21: Beta

Lisa Yan

May 22, 2020

Quick slide reference

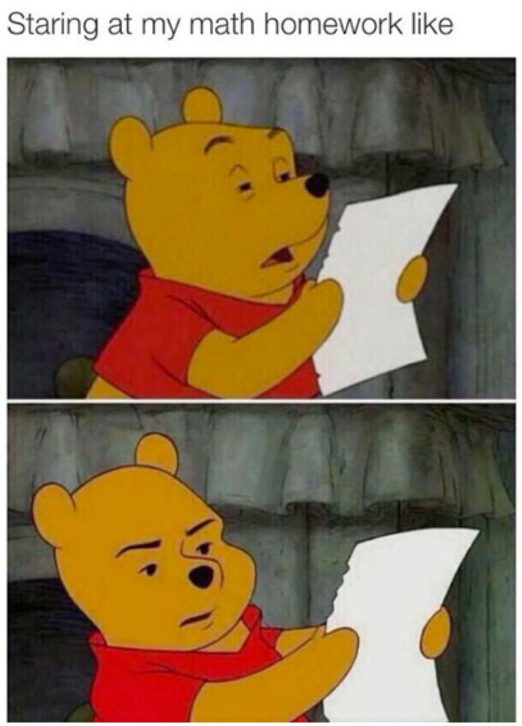
MLE: Multinomial

Okay, just one more MLE with the Multinomial

Consider a sample of n i.i.d. random variables where

- Each element is drawn from one of m outcomes.
  $P(\text{outcome } i) = p_i$, where $\sum_{i=1}^{m} p_i = 1$
- X_i = # of trials with outcome i, where $\sum_{i=1}^{m} X_i = n$

Let's give an example!



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- X_i = # of trials with outcome i, where $\sum_{i=1}^{m} X_i = n$

Example: Suppose each RV is outcome of 6-sided die. $m = 6, \sum_{i=1}^{6} p_i = 1$

- Roll the dice $n = 12$ times.
- Observe data: 3 ones, 2 twos, 0 threes, 3 fours, 1 fives, 3 sixes

$$X_1 = 3, X_2 = 2, X_3 = 0,$$
$$X_4 = 3, X_5 = 1, X_6 = 3$$

Check: $X_1 + X_2 + \cdots + X_6 = 12$

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- X_i = # of trials with outcome i, where $\sum_{i=1}^{m} X_i = n$

1. What is the likelihood of observing the sample $(X_1, X_2, \ldots, X_m)$, given the probabilities $p_1, p_2, \ldots, p_m$?

A. $$\frac{n!}{X_1!\,X_2!\cdots X_m!} p_1^{X_1} p_2^{X_2} \cdots p_m^{X_m}$$

B. $$p_1^{X_1} p_2^{X_2} \cdots p_m^{X_m}$$

C. $$\frac{n!}{X_1!\,X_2!\cdots X_m!} X_1^{p_1} X_2^{p_2} \cdots X_m^{p_m}$$

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C. $\dfrac{n!}{X_1!\,X_2!\cdots X_m!} X_1^{p_1} X_2^{p_2} \cdots X_m^{p_m}$

$\binom{n}{x_1, x_2, \ldots, x_m} p_1^{x_1} p_2^{x_2} \cdots p_m^{x_m}$

Okay, just one more MLE with the Multinomial

Consider a sample of n i.i.d. random variables where

- Each element is drawn from one of m outcomes. $P(\text{outcome } i) = p_i$, where $\sum_{i=1}^{m} p_i = 1$
- X_i = # of trials with outcome i, where $\sum_{i=1}^{m} X_i = n$

1. What is the likelihood of observing the sample $(X_1, X_2, \ldots, X_m)$, given the probabilities $p_1, p_2, \ldots, p_m$?

