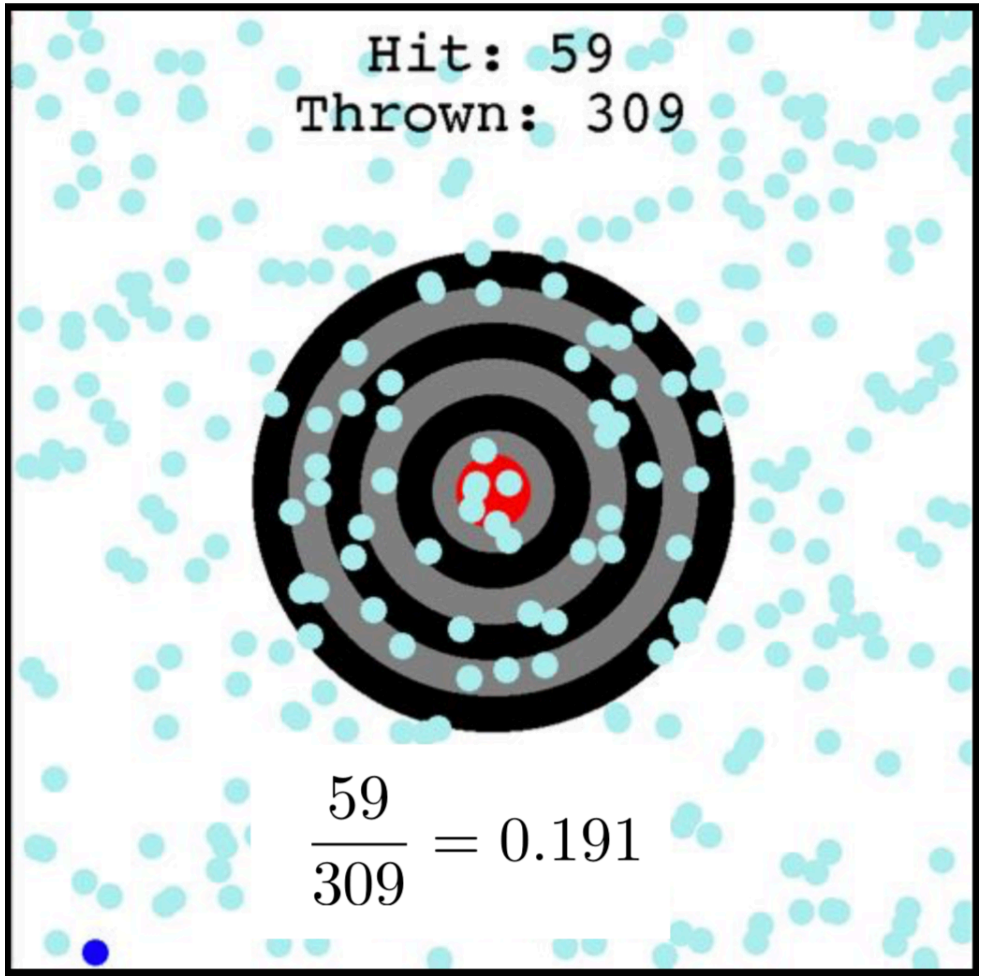
Lisa Yan

May 11, 2020

Quick slide reference

Continuous joint distributions

Remember target?



Good times…

CS109 logo with darts

The CS109 logo was created by throwing 500,000 darts according to a joint distribution.

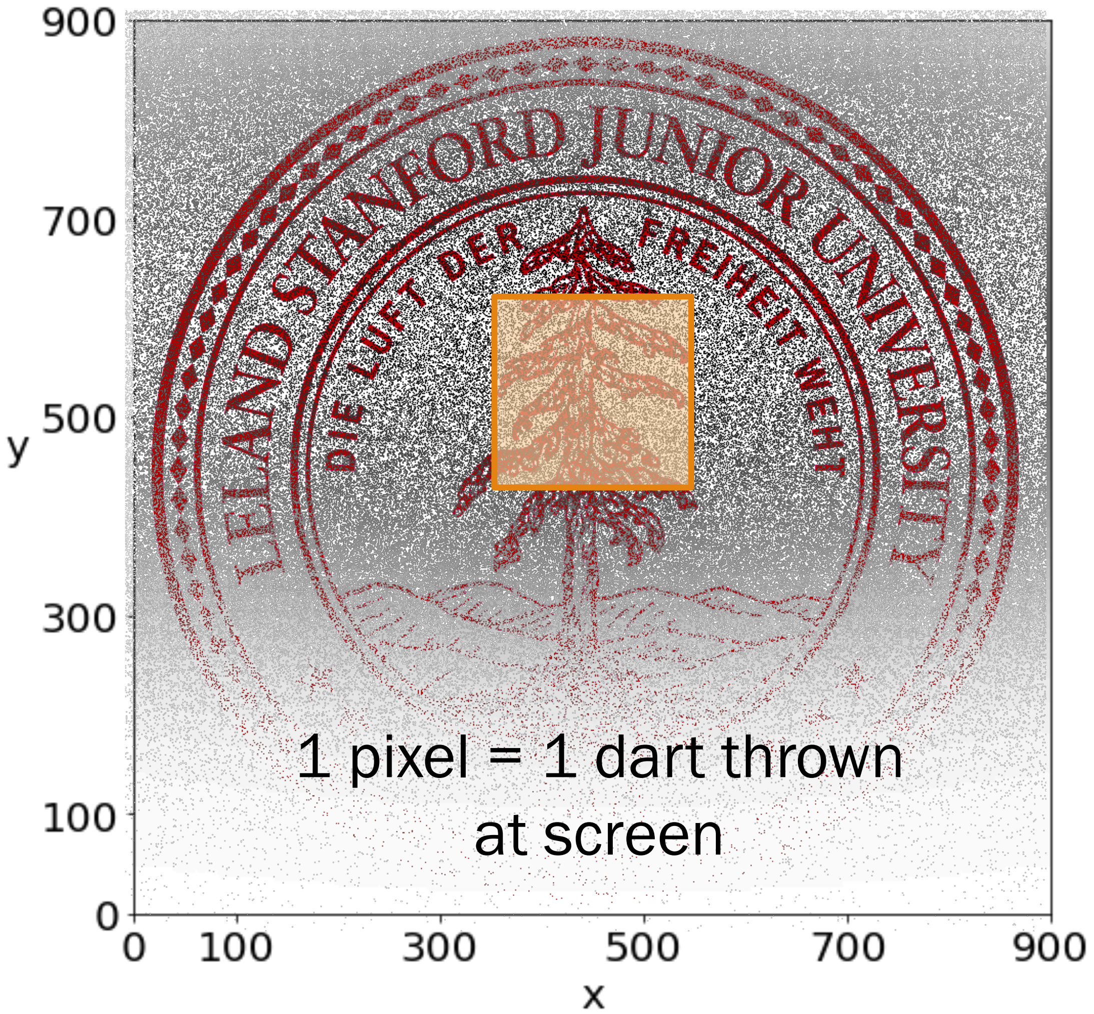
If we throw another dart according to the same distribution, what is

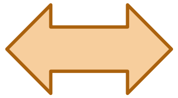
P(dart hits within r pixels of center)?

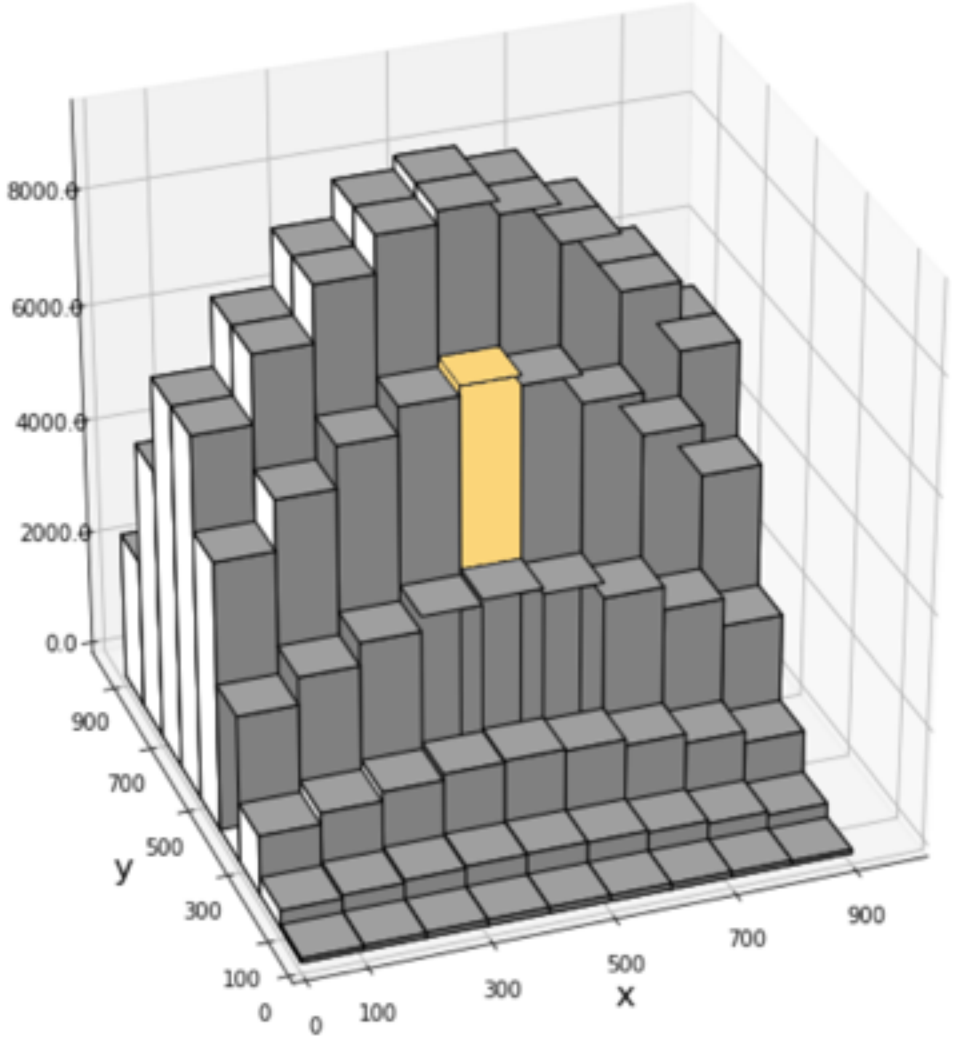
Quick check: What is the probability that a dart hits at (456.2344132343, 532.1865739012)?

CS109 logo with darts

P(dart hits within r pixels of center)?



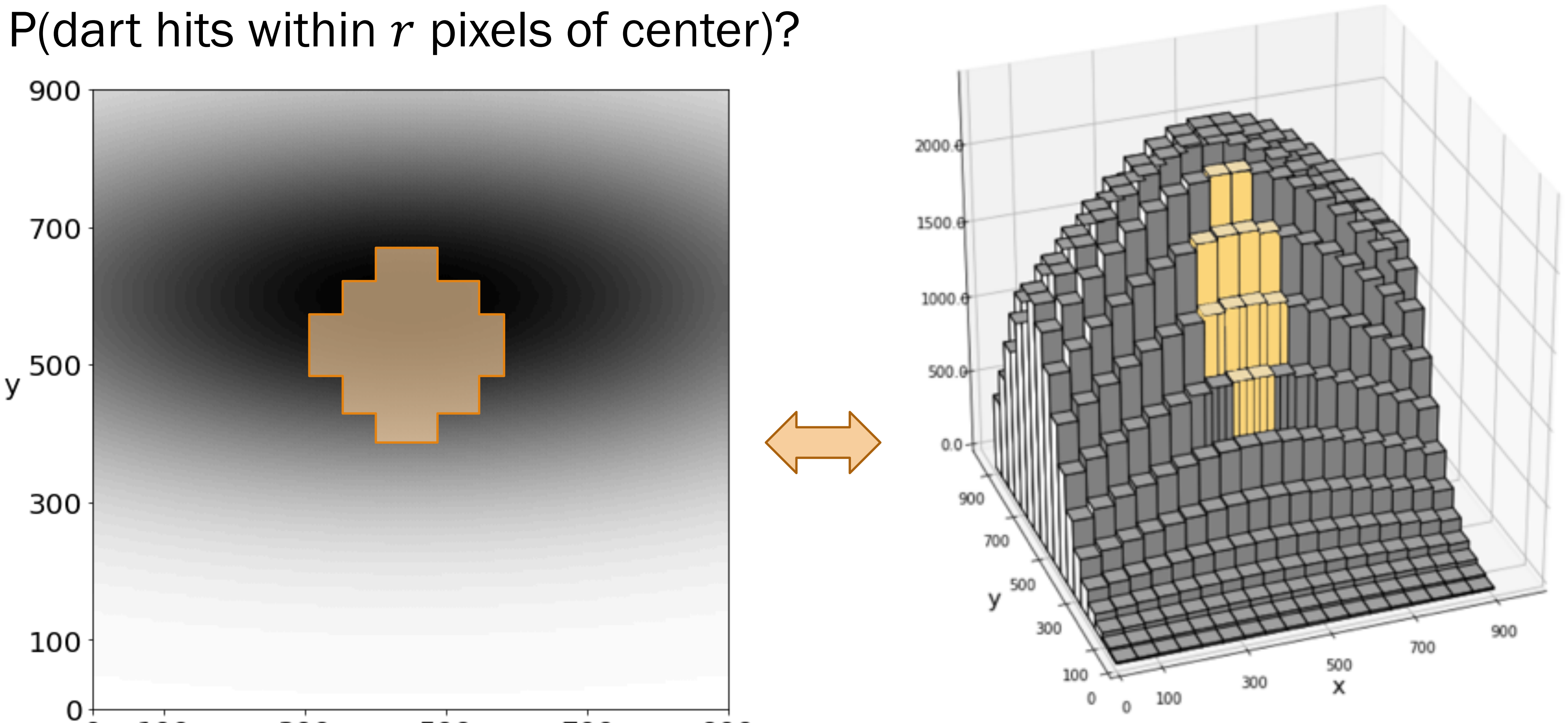




Possible dart counts (in 100x100 boxes)

CS109 logo with darts

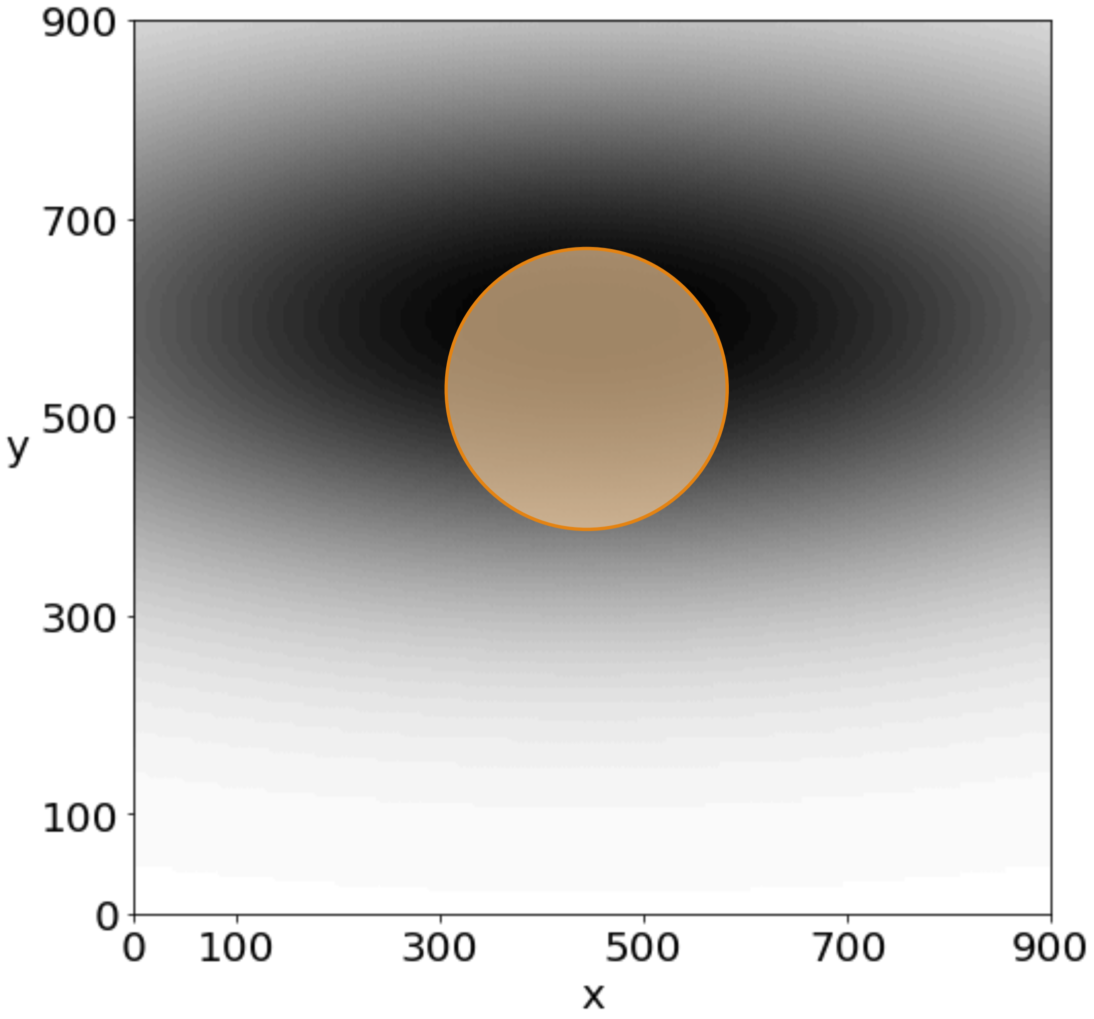
P(dart hits within r pixels of center)?