$\theta = (p_1, \ldots, p_m)$

$$L(\theta) = \frac{n!}{X_1!\,X_2!\cdots X_m!} p_1^{X_1} p_2^{X_2} \cdots p_m^{X_m}$$

$\log p_1^{X_1} \Rightarrow X_1 \log p_1$

2. What is θ_{MLE}?

$$LL(\theta) = \log(n!) - \sum_{i=1}^{m} \log(X_i!) + \sum_{i} X_i \log(p_i), \text{ such that } \sum_{i=1}^{m} p_i = 1$$

$i = 1, \ldots, m$

Optimize with Lagrange multipliers in extra slides → θ_{MLE}: $p_i = \frac{X_i}{n}$

Intuitively, probability p_i = proportion of outcomes

When MLEs attack!

MLE for Multinomial: $p_i = \frac{X_i}{n}$

Consider a 6-sided die.

- Roll the dice $n = 12$ times.
- Observe: 3 ones, 2 twos, 0 threes, 3 fours, 1 fives, 3 sixes

What is θ_{MLE}?

$p_1 = \frac{3}{12}$

$p_2 = \frac{2}{12}$

$p_3 = \frac{0}{12}$

$p_4 = \frac{3}{12}$

$p_5 = \frac{1}{12}$

$p_6 = \frac{3}{12}$

When MLEs attack!

MLE for Multinomial: $p_i = \frac{X_i}{n}$

Consider a 6-sided die.

- Roll the dice $n = 12$ times.
- Observe: 3 ones, 2 twos, 0 threes, 3 fours, 1 fives, 3 sixes

θ_{MLE}:

$$p_1 = 3/12$$
$$p_2 = 2/12$$
$$p_3 = 0/12$$
$$p_4 = 3/12$$
$$p_5 = 1/12$$
$$p_6 = 3/12$$

- MLE: you'll **never…EVER…** roll a three.
- Do you really believe that?

Today: A new definition of probability!

Bayesian Statistics

When MLEs attack!

Consider a 6-sided die.

- Roll the dice $n = 12$ times.
- Observe: 3 ones, 2 twos, 0 threes, 3 fours, 1 fives, 3 sixes

θ_{MLE}:

$$p_1 = 3/12$$
$$p_2 = 2/12$$
$$p_3 = 0/12$$ ⚠️
$$p_4 = 3/12$$
$$p_5 = 1/12$$
$$p_6 = 3/12$$

- MLE: you'll **never…<u>EVER</u>…** roll a three.
- Do you really believe that?

🤔

Roll more!
Prob. = frequency
in limit

Frequentist

But what if you cannot observe anymore rolls?

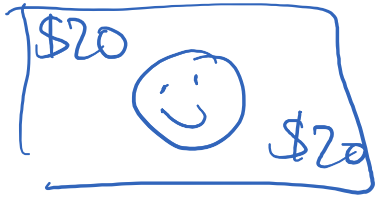
Today's plan

Today we are going to learn something unintuitive, beautiful, and useful!

We are going to think of probabilities as random variables.

Let's play a game

Roll 2 dice. If *neither* roll is a 6,
you win (event W). Else, I win (event W^C).



- Before you play, what's the probability that you win? $P(W)$
- Play once. What's the probability that you win? $P(W)$
- Play three more times. What's the probability that you win? $P(W)$

Frequentist

$$P(W) = \left(\frac{5}{6}\right)^2$$

Bayesian

wait hold up this situation is whack tho

Bayesian statistics: Update your prior beliefs of **probability.**

Bayesian probability

Bayesian statistics: Probability is a reasonable expectation representing a state of knowledge.

Mixing discrete and continuous random variables, combined with Bayes' Theorem, allows us to reason about probabilities as random variables.

A new definition of probability

Flip a coin $n + m$ times, come up with n heads.

We don't know the probability θ that the coin comes up heads.

The world's first coin

Frequentist

θ is a single value.

$$\theta = \lim_{n+m\to\infty} \frac{n}{n+m} \approx \frac{n}{n+m}$$

Bayesian

θ is a **random variable**.

θ's continuous support: (0, 1)

Mixing discrete and continuous

Let X be a continuous random variable, and
N be a discrete random variable.

Bayes' Theorem:

$$f_{X|N}(x|n) = \frac{p_{N|X}(n|x)f_X(x)}{p_N(n)}$$

Intuition:

$$P(X = x|N = n) = \frac{P(N = n|X = x)\overbrace{P(X = x)}}{P(N = n)}$$

$$\int_{x-\varepsilon_X/2}^{x+\varepsilon_X/2} f_X(x)\,dx \approx \varepsilon_X f_X(x)$$

$$f_{X|N}(x|n)\cancel{\varepsilon_X} = \frac{P(N = n|X = x)\overbrace{f_X(x)\cancel{\varepsilon_X}}}{P(N = n)}$$

$$f_{X|N}(x|n) = \frac{p_{N|X}(n|x)f_X(x)}{p_N(n)}$$

All your Bayes are belong to us

Let X, Y be **continuous** and M, N be **discrete** random variables.

OG Bayes: $$p_{M|N}(m|n) = \frac{p_{N|M}(n|m)p_M(m)}{p_N(n)}$$

Mix Bayes #1: $$f_{X|N}(x|n) = \frac{p_{N|X}(n|x)f_X(x)}{p_N(n)}$$

Mix Bayes #2: $$p_{N|X}(n|x) = \frac{f_{X|N}(x|n)p_N(n)}{f_X(x)}$$

All continuous: $$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)}$$

Mixing discrete and continuous

Let θ be a random variable for the probability your coin comes up heads, and N be the number of heads you observe in an experiment. n+m trials

posterior, likelihood, prior

$$\underbrace{f_{\theta|N}(x|n)}_{\text{posterior}} = \frac{\overbrace{p_{N|\theta}(n|x)}^{\text{likelihood}}\overbrace{f_\theta(x)}^{\text{prior}}}{p_N(n)}$$

normalization constant

- Prior belief of parameter θ — $f_\theta(x)$
- Likelihood of $N = n$ heads, given parameter $\theta = x$. — $p_{N|\theta}(n|x)$
- Posterior updated belief of parameter θ. — $f_{\theta|N}(x|n)$

More in live lecture!

Beta RV

- MLE Multinomial θ is a value
- Bayesian θ is a RV
- Beta RV probability of a probability

Beta random variable

def A **Beta** random variable X is defined as follows:

$$X \sim \text{Beta}(a, b)$$

$a > 0, b > 0$

Support of X: $(0, 1)$

PDF $f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$, normalizing constant