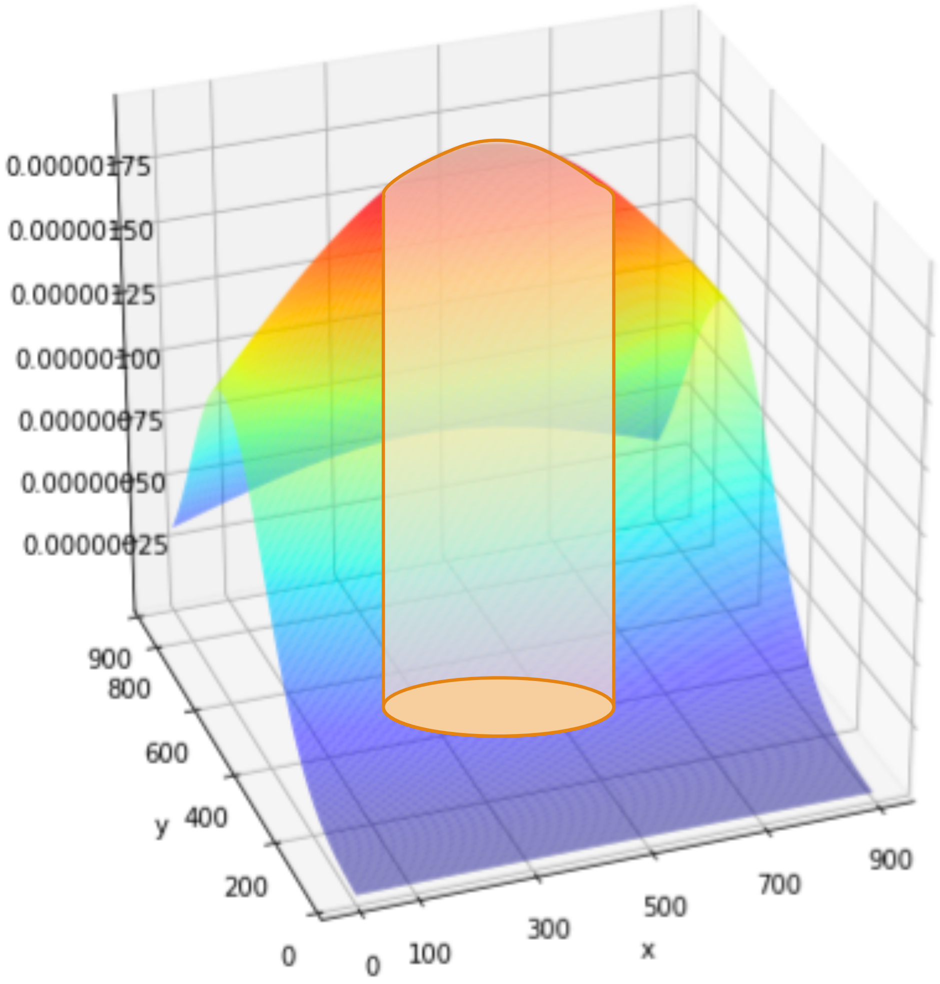


Possible dart counts (in 50x50 boxes)

CS109 logo with darts

P(dart hits within r pixels of center)?





Possible dart counts

(in infinitesimally small boxes)

Continuous joint probability density functions

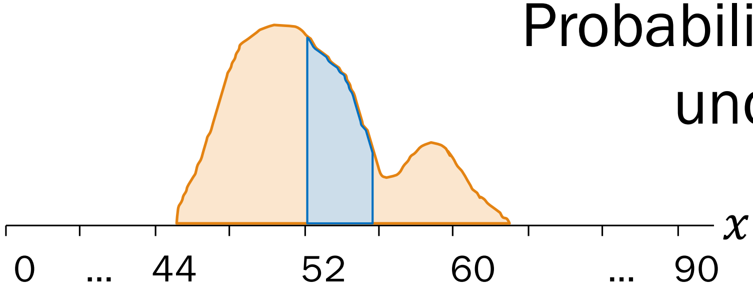
If two random variables X and Y are jointly continuous, then there exists a joint probability density function $f_{X,Y}$ defined over $-\infty < x, y < \infty$ such that:

$$P\left(a_1 \leq X \leq a_{2,} \;\; b_1 \leq Y \leq b_2\right) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x,y) dy\, dx$$

From one continuous RV to jointly continuous RVs

Single continuous RV X

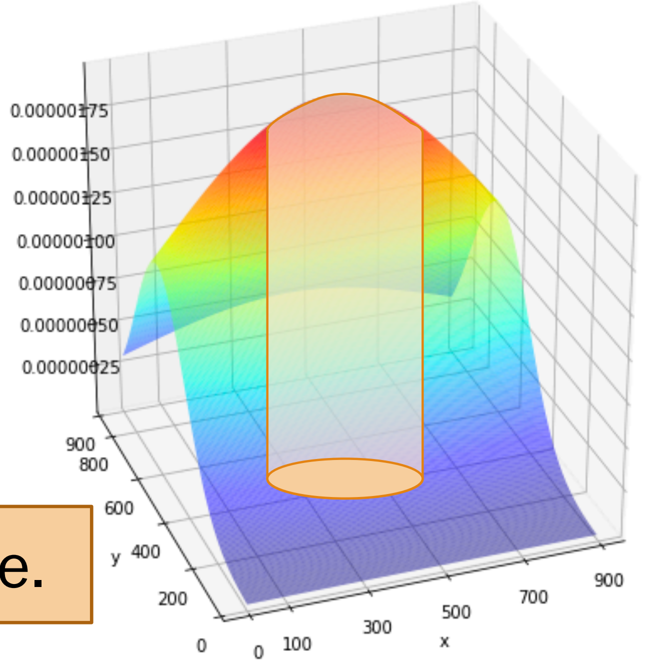
- PDF f_X such that $\int_{-\infty}^{\infty} f_X(x)dx = 1$
- Integrate to get probabilities



Probability = **area** under curve

Jointly continuous RVs X and Y

- PDF $f_{X,Y}$ such that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy\,dx = 1$
- Double integrate to get probabilities



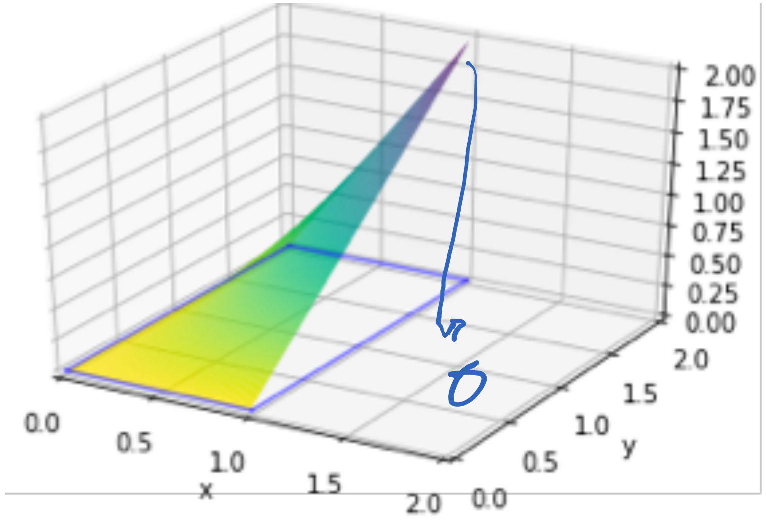
Probability for jointly continuous RVs is **volume** under a surface.

Double integrals without tears

Let X and Y be two continuous random variables.

- Support: $0 \leq X \leq 1$, $0 \leq Y \leq 2$.

Is $g(x, y) = xy$ a valid joint PDF over X and Y?



Write down the definite double integral that must integrate to 1:

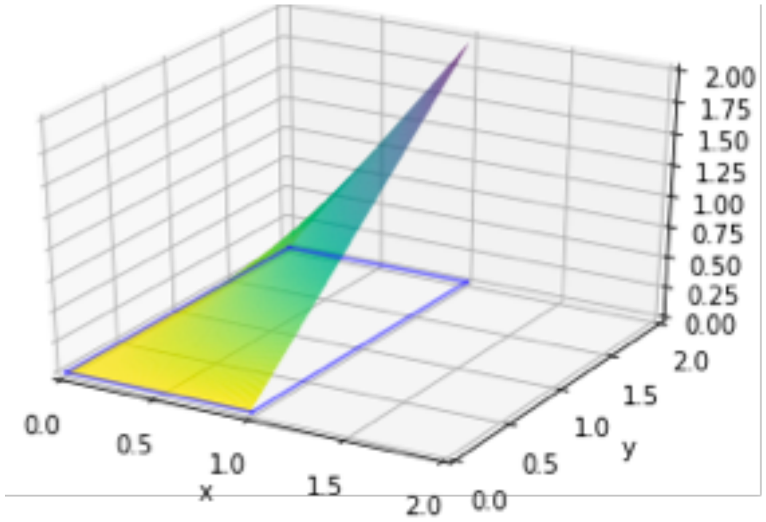
Stanford University

Double integrals without tears

Let X and Y be two continuous random variables.

- Support: $0 \le X \le 1$, $0 \le Y \le 2$.

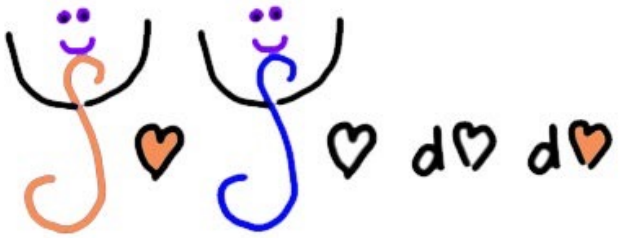
Is $g(x, y) = xy$ a valid joint PDF over X and Y?



Write down the definite double integral that must integrate to 1:

$$\int_{y=0}^{2} \int_{x=0}^{1} xy \, dx \, dy = 1 \quad \text{or} \quad \int_{x=0}^{1} \int_{y=0}^{2} xy \, dy \, dx = 1$$

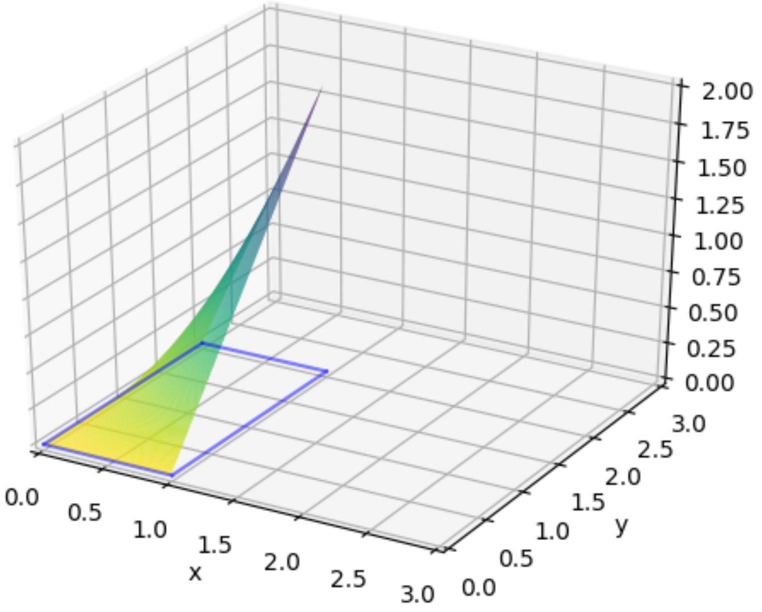
(used in next slide)



Double integrals without tears

Let X and Y be two continuous random variables.

- Support: $0 \le X \le 1$, $0 \le Y \le 2$.



Is $g(x,y) = xy$ a valid joint PDF over X and Y?

0. Set up integral:

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)dx\,dy = \int_{y=0}^{2}\int_{x=0}^{1} xy\,dx\,dy$$

1. Evaluate inside integral by treating y as a constant:

$$\int_{y=0}^{2}\left(\int_{x=0}^{1} xy\,dx\right)dy = \int_{y=0}^{2} y\left(\int_{x=0}^{1} x\,dx\right)dy = \int_{y=0}^{2} y\left[\frac{x^2}{2}\right]_0^1 dy = \int_{y=0}^{2} y\frac{1}{2}dy$$

2. Evaluate remaining (single) integral:

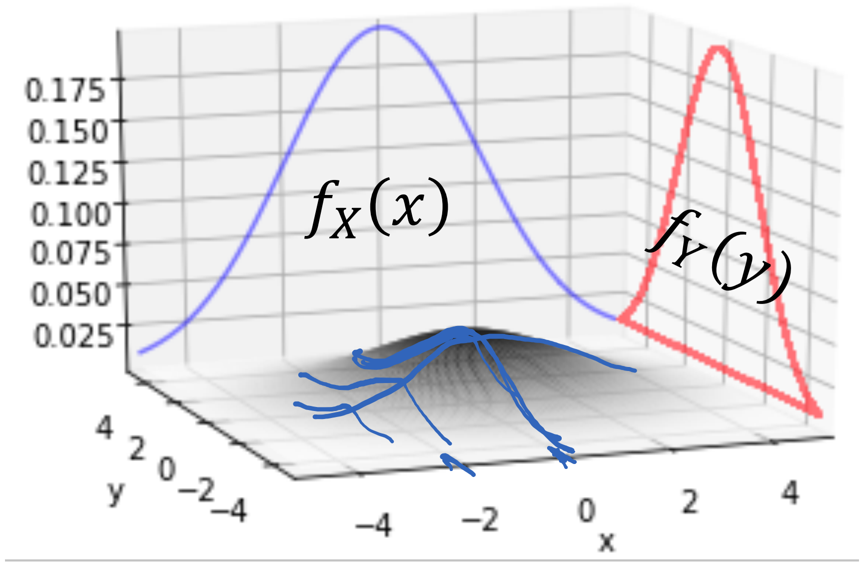
$$\int_{y=0}^{2} y\frac{1}{2}dy = \left[\frac{y^2}{4}\right]_{y=0}^{2} = 1 - 0 = 1$$

Yes, $g(x,y)$ is a valid joint PDF because it integrates to 1.

Marginal distributions

Suppose X and Y are continuous random variables with joint PDF:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx = 1$$



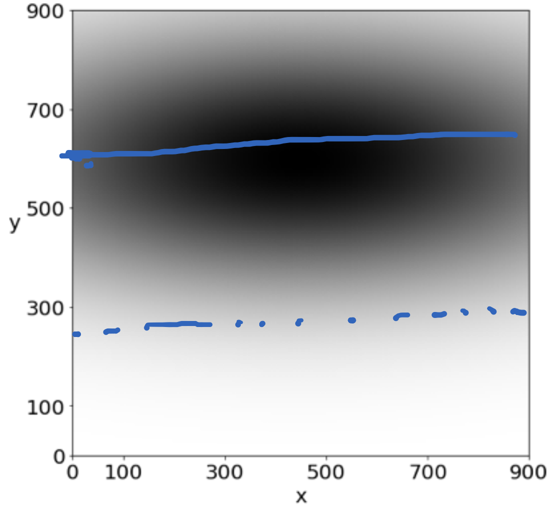
The marginal density functions (marginal PDFs) are therefore:

$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a,y)\,dy \qquad f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x,b)\,dx$$

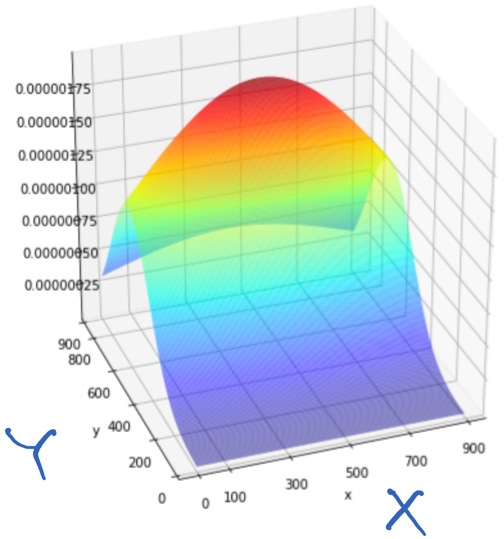
$$p_X(a) = \sum_y p_{X,Y}(a,y)$$

Back to darts!

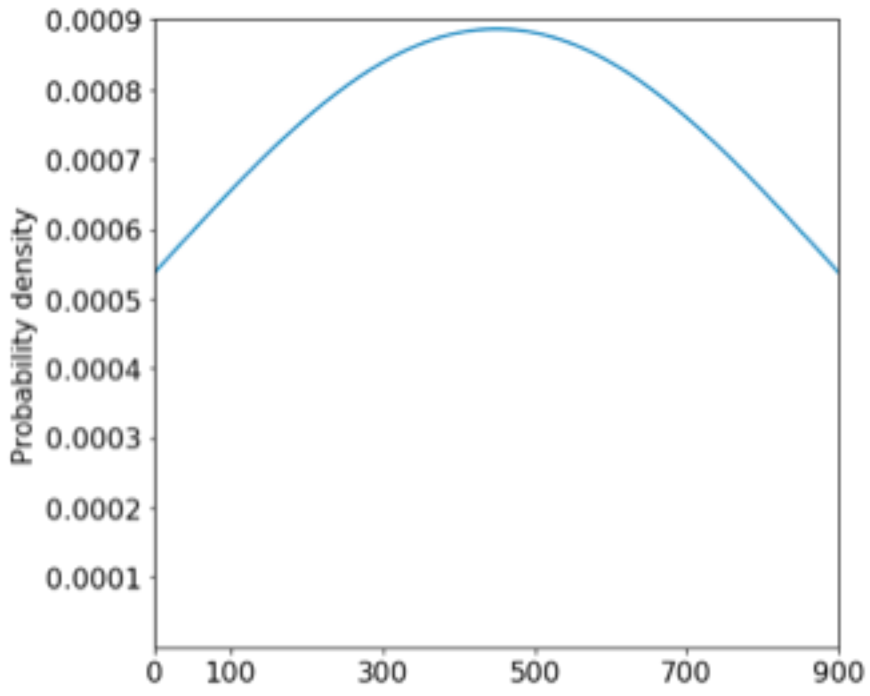
(top-down)

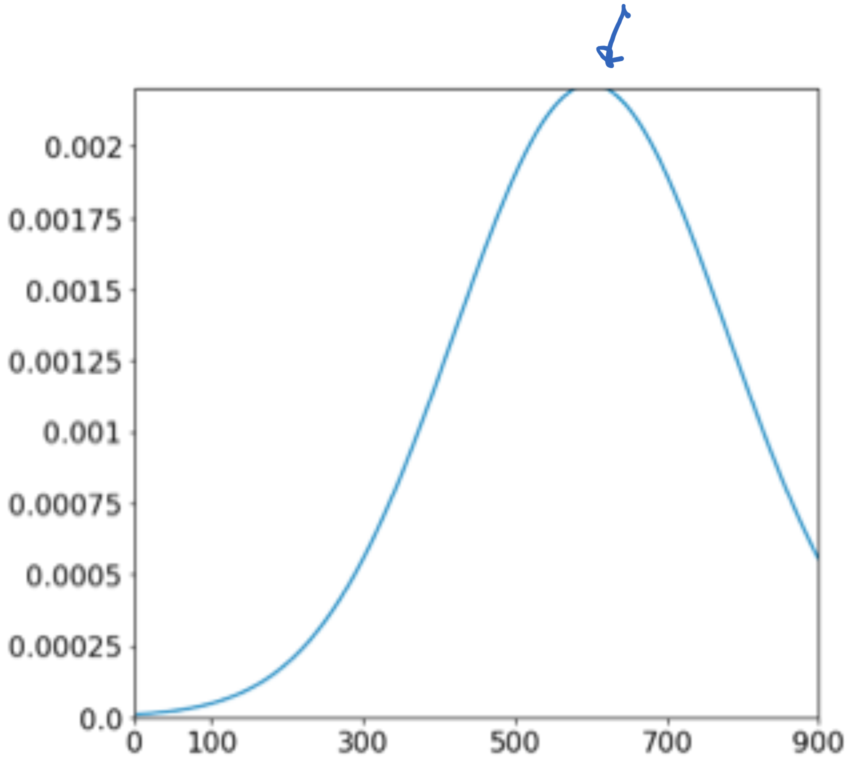


(side view)



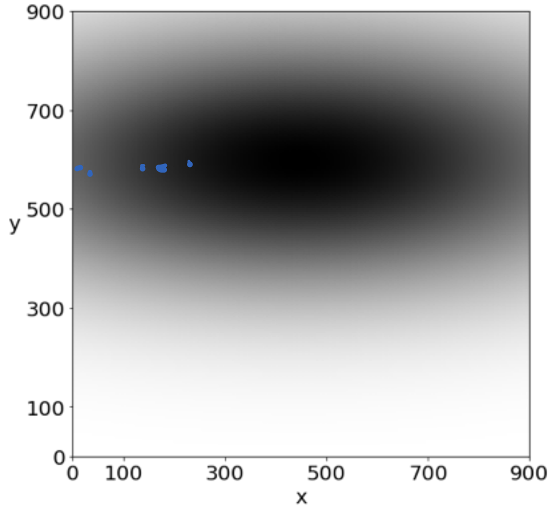
Match X and Y to their respective marginal PDFs:





Back to darts!

(top-down)



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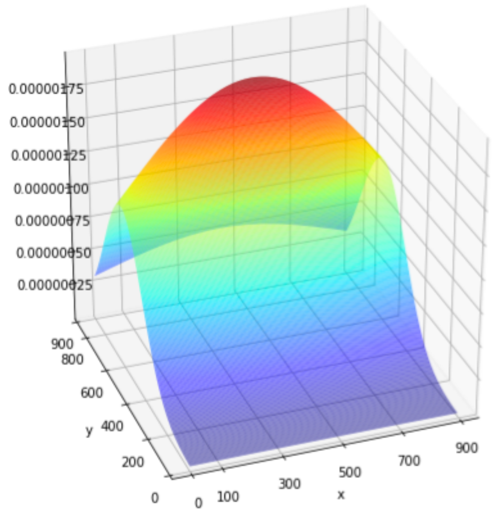
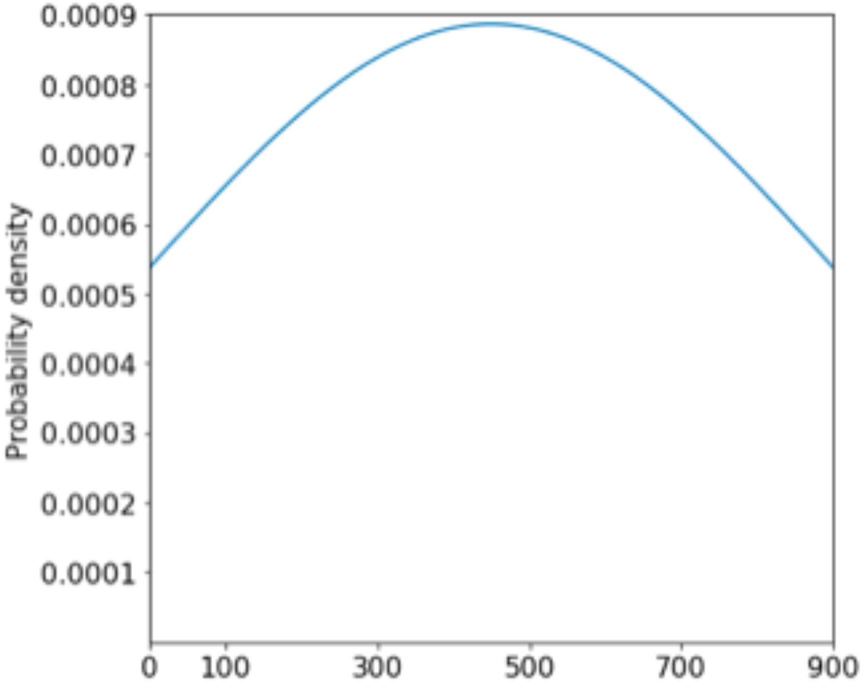


Match X and Y to their respective marginal PDFs:



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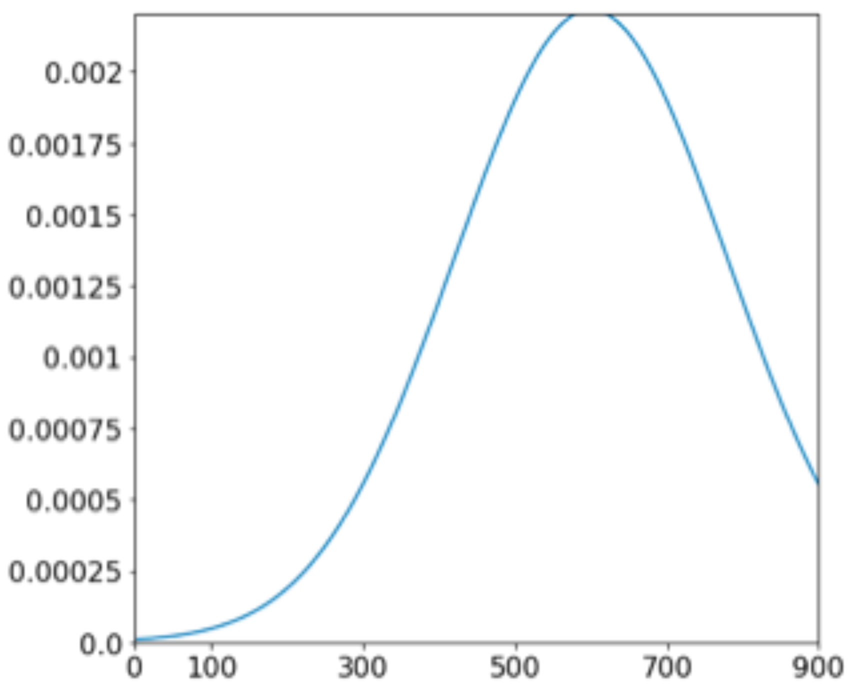

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Extra slides

If you want more practice with double integrals,

I've included two exercises at the end of this lecture.

Joint CDFs

An observation: Connecting CDF to PDF

For a continuous random variable X with PDF f, the CDF (cumulative distribution function) is

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

The density f is therefore the derivative of the CDF, F:

$$f(a) = \frac{d}{da}F(a)$$

(Fundamental Theorem of Calculus)

Joint cumulative distribution function

For two random variables X and Y, there can be a **joint cumulative distribution function** $F_{X,Y}$:

$$F_{X,Y}(a,b) = P(X \leq a, Y \leq b)$$

For discrete X and Y:

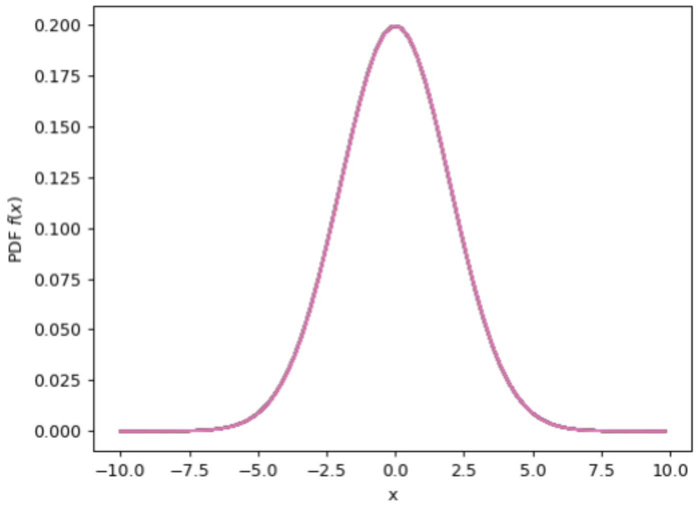
$$F_{X,Y}(a,b) = \sum_{x \leq a} \sum_{y \leq b} p_{X,Y}(x,y)$$