Expectation $E[X] = \frac{a}{a+b}$

Variance $\text{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}$

$$\int_0^1 \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} dx = 1$$

$$\frac{1}{B(a,b)} \int_0^1 x^{a-1}(1-x)^{b-1} dx = 1$$

Beta RV with different a, b

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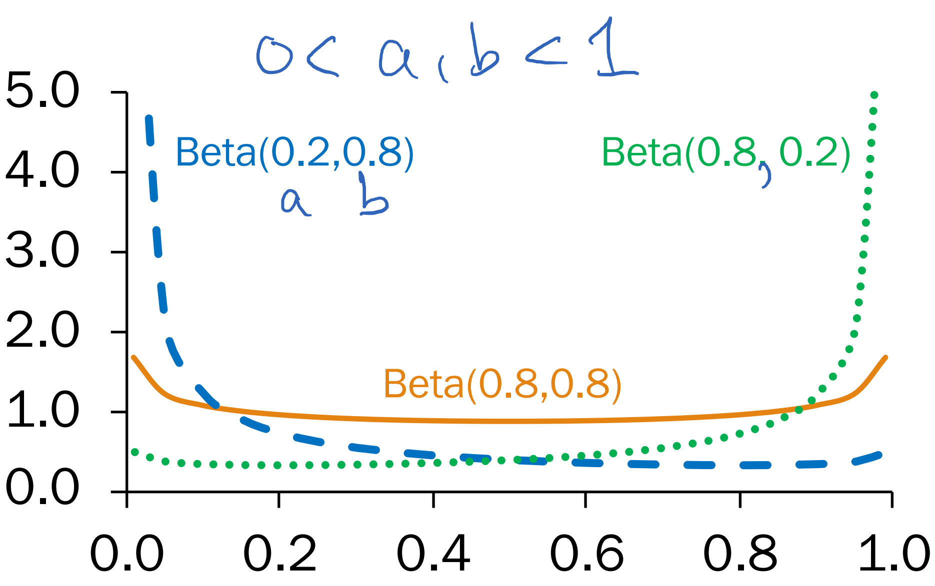
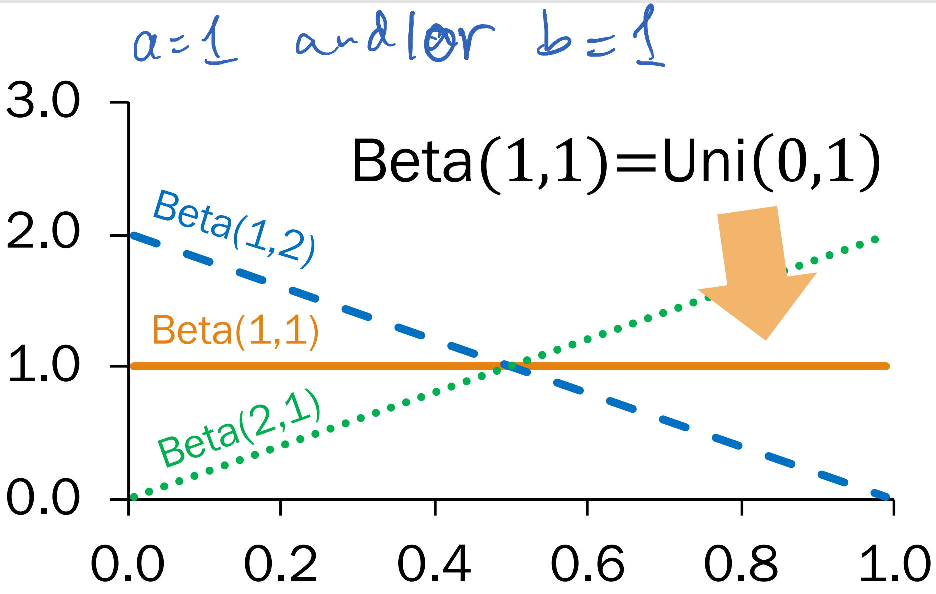




+ a third case (next slide)