For continuous X and Y:

$$F_{X,Y}(a,b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x,y)dy\,dx$$

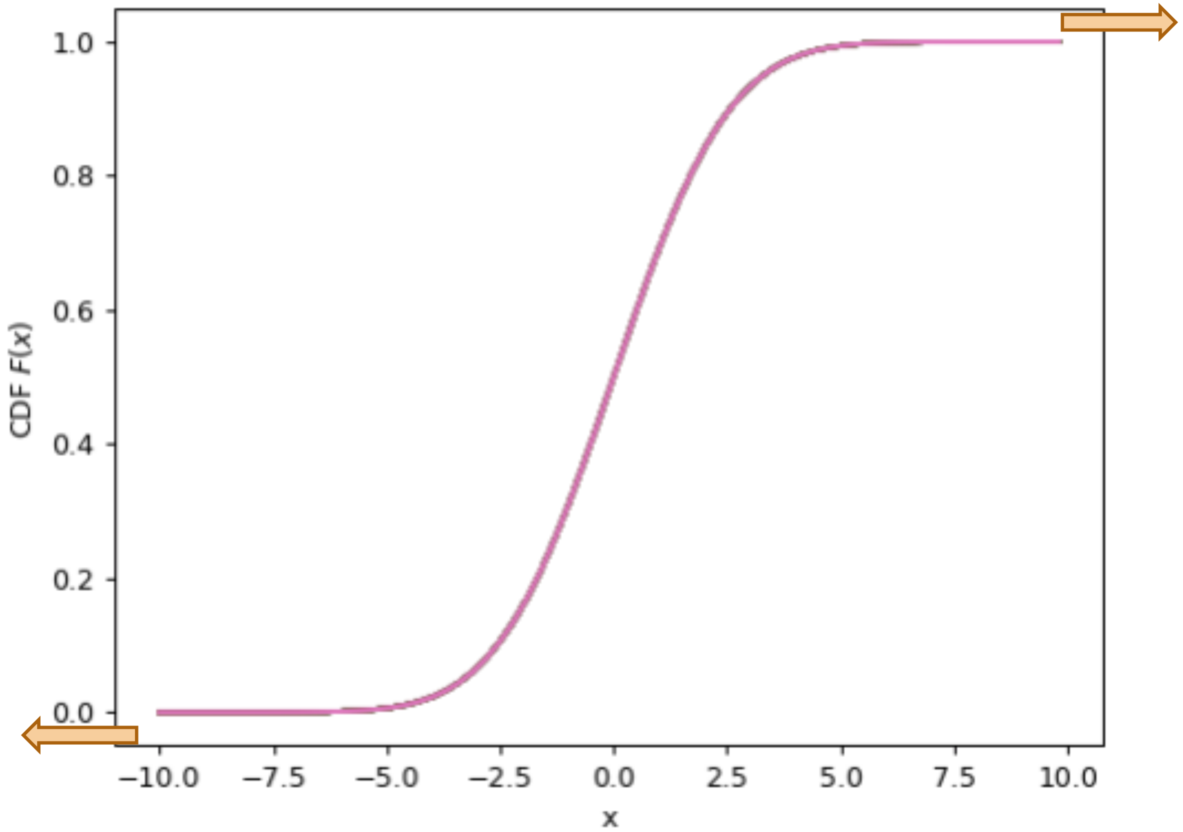
$$f_{X,Y}(a,b) = \frac{\partial^2}{\partial a\,\partial b} F_{X,Y}(a,b)$$

Single variable CDF, graphically



$f_X(x)$

$\lim_{x \to +\infty} F_X(x) = 1$

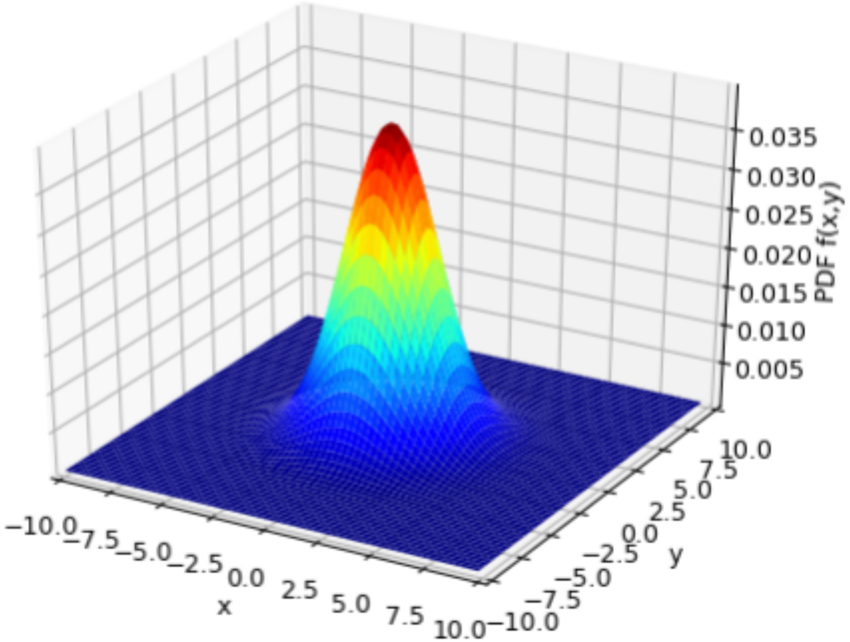


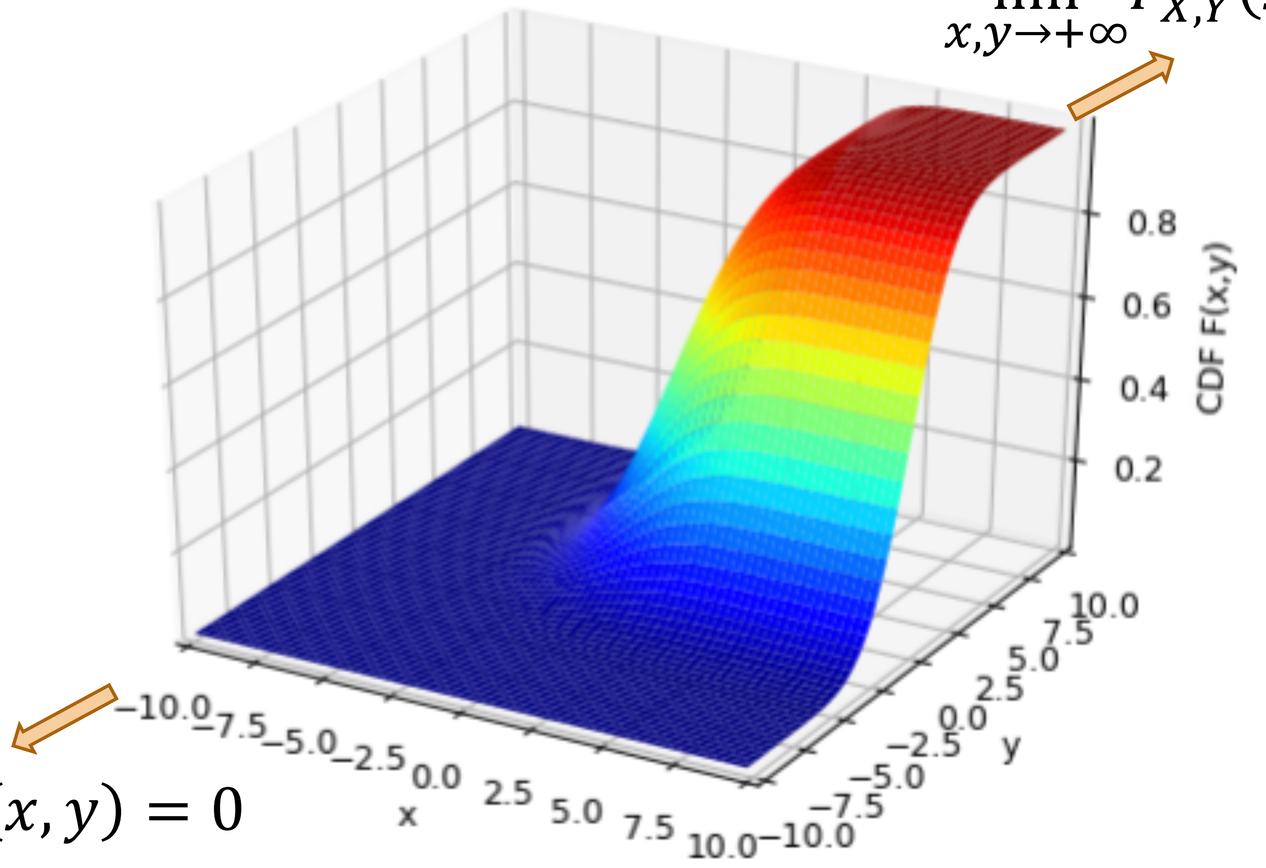
$\lim_{x \to -\infty} F_X(x) = 0$

$F_X(x) = P(X \leq x)$

Joint CDF, graphically

$$\lim_{x,y\to+\infty} F_{X,Y}(x,y) = 1$$





$$\lim_{x,y\to-\infty} F_{X,Y}(x,y) = 0$$

$f_{X,Y}(x,y)$

$F_{X,Y}(x,y) = P(X \le x, Y \le y)$

Independent Continuous RVs

Independent continuous RVs

Two continuous random variables X and Y are **independent** if:

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

$\forall x, y$

Equivalently:

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$
$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

Proof of PDF:

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x\, \partial y} F_{X,Y}(x, y) = \frac{\partial^2}{\partial x\, \partial y} F_X(x)F_Y(y)$$

$\forall x, y$

$$= \frac{\partial}{\partial x} \frac{\partial}{\partial y} F_X(x)F_Y(y) \qquad = \frac{\partial}{\partial x} F_X(x) \frac{\partial}{\partial y} F_Y(y)$$

$$= f_X(x)f_Y(y)$$

Independent continuous RVs

Two continuous random variables X and Y are **independent** if:

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

Equivalently:

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$
$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

More generally, X and Y are independent if joint density factors separately:

$$f_{X,Y}(x, y) = g(x)h(y), \text{ where } -\infty < x, y < \infty$$

Pop quiz! (just kidding)

$f_{X,Y}(x,y) = g(x)h(y)$, where $-\infty < x, y < \infty$ → independent X and Y

Are X and Y independent in the following cases?

1. $f_{X,Y}(x,y) = 6e^{-3x}e^{-2y}$
   where $0 < x, y < \infty$

2. $f_{X,Y}(x,y) = 4xy$
   where $0 < x, y < 1$

3. $f_{X,Y}(x,y) = 24xy$
   where $0 < x + y < 1$

🤔

Pop quiz! (just kidding)

$f_{X,Y}(x,y) = g(x)h(y)$, where $-\infty < x, y < \infty$ ⇒ independent X and Y

Are X and Y independent in the following cases?

1. $f_{X,Y}(x,y) = 6e^{-3x}e^{-2y}$ where $0 < x, y < \infty$

   Separable functions: $g(x) = 3e^{-3x}$, $h(y) = 2e^{-2y}$

2. $f_{X,Y}(x,y) = 4xy$ where $0 < x, y < 1$

   Separable functions: $g(x) = 2x$, $h(y) = 2y$

   $g(x) = 4x$, $h(y) = y$

3. $f_{X,Y}(x,y) = 24xy$ where $0 < x + y < 1$

   Cannot capture constraint on $x + y$ into factorization!

$$\int_0^1\int_0^1 4xy = 1 = \underbrace{\int_0^1 c\cdot 4x\,dx}_{1} \cdot \underbrace{\int_0^1 \frac{1}{c}\cdot y\,dy}_{1}$$

If you can factor densities over all of the support, you have independence.

Bivariate Normal Distribution

Bivariate Normal Distribution

$X \sim N(\mu, \sigma^2)$ $\quad f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

X_1 and X_2 follow a bivariate normal distribution if their joint PDF f is

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}$$

Can show that $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ (Ross chapter 6, example 5d)

Often written as:

$$\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

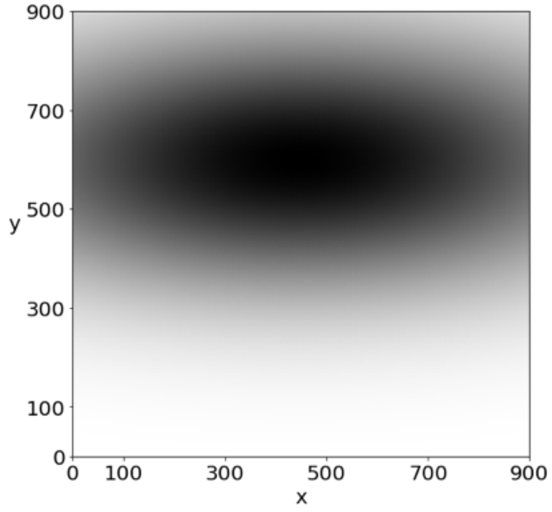
- Vector $\boldsymbol{X} = (X_1, X_2)$
- Mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)$, Covariance matrix: $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \mathrm{Cov}(X_1, X_1) & \mathrm{Cov}(X_1, X_2) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Cov}(X_2, X_2) \end{bmatrix}$

Recall correlation: $\rho = \frac{\mathrm{Cov}(X_1, X_2)}{\sigma_1\sigma_2}$

We will focus on understanding the **shape** of a bivariate Normal RV.

Back to darts

(top-down)



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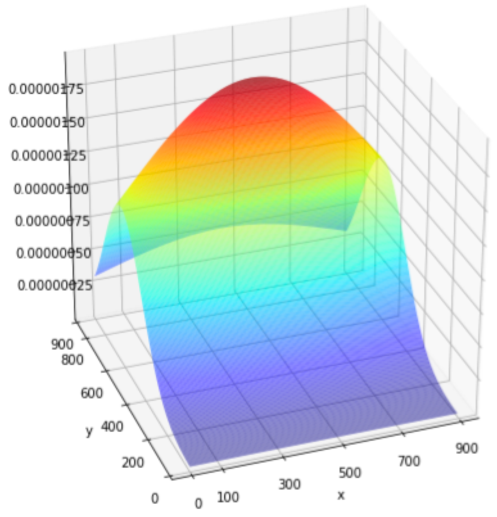


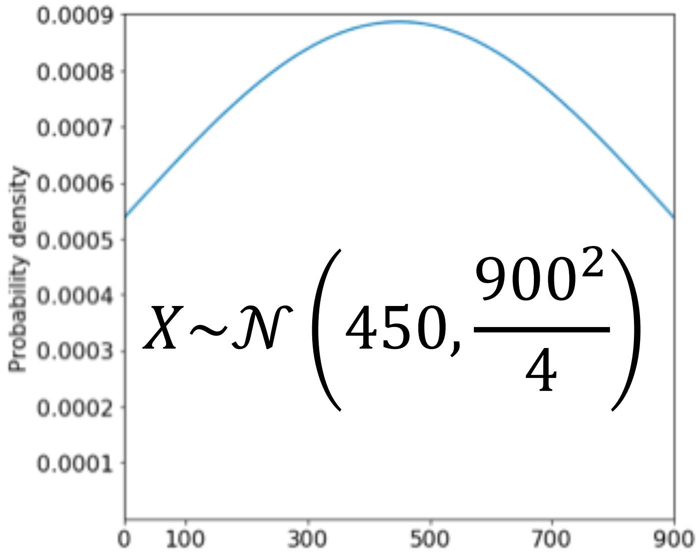
These darts were actually thrown according to a bivariate normal distribution:

$$(X, Y) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

$$\boldsymbol{\mu} = (450, 600)$$

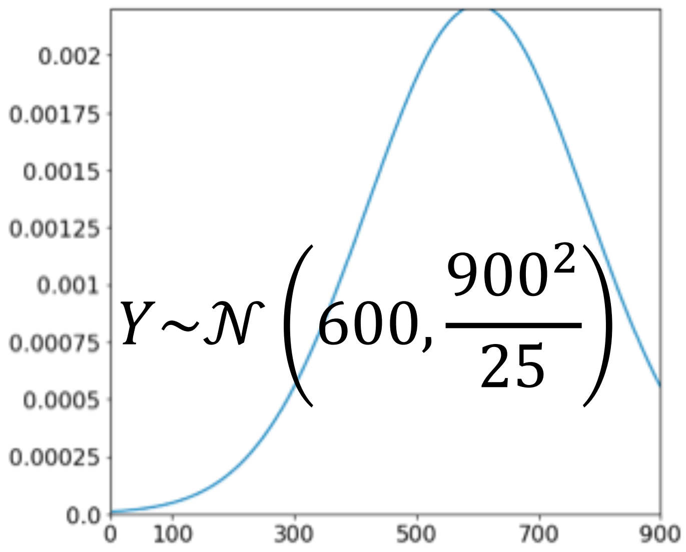
$$\boldsymbol{\Sigma} = \begin{bmatrix} 900^2/4 & 0 \\ 0 & 900^2/25 \end{bmatrix}$$

Marginal PDFs:



$$X \sim \mathcal{N}\left(450, \frac{900^2}{4}\right)$$

pixel x



$$Y \sim \mathcal{N}\left(600, \frac{900^2}{25}\right)$$

pixel y

A diagonal covariance matrix

Let $\boldsymbol{X} = (X_1, X_2)$ follow a bivariate normal distribution $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = (\mu_1, \mu_2), \qquad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

Cov$(X_1, X_2) = \rho\sigma_1\sigma_2$
$= 0$

Are X_1 and X_2 independent?