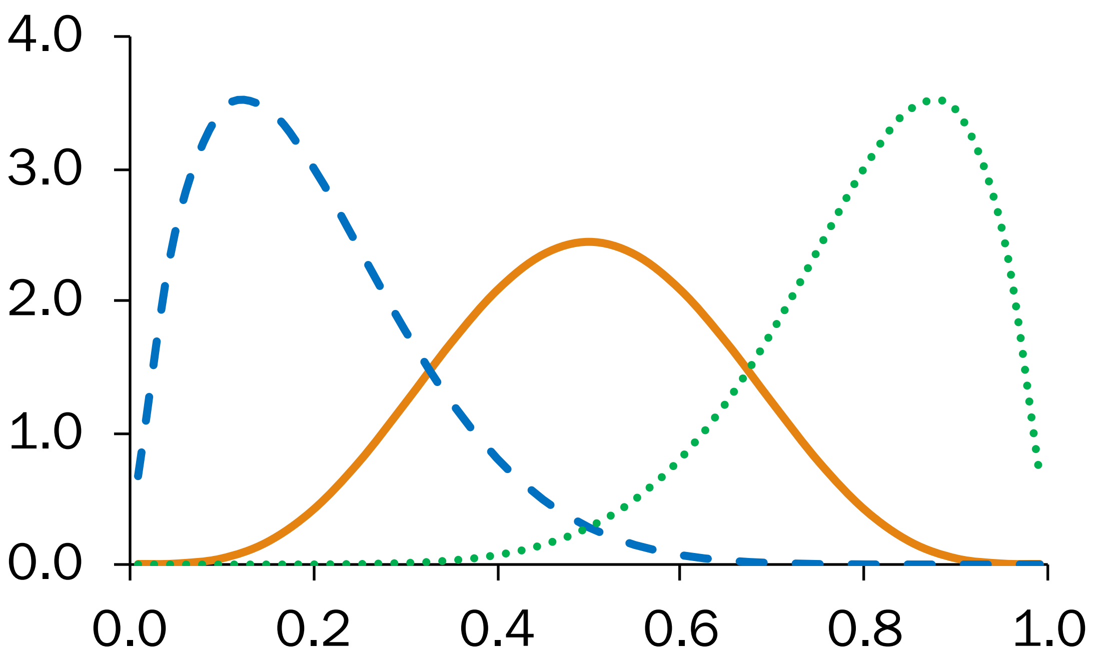
Note: PDF symmetric when $a = b$

Beta RV with different a, b

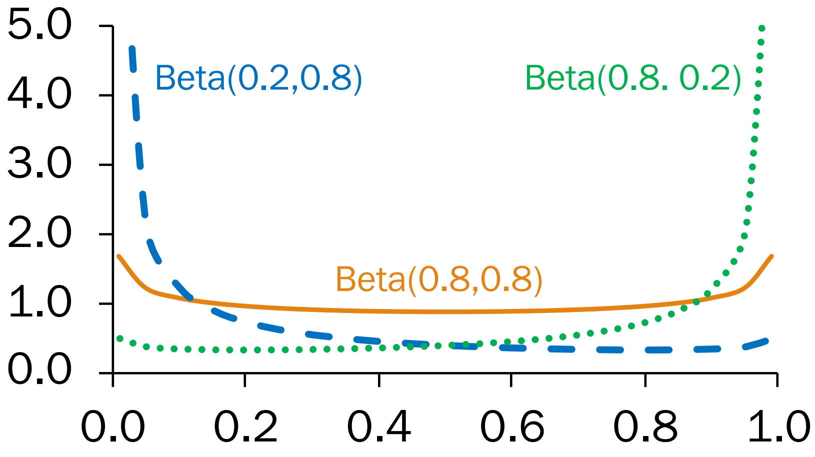
$X \sim \text{Beta}(a, b)$

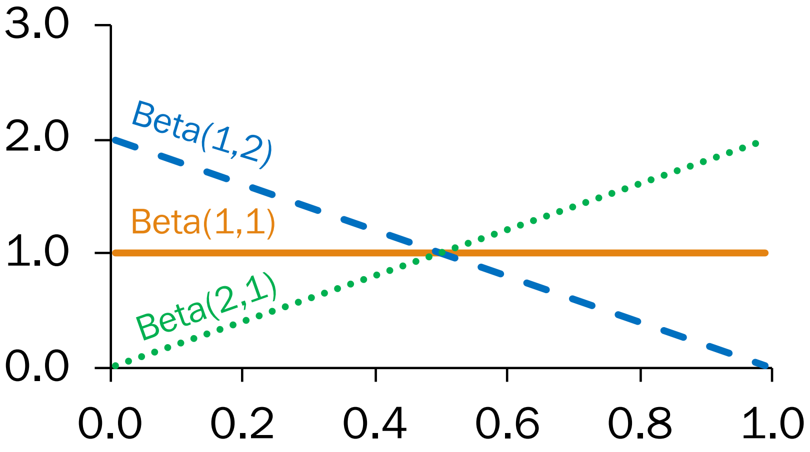
Match PDF to distribution:

a, b positive integers > 1



A. Beta(5,5)
B. Beta(2,8)
C. Beta(8,2)





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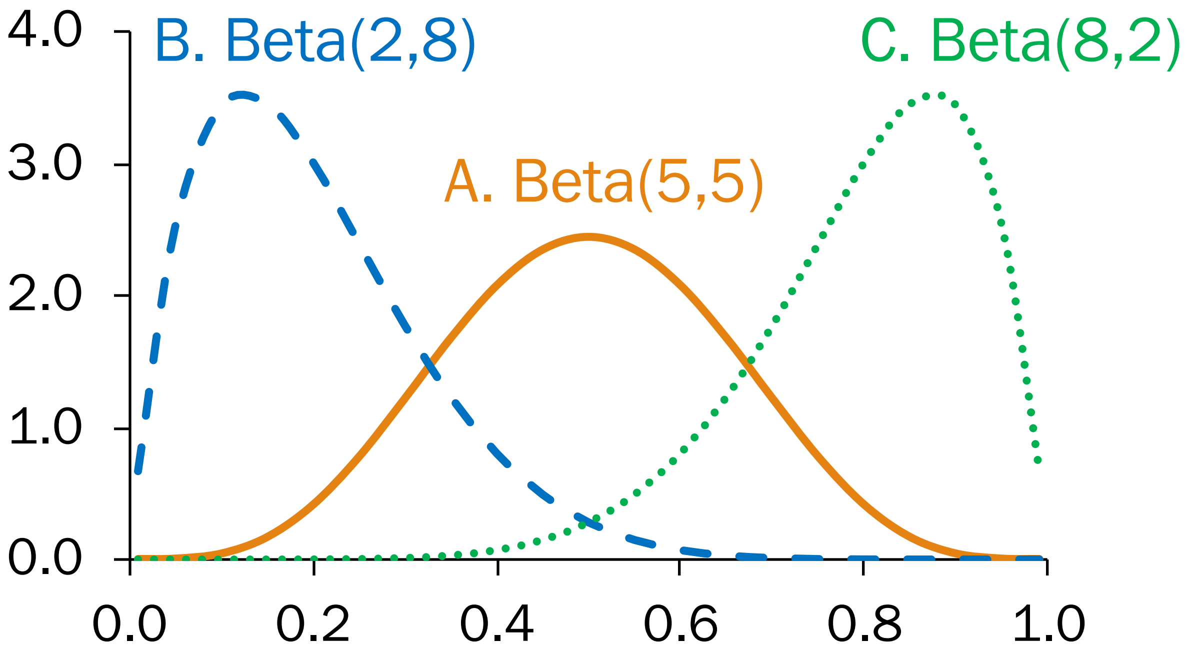

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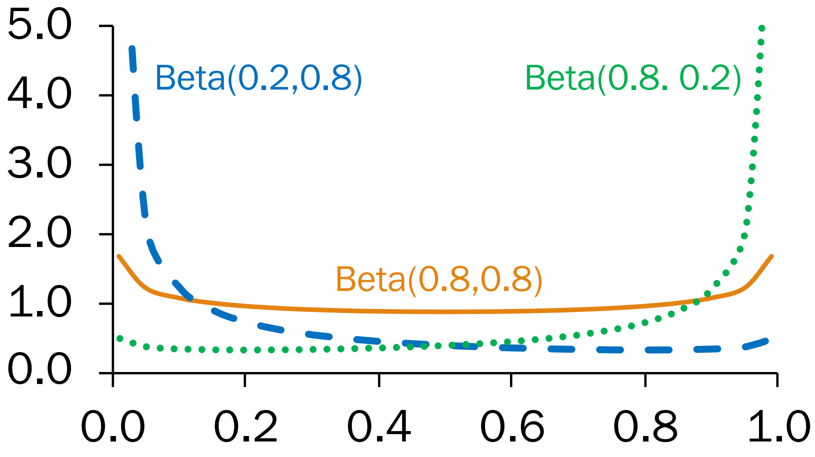