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}$$

$$= \frac{1}{\sigma_1\sqrt{2\pi}} e^{-(x_1-\mu_1)^2/2\sigma_1^2} \frac{1}{\sigma_2\sqrt{2\pi}} e^{-(x_2-\mu_2)^2/2\sigma_2^2}$$

(Note covariance: $\rho\sigma_1\sigma_2 = 0$)

$\rho = 0$

X_1 and X_2 are **independent** with marginal distributions $X_1 \sim \mathcal{N}(\mu_1\ \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2\ \sigma_2^2)$

(live)

16: Continuous Joint Distributions (I)

Lisa Yan

May 11, 2020

E[109]

Jointly continuous RVs

X and Y are jointly continuous if they have a joint PDF:

$$f_{X,Y}(x,y) \text{ such that } \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx = 1$$

Most things we've learned about discrete joint distributions translate:

Marginal distributions	$p_X(a) = \sum_y p_{X,Y}(a,y)$	$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a,y)dy$
Independent RVs	$p_{X,Y}(x,y) = p_X(x)p_Y(y)$	$f_{X,Y}(x,y) = f_X(x)f_Y(y)$
Expectation (e.g., LOTUS)	$E[g(X,Y)] = \sum_x \sum_y g(x,y)p_{X,Y}(x,y)$	$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)dy\,dx$

...etc.

Think

Slide 35 has a question to go over by yourself.

Post any clarifications here!

https://us.edstem.org/courses/109/discussion/60584

Think by yourself: 2 min

(by yourself)

Warmup exercise

X and Y have the following joint PDF:

$$f_{X,Y}(x,y) = 3e^{-3x}$$
where $0 < x < \infty, 1 < y < 2$

1. Are X and Y independent?
2. What is the marginal PDF of X? Of Y?
3. What is $E[X + Y]$?

(by yourself)

Warmup exercise

X and Y have the following joint PDF:

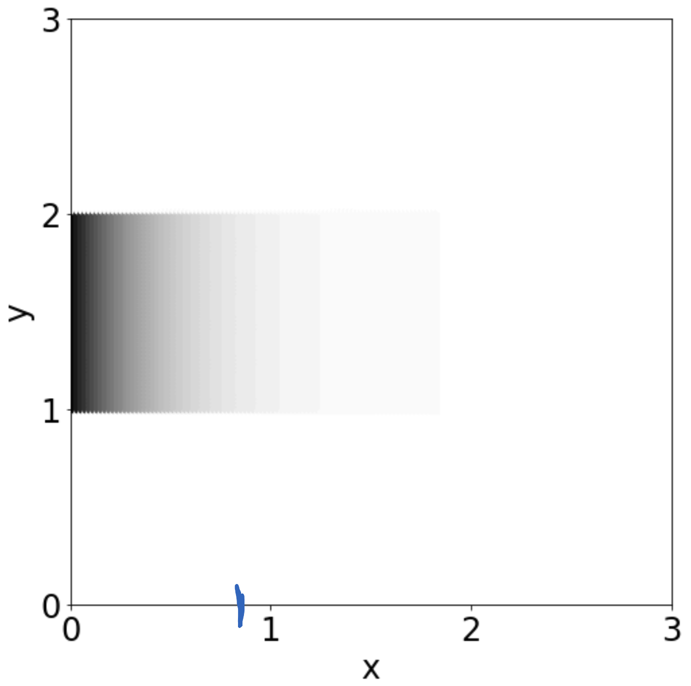
$$f_{X,Y}(x,y) = 3e^{-3x}$$
where $0 < x < \infty, 1 < y < 2$

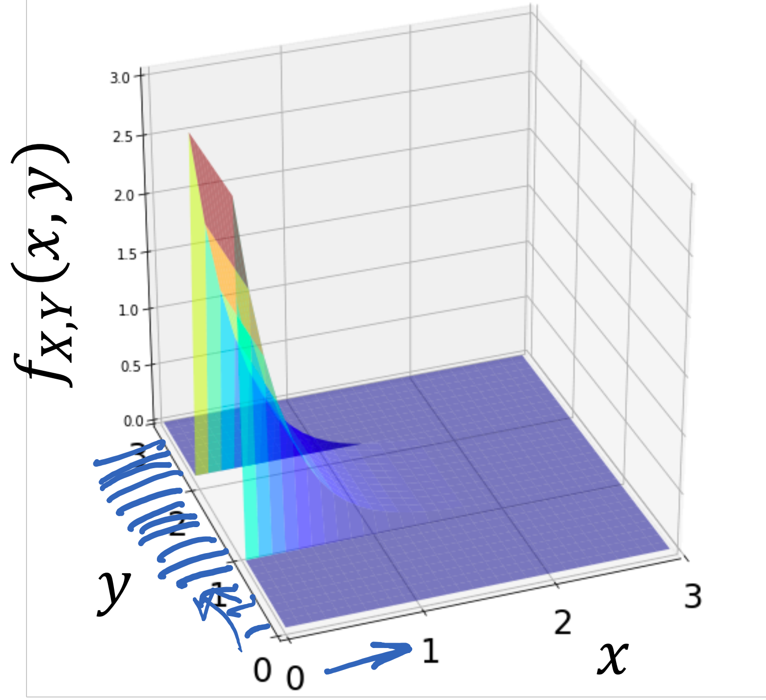
1. Are X and Y independent?

$g(x) = 3Ce^{-3x}, 0 < x < \infty$
$h(y) = 1/C, \quad 1 < y < 2$
C is a constant

2. What is the marginal PDF of X? Of Y?

3. What is $E[X+Y]$?





Warmup exercise

X and Y have the following joint PDF:

$$f_{X,Y}(x,y) = 3e^{-3x}$$
where $0 < x < \infty, 1 < y < 2$

1. Are X and Y independent?

$g(x) = 3Ce^{-3x}, 0 < x < \infty$
$h(y) = 1/C, \quad 1 < y < 2$
C is a constant

2. What is the marginal PDF of X? Of Y?

$x \geq 0 \quad f_X(x) = 3e^{-3x} \quad X \sim \text{Exp}(\lambda = 3)$
$f_Y(y) = 1 \quad Y \sim \text{Unif}(a=1, b=2)$
$1 < y < 2$

3. What is $E[X+Y]$?

Strategy 1: $E[g(X,Y)] = \int_1^2 \int_0^\infty g(x,y) f_{X,Y}(x,y)\,dx\,dy$

Strategy 2: $E[X+Y] = E[X] + E[Y]$
$1/3 + 3/2$

Breakout Rooms

Check out the question on the next slide (Slide 39). Post any clarifications here!

https://us.edstem.org/courses/109/discussion/60584

Breakout rooms: 4 min. Introduce yourself!

The joy of meetings

Two people set up a meeting time. Each arrives independently at a time uniformly distributed between 12pm and 12:30pm.

Define X = # minutes past 12pm that person 1 arrives. $X \sim \text{Unif}(0, 30)$
Y = # minutes past 12pm that person 2 arrives. $Y \sim \text{Unif}(0, 30)$

What is the probability that the first to arrive waits >10 mins for the other?

Compute: $P(X + 10 < Y) + P(Y + 10 < X) = 2P(X + 10 < Y)$ (by symmetry)

1. What is "symmetry" here?
2. How do we integrate to compute this probability?

$f_{X,Y}(x,y) = \left(\frac{1}{30}\right)^2$

$$\iint_{\substack{x+10<y \\ 0\le x,y\le 30}} \left(\frac{1}{30}\right)^2 dx\,dy + \iint_{\substack{y+10<x \\ 0\le x,y\le 30}} \left(\frac{1}{30}\right)^2 dx\,dy$$

The joy of meetings

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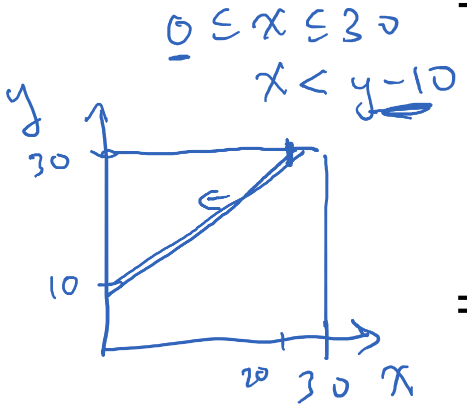


$$= 2 \cdot \iint_{x+10<y} f_{X,Y}(x,y)\,dx\,dy = 2 \cdot \iint_{\substack{x+10<y,\\ 0\le x,y,\le 30}} (1/30)^2\,dx\,dy$$ (independence)

$$= \frac{2}{30^2} \int_{10}^{30} \int_{0}^{y-10} dx\,dy = \frac{2}{30^2} \int_{10}^{30} (y-10)\,dy = \cdots = \frac{4}{9}$$

Su	M	Tu	W	Th	F	Sa

Interlude for jokes/announcements

Announcements

<u>Mid-quarter feedback form</u>

[link]

Open until: this Friday

<u>Problem Set 4</u>

Due: Monday 5/18 10am

Covers: Up to and including ~~today~~ Friday

Announcements: CS109 contest

Do something cool and creative with probability

Replaces one "passing" work requirement

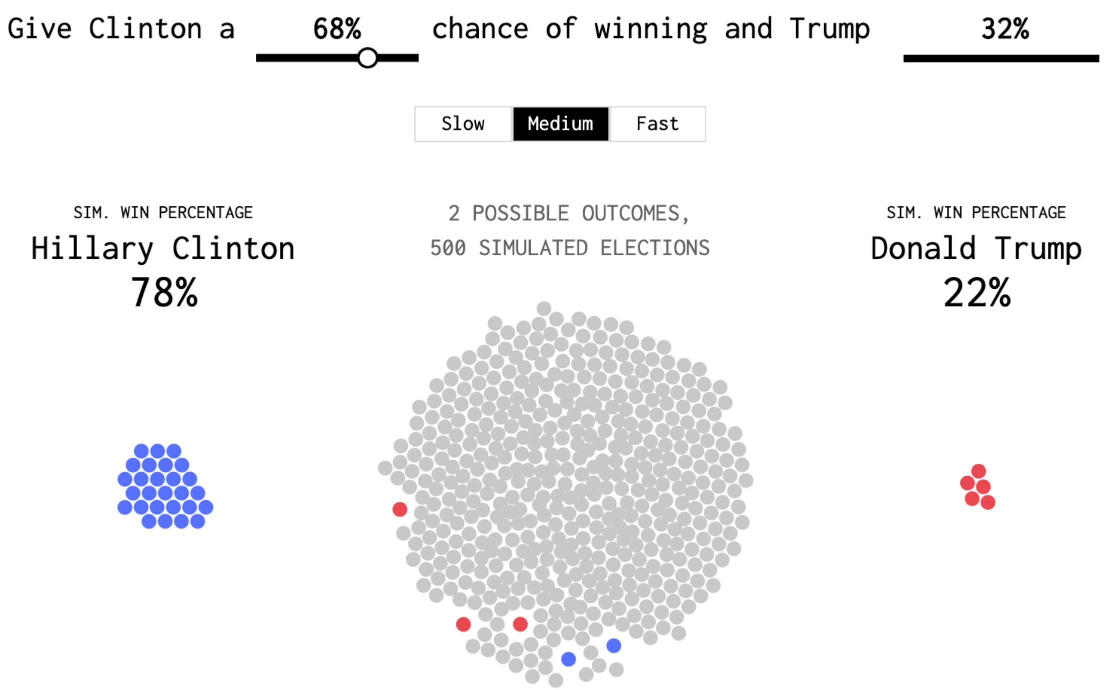
Optional Proposal: Sat. 5/23 11:59pm
Due: Monday 6/8, 11:59pm

https://web.stanford.edu/class/cs109/psets/cs109_contest.pdf

Interesting probability news

What That Election Probability Means

Even when you shift the probability far left or far right, the opposing candidate still gets some wins. That doesn't mean a forecast was wrong. That's just randomness and uncertainty at play. **The probability estimates the percentage of times you get an outcome if you were to do something multiple times.**



https://flowingdata.com/2016/07/28/what-that-election-probability-means/

CS109 Current Events Spreadsheet

Bivariate Normal Distribution

The bivariate normal distribution of $\boldsymbol{X} = (X_1, X_2)$:

$$\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

- Mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)$
- Covariance matrix: $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ Cov$(X_1, X_2) =$ Cov$(X_2, X_1) = \rho\sigma_1\sigma_2$
- Marginal distributions: $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$
- For bivariate normals in particular, Cov$(X_1, X_2) = 0$ implies X_1, X_2 independent.

We will focus on understanding the **shape** of a bivariate Normal RV.

Breakout Rooms

Check out the question on the next slide (Slide 47). Post any clarifications here!

https://us.edstem.org/courses/109/discussion/60584

Breakout rooms: 3 min. Introduce yourself!

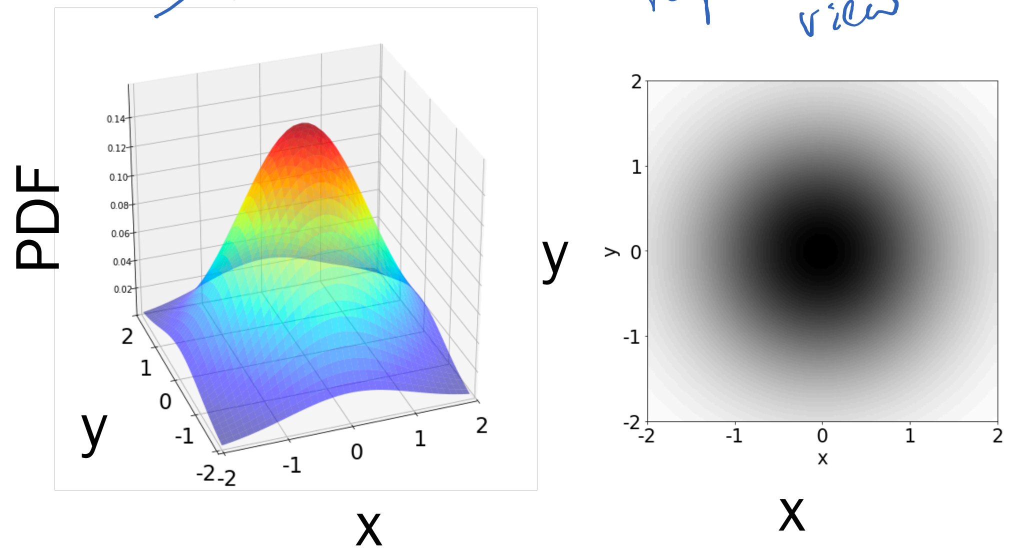
(X, Y) Matching (all have $\boldsymbol{\mu} = (0, 0)$)

A. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

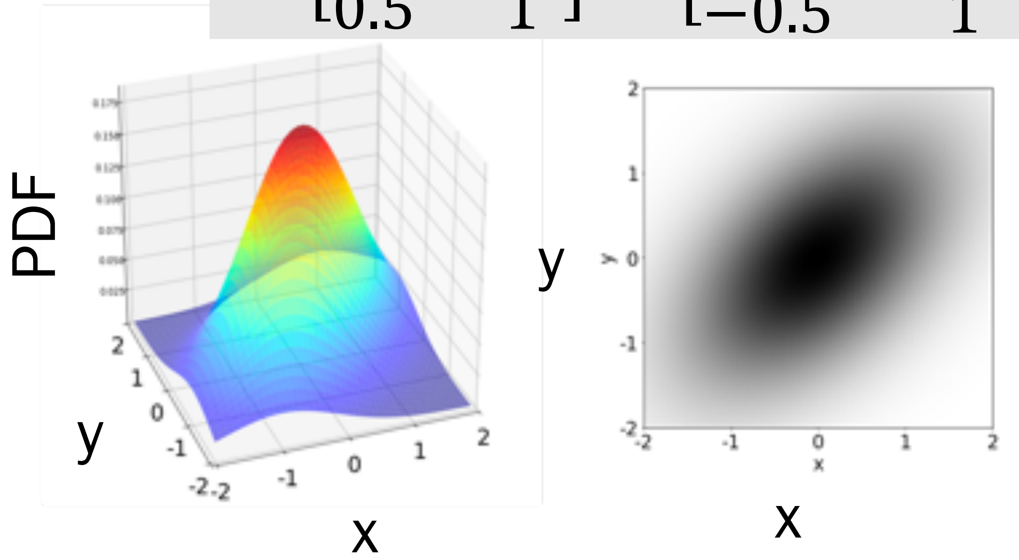
C. $\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$

3-D view   top-down view

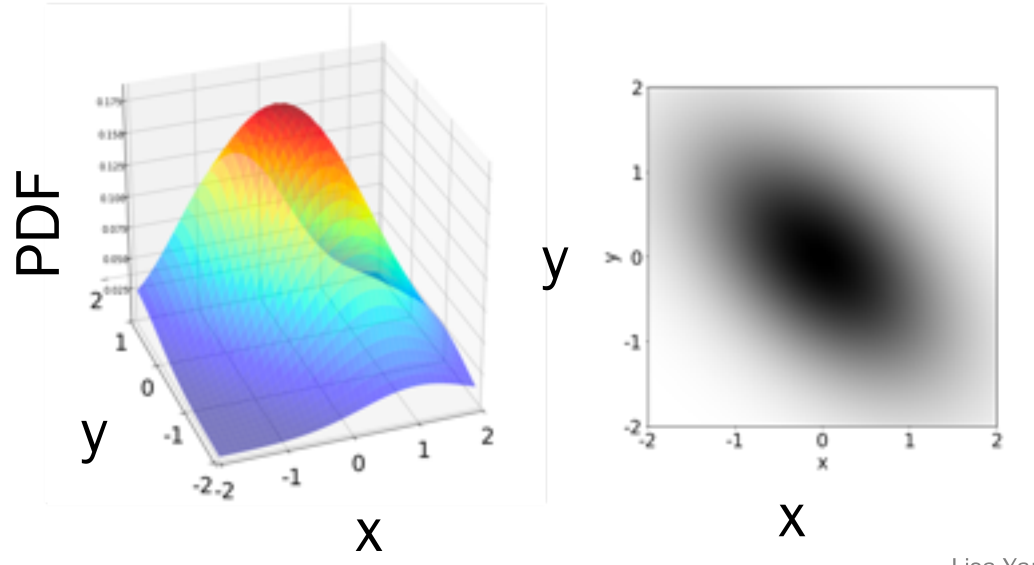
1.



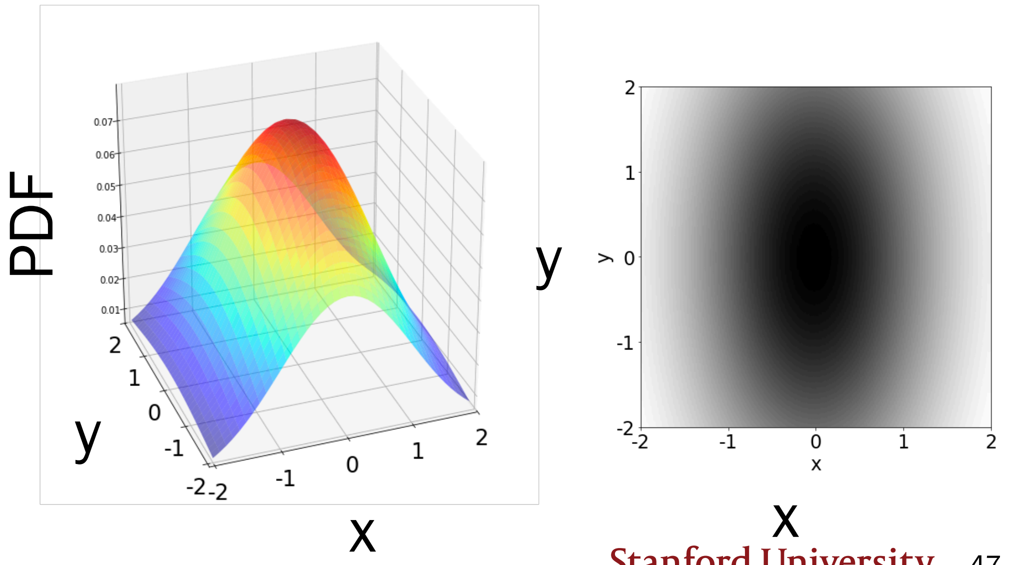
2.



3.



4.



(X, Y) Matching (all have $\boldsymbol{\mu} = (0, 0)$)

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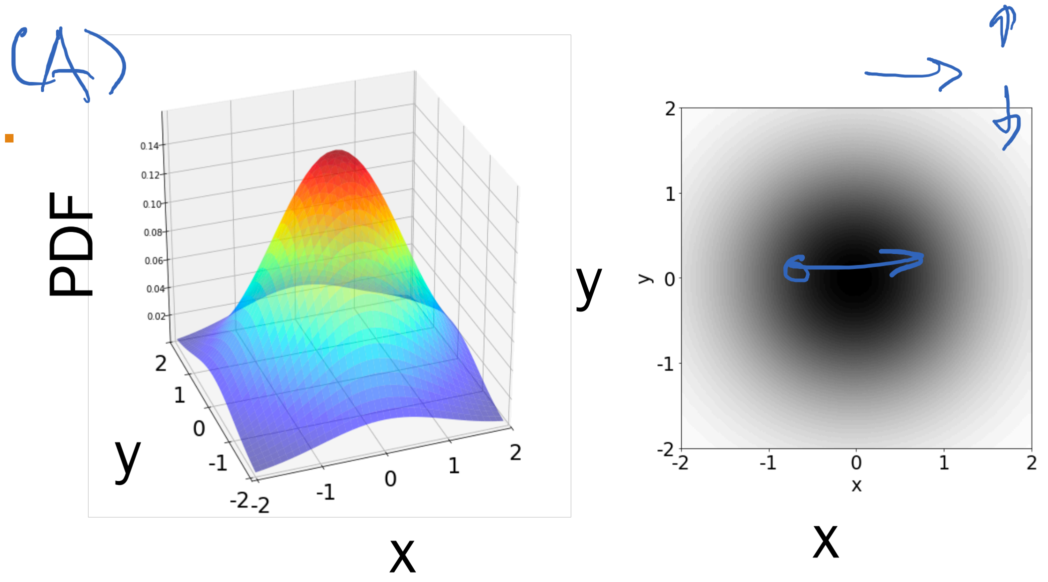
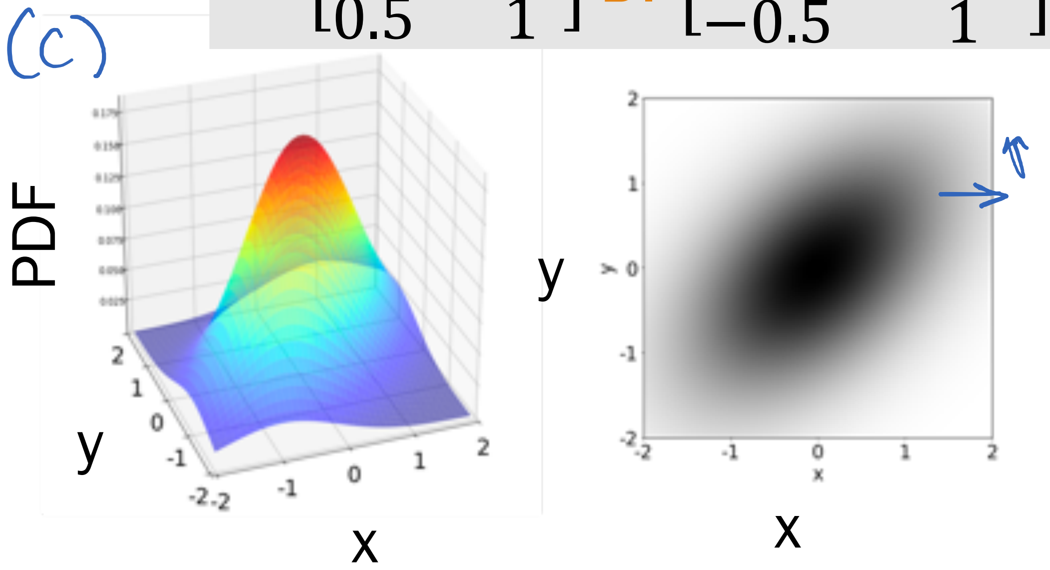
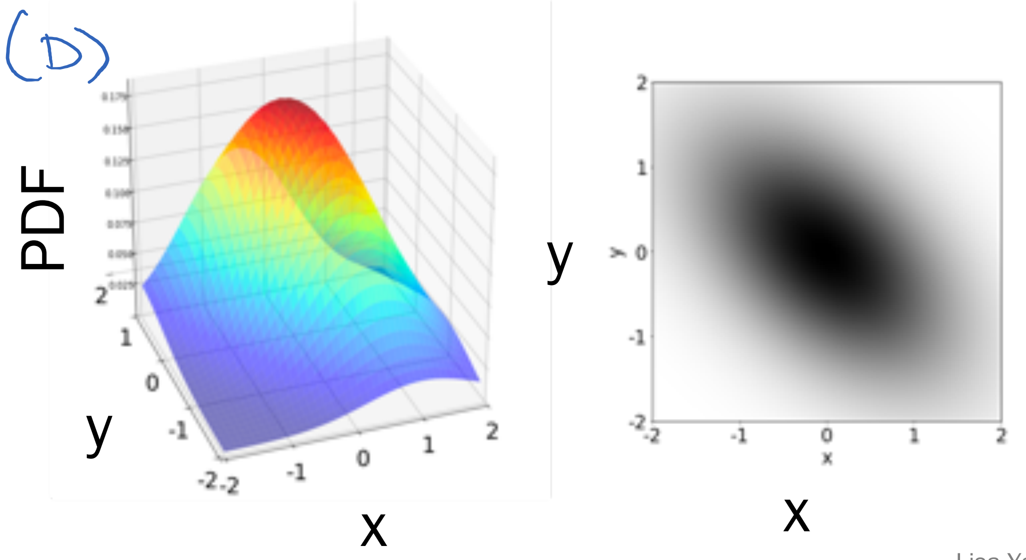
1.



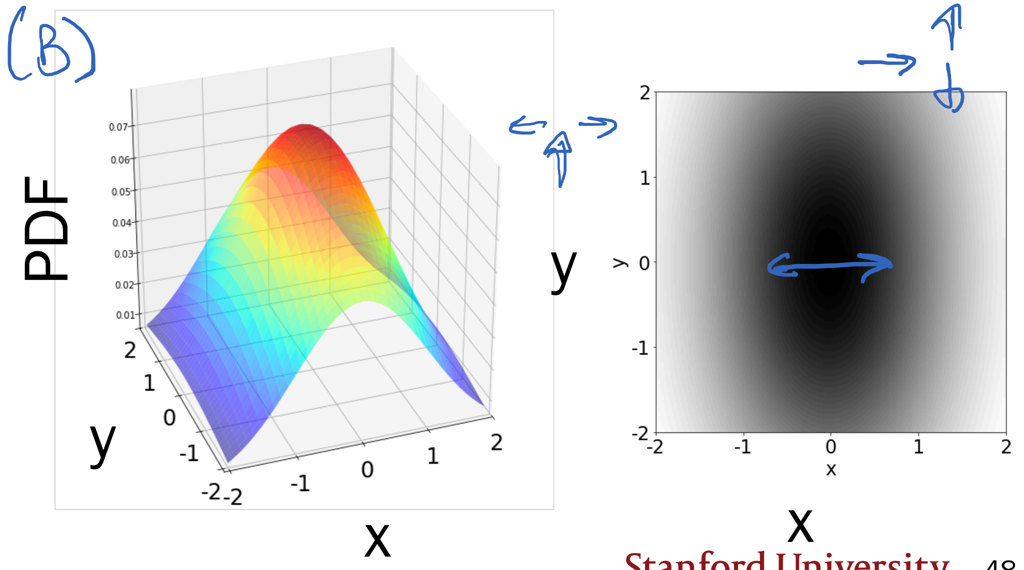
2.



3.

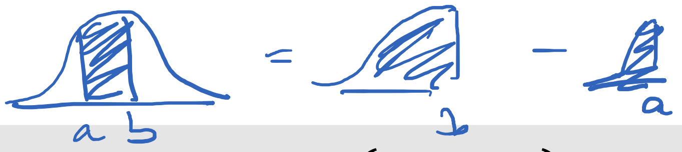


4.