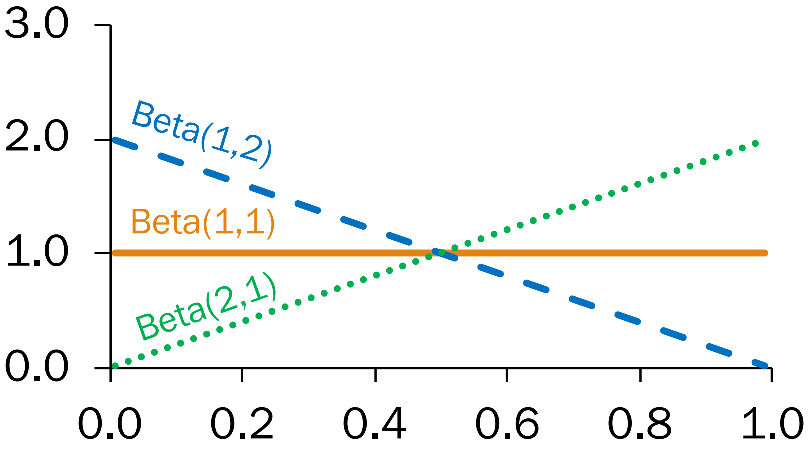

In CS109, we focus on Betas where a, b are both positive integers.

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Beta can be a distribution of probabilities.

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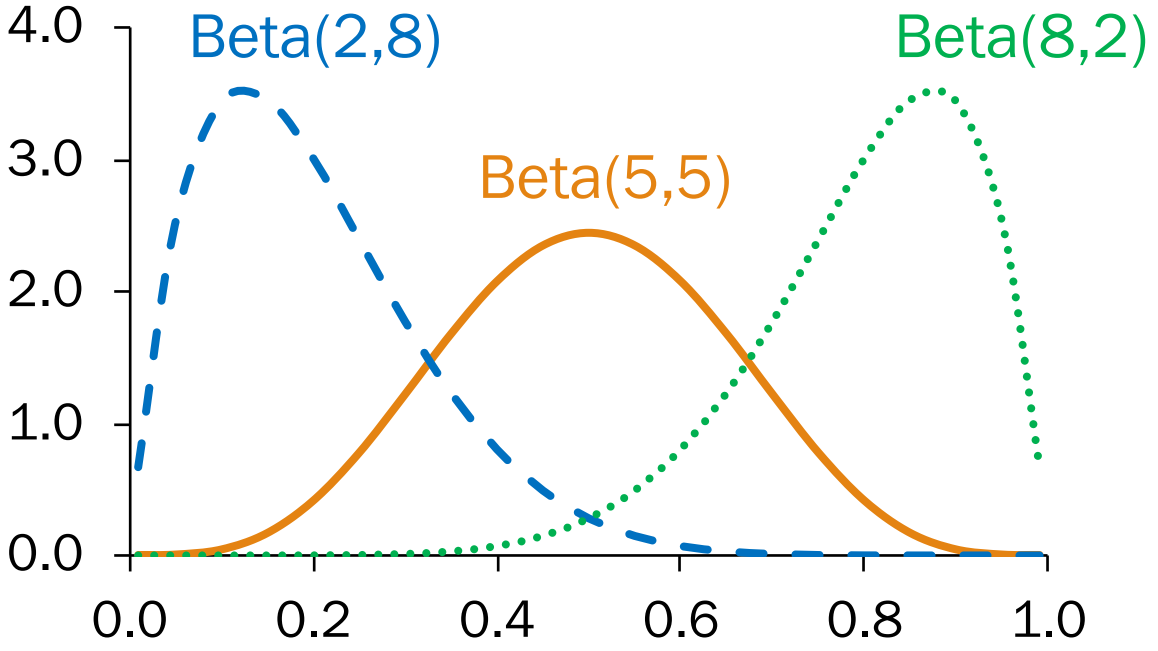

Beta parameters a, b could come from an experiment…

But which one?
Stay tuned…

21: Beta (live)

Lisa Yan

May 22, 2020

Flipping a coin with unknown probability

A new definition of probability

Flip a coin $n + m$ times, comes up with n heads.

We don't know the probability θ that the coin comes up heads.