Probabilities from joint CDFs

Recall for a single RV X with CDF F_X:



CDF: $P(X \leq x) = F_X(x)$

$$P(a < X \leq b) = F_X(b) - F(a)$$

For two RVs X and Y with joint CDF $F_{X,Y}$:

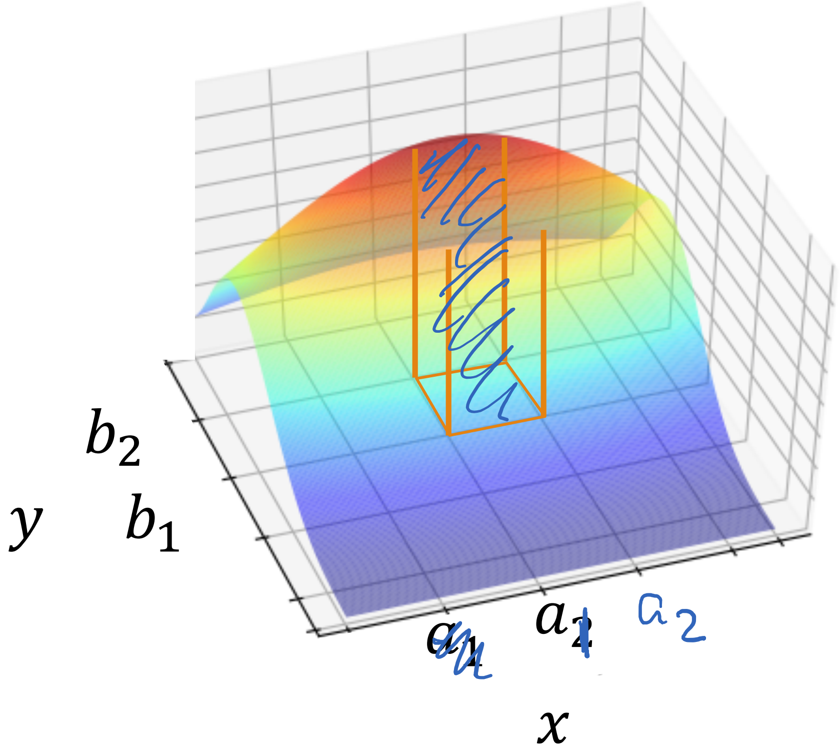
Joint CDF: $P(X \leq x, Y \leq y) = F_{X,Y}(x,y)$

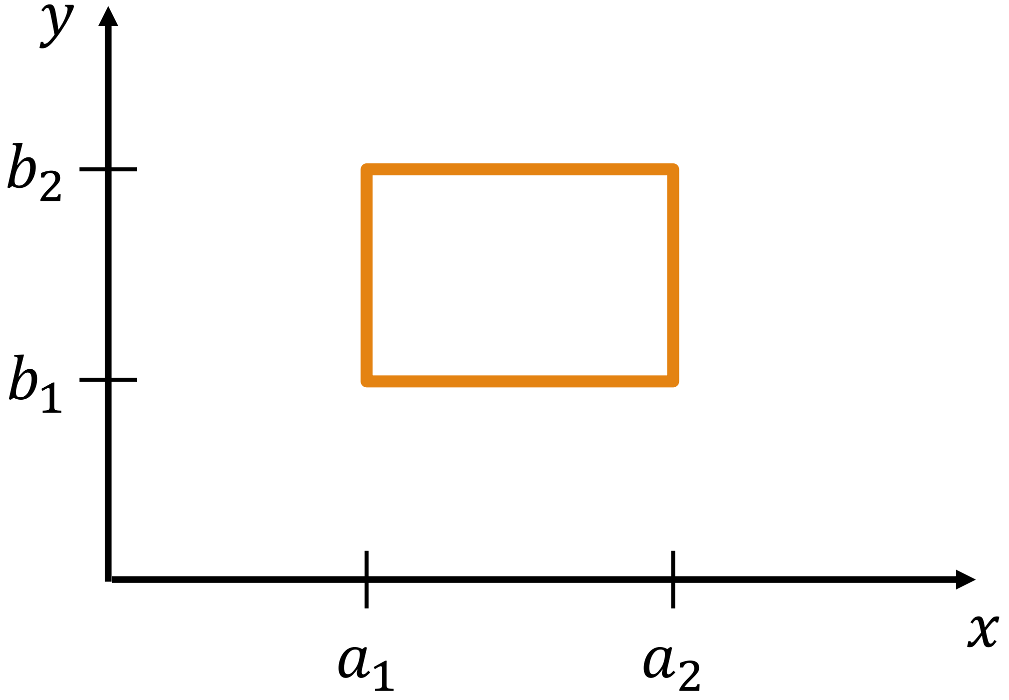
$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =$$
$$F_{X,Y}(a_2,b_2) - F_{X,Y}(a_1,b_2) - F_{X,Y}(a_2,b_1) + F_{X,Y}(a_1,b_1)$$

Note strict inequalities; these properties hold for both discrete and continuous RVs.

Probabilities from joint CDFs

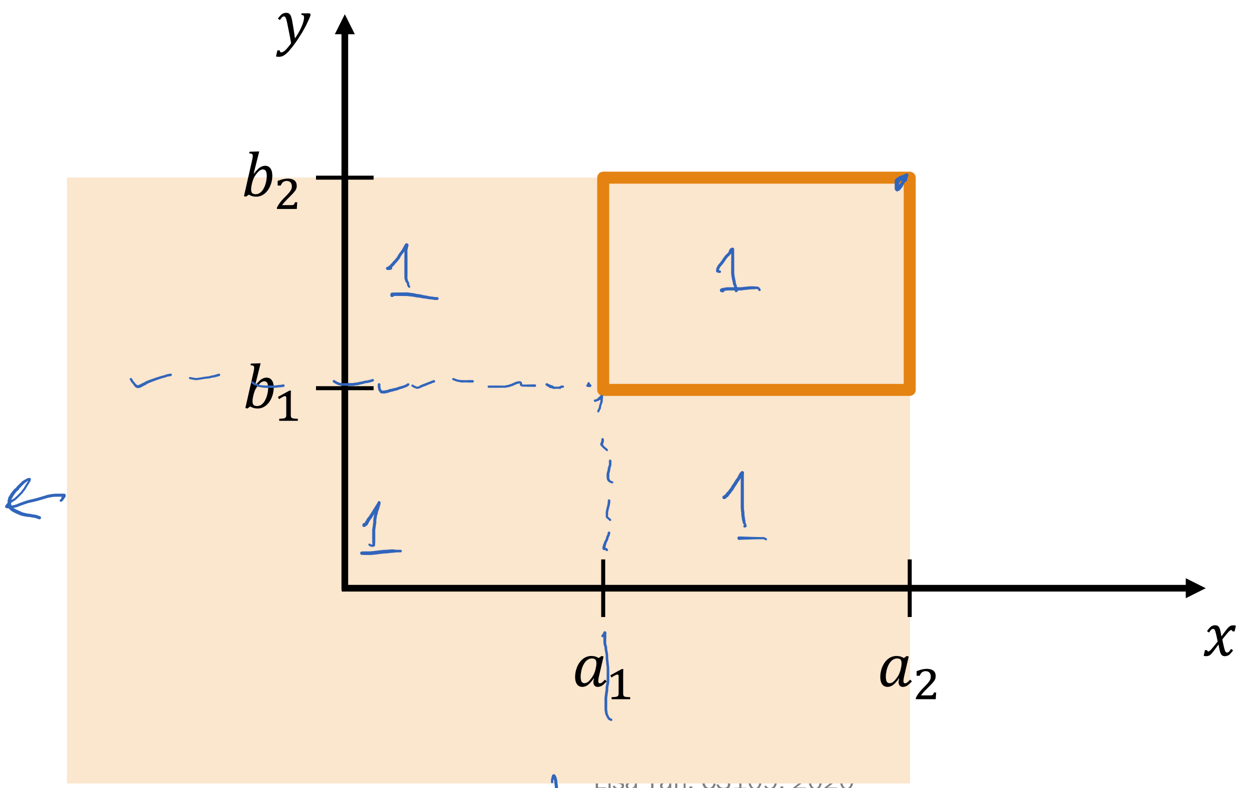
$$P(a_1 < X \le a_2, b_1 < Y \le b_2) =$$
$$F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$





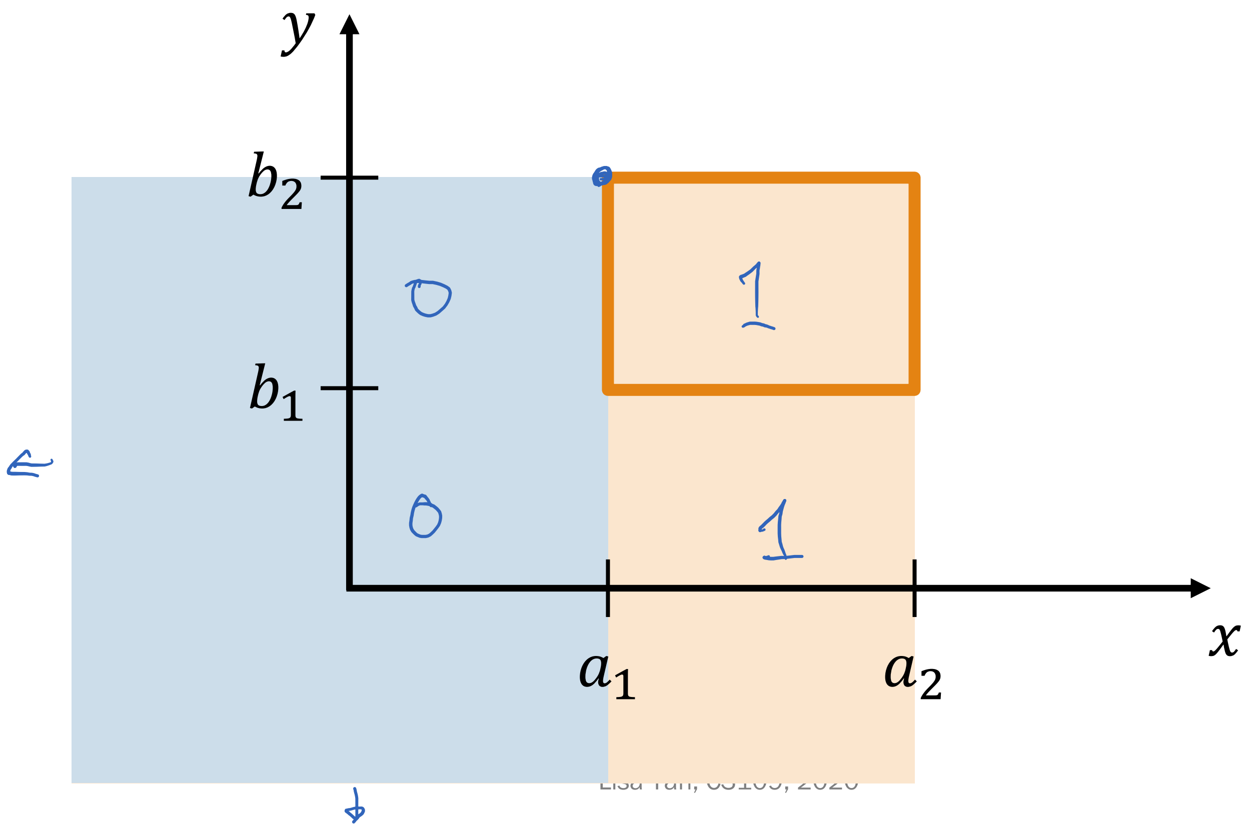
Probabilities from joint CDFs

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =$$
$$F_{X,Y}(a_2,b_2) - F_{X,Y}(a_1,b_2) - F_{X,Y}(a_2,b_1) + F_{X,Y}(a_1,b_1)$$



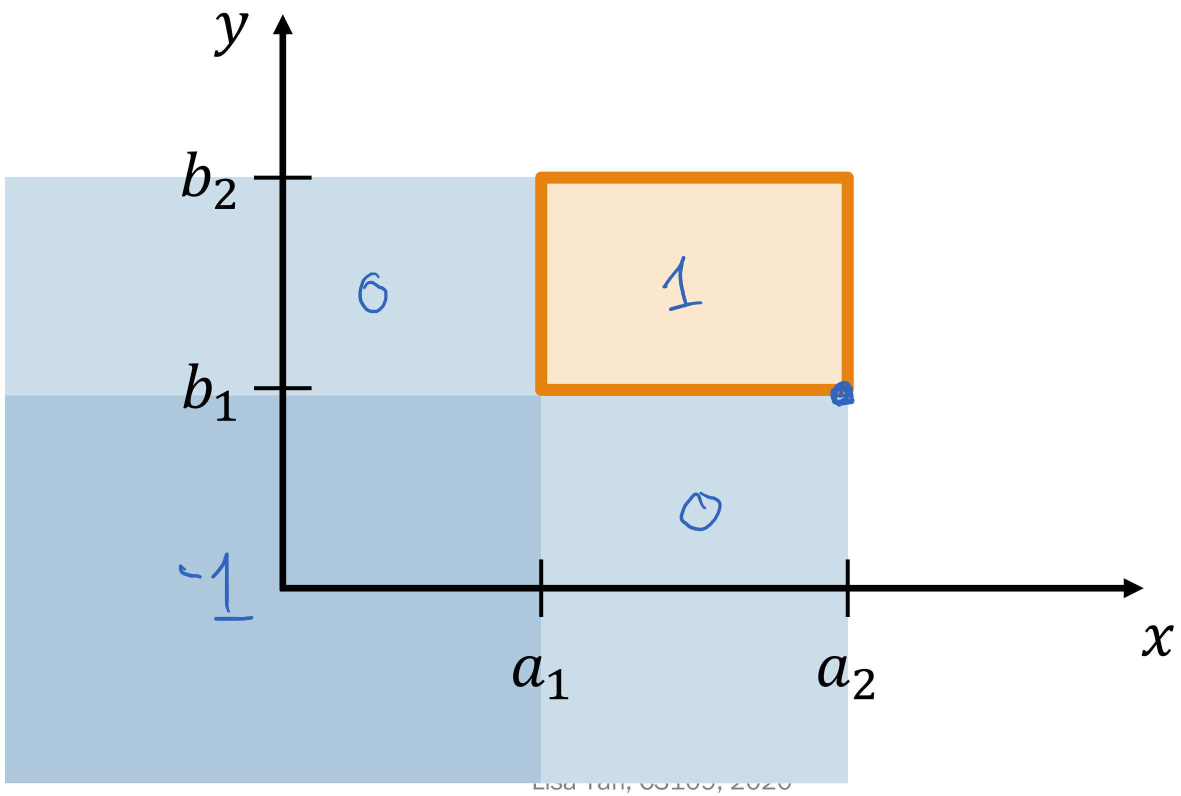
Probabilities from joint CDFs

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =$$
$$F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$



Probabilities from joint CDFs

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =$$
$$F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$



Probabilities from joint CDFs

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =$$
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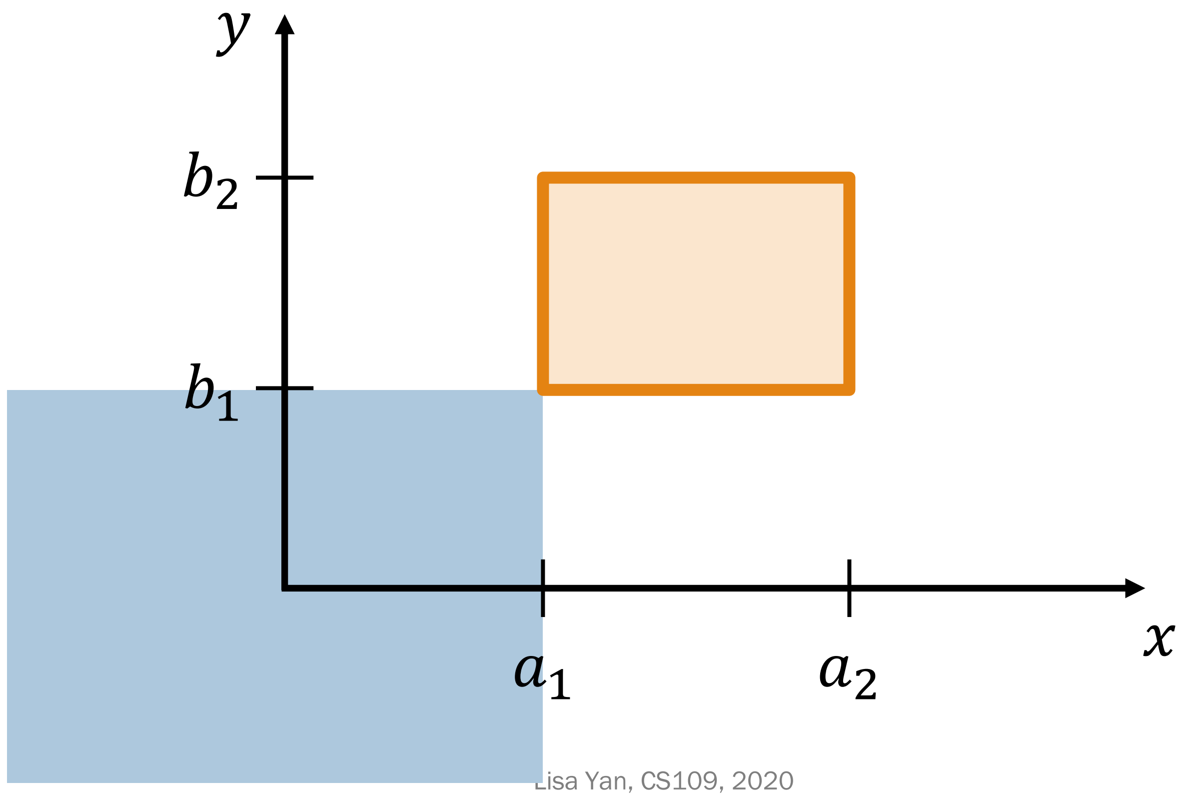
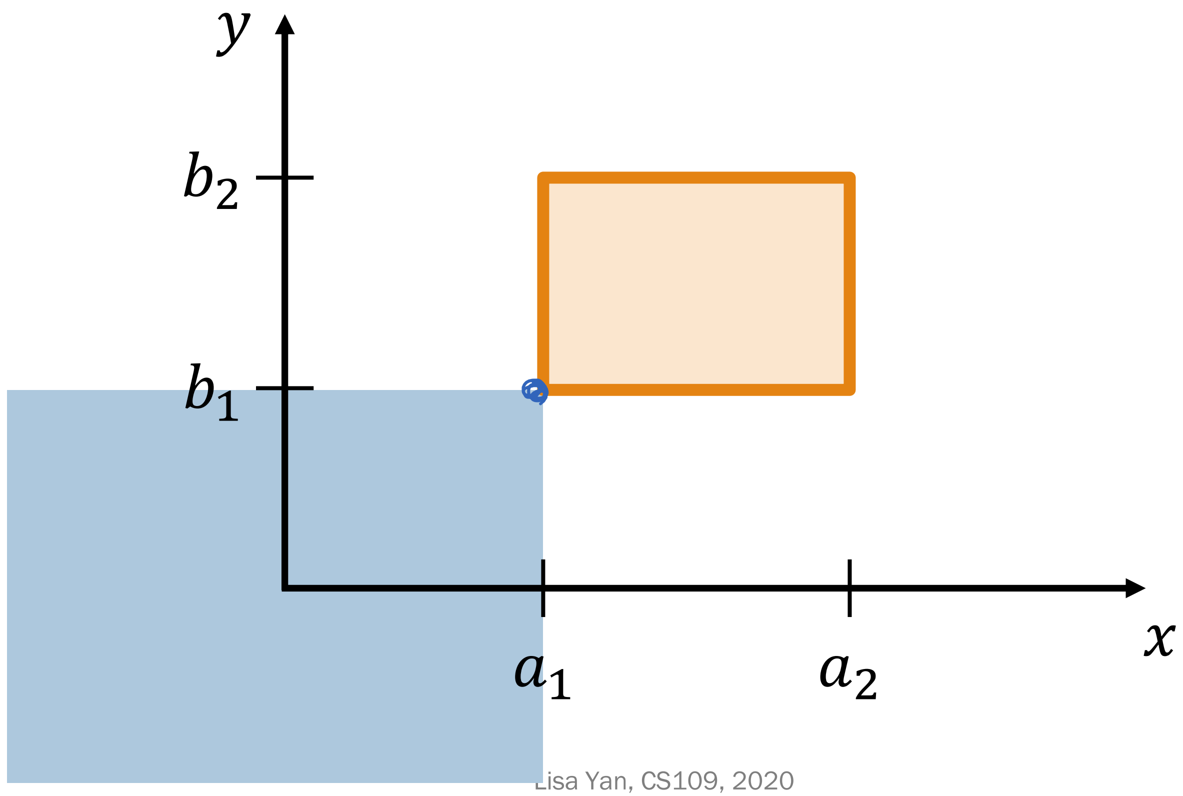


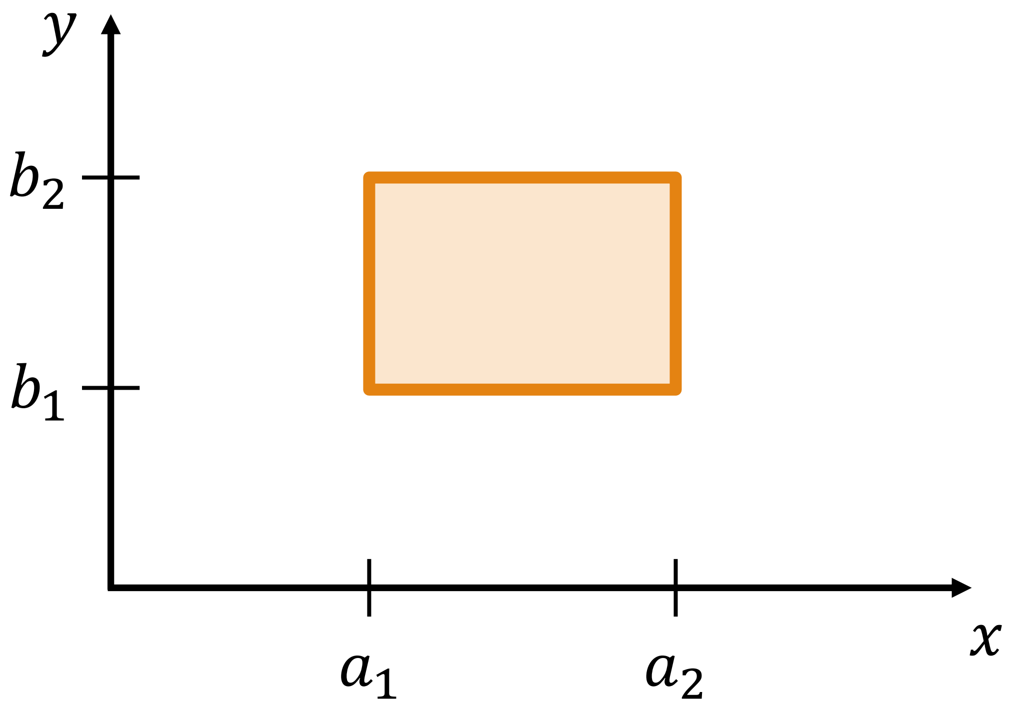
Probabilities from joint CDFs

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =$$
$$F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$



Probabilities from joint CDFs

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =$$
$$F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$



Probability with Instagram!

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$

(for next time)

In image processing, a Gaussian blur is the result of blurring an image by a Gaussian function. It is a widely used effect in graphics software, typically to reduce image noise.

Gaussian blur (for next time)

$P(a_1 < X \le a_2, b_1 < Y \le b_2) =$
$F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$

In a Gaussian blur, for every pixel:

- Weight each pixel by the probability that X and Y are both within the pixel bounds
- The weighting function is a Bivariate Gaussian (Normal) standard deviation parameter σ

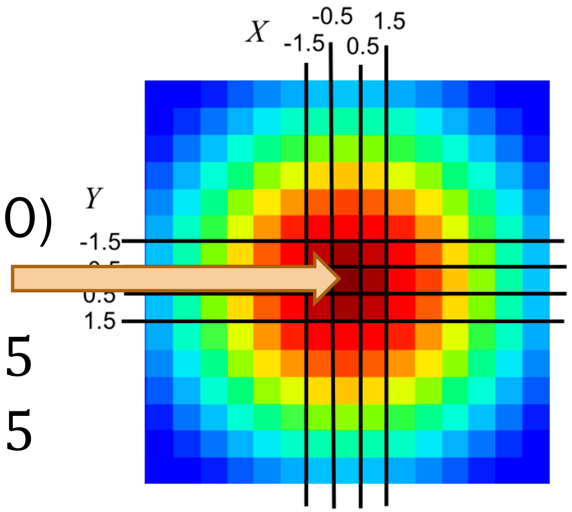
Weight matrix:

Center pixel: (0, 0)

Pixel bounds:

$-0.5 < x \le 0.5$

$-0.5 < y \le 0.5$



Gaussian blurring with $\sigma = 3$:

$$f_{X,Y}(x, y) = \frac{1}{2\pi \cdot 3^2} e^{-(x^2+y^2)/2\cdot 3^2}$$

→ Independent $X \sim \mathcal{N}(0, 3^2), Y \sim \mathcal{N}(0, 3^2)$

→ Joint CDF: $F_{X,Y}(x, y) = \Phi\left(\frac{x}{3}\right)\Phi\left(\frac{y}{3}\right)$

What is the weight of the center pixel?

$P(-0.5 < X \le 0.5, -0.5 < Y \le 0.5) =$

$= 0.206$

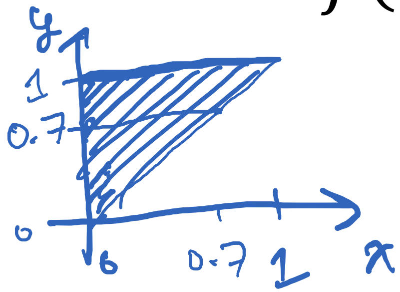
Extra

1. Integral practice

Let X and Y be two continuous random variables with joint PDF:

$$f(x,y) = \begin{cases} 4xy & 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

What is $P(X \le Y)$?



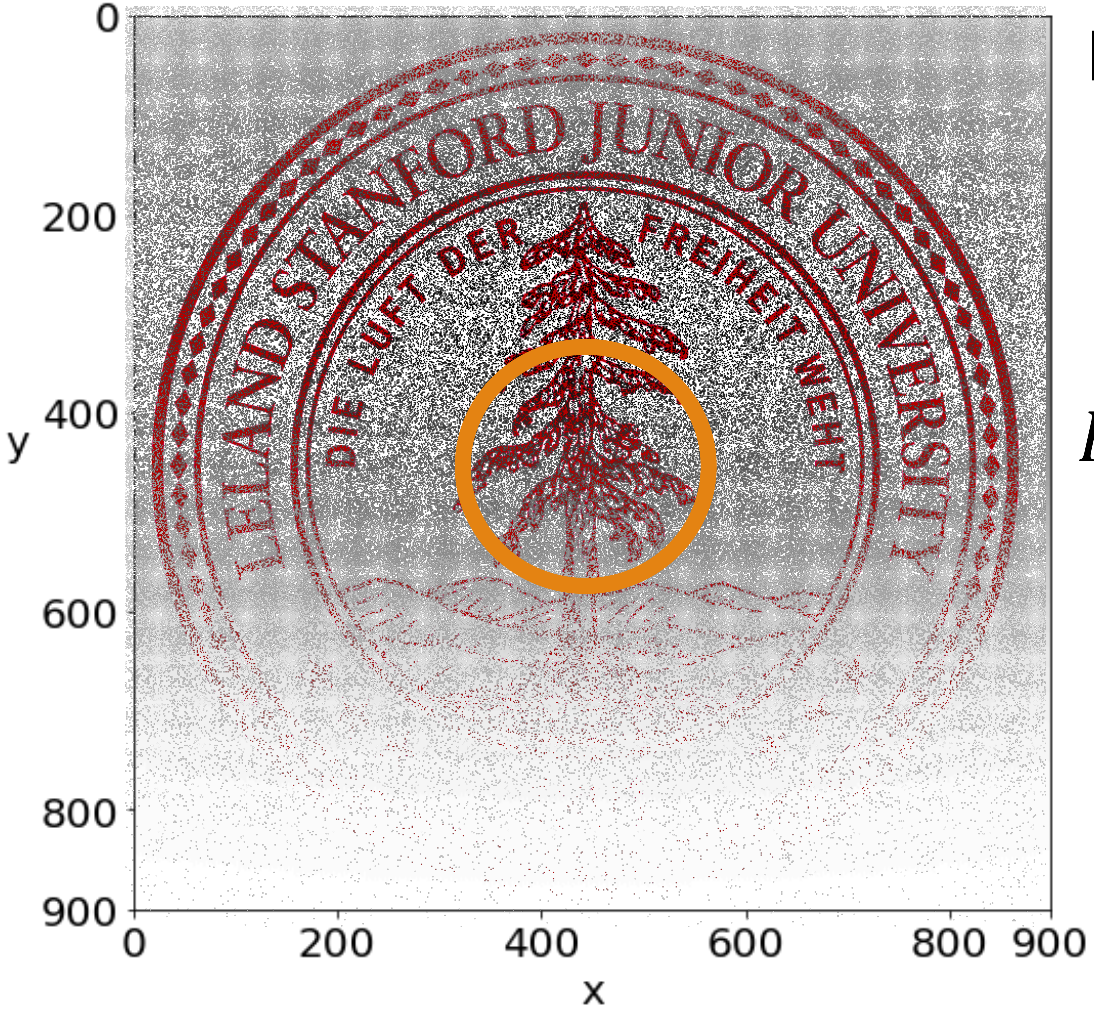
$$P(X \le Y) = \iint\limits_{\substack{x \le y, \\ 0 \le x, y \le 1}} 4xy \, dx \, dy = \int_{y=0}^{1} \int_{x \le y} 4xy \, dx \, dy = \int_{y=0}^{1} \int_{x=0}^{y} 4xy \, dx \, dy$$

$$= \int_{y=0}^{1} 4y \left[\frac{x^2}{2} \right]_0^y dy = \int_{y=0}^{1} 2y^3 dy = \left[\frac{2}{4} y^4 \right]_0^1 = \frac{1}{2}$$

$P(X \le Y) + P(X > Y) = 1$

$P(X \le Y) + P(X \ge Y) = 1 \rightarrow 2P(X \le Y) = 1 \rightarrow P(X \le Y) = 1/2$

2. How do you integrate over a circle?



P(dart hits within $r = 10$ pixels of center)?

$$P(X^2 + Y^2 \leq 10^2) = \iint_{x^2 + y^2 \leq 10^2} f_{X,Y}(x, y)\, dy\, dx$$

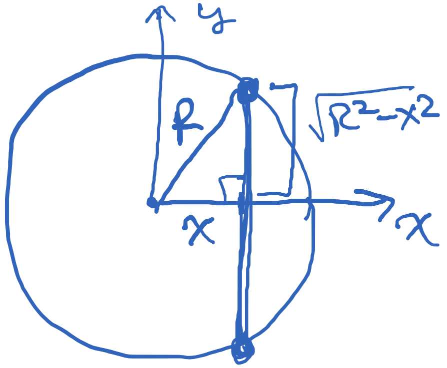
Let's try an example that doesn't involve integrating a Normal RV ☺

2. Imperfection on Disk

You have a disk surface, a circle of radius R. Suppose you have a single point imperfection uniformly distributed on the disk.

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi R^2} & x^2 + y^2 \le R^2 \\ 0 & \text{otherwise} \end{cases}$$

What are the marginal distributions of X and Y? Are X and Y independent?



$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \frac{1}{\pi R^2} \int_{x^2+y^2 \le R^2} dy$$

where $-R \le x \le R$

$$= \frac{1}{\pi R^2} \int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy = \frac{2\sqrt{R^2-x^2}}{\pi R^2}$$

$$f_Y(y) = \frac{1}{\pi R^2} \int_{x^2+y^2 \le R^2} dx$$

$$f_Y(y) = \frac{2\sqrt{R^2-y^2}}{\pi R^2}$$ where $-R \le y \le R$, by symmetry

No, X and Y are **dependent.**

$$f_{X,Y}(x,y) \ne f_X(x) f_Y(y)$$