The world's first coin

Frequentist

θ is a single value.

$$\theta = \lim_{n+m\to\infty} \frac{n}{n+m} \approx \frac{n}{n+m}$$

Bayesian

θ is a **random variable**.

θ's continuous support: (0, 1)

Flip a coin with unknown probability

Flip a coin $n + m$ times, observe n heads.

- Before our experiment, θ (the probability that the coin comes up heads) can be any probability.
- Let N = number of heads.
- Given $\theta = x$, coin flips are independent.

What is our updated belief of θ after we observe $N = n$?

What are reasonable distributions of the following?

1. θ
2. $N|\theta = x$
3. $\theta|N$

Flip a coin with unknown probability

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What is our updated belief of θ after we observe $N = n$?

What are reasonable distributions of the following?

1. θ — Bayesian prior $\theta \sim \text{Uni}(0,1)$
2. $N|\theta = x$ — Likelihood $(N|\theta = x) \sim \text{Bin}(n + m, x)$
3. $\theta|N = n$ — Bayesian posterior. Use Bayes'

Flip a coin with unknown probability

Flip a coin $n + m$ times, observe n heads.

- Before our experiment, X (the probability that the coin comes up heads) can be any probability.
- Let N = number of heads.
- Given $X = x$, coin flips are independent.

What is our updated belief of X after we observe $N = n$?

Prior: $\theta \sim \text{Uni}(0,1)$

Likelihood: $N|\theta = x \sim \text{Bin}(n+m, x)$

Posterior: $f_{\theta|N}(\theta|n)$

$$f_{\theta|N}(x|n) = \frac{p_{N|\theta}(n|x) f_\theta(x)}{p_N(n)} = \frac{\binom{n+m}{n} x^n (1-x)^m \cdot 1}{p_N(n)}$$

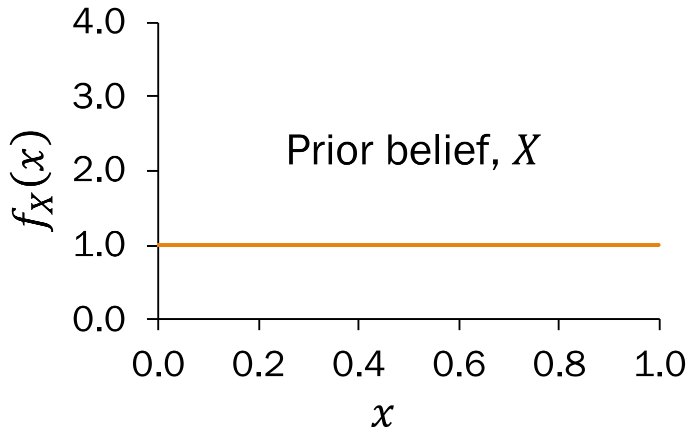
$$\int_0^1 \frac{1}{c} x^n (1-x)^m = 1$$

$$= \underbrace{\boxed{\frac{\binom{n+m}{n}}{p_N(n)}}}_{\substack{\text{constant,}\\ \text{doesn't depend on } x}} x^n (1-x)^m = \frac{1}{c} x^n (1-x)^m, \text{ where } c = \int_0^1 x^n (1-x)^m dx$$

$$= \frac{\binom{n+m}{n}}{p_N(n)}$$

Let's try it out

1. Start with a $\theta \sim \text{Uni}(0,1)$ over probability that a coin lands heads.



2. Flip a coin 8 times. Observe $n = 7$ heads and $m = 1$ tail

okay

3. What is our posterior belief of the probability θ?

$$f_{\theta|N}(x|n) = \frac{1}{c}\, x^7(1-x)^1$$

$n=7$

c normalizes to valid PDF

Wait a minute...

Beta RV with different a, b

Review

$$X \sim \text{Beta}(a, b)$$

$a > 0, b > 0$

Support of X: $(0, 1)$

PDF $f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$, normalizing constant

$$f_{\theta|N}(x|n) = \frac{1}{c}\, x^7(1-x)^1$$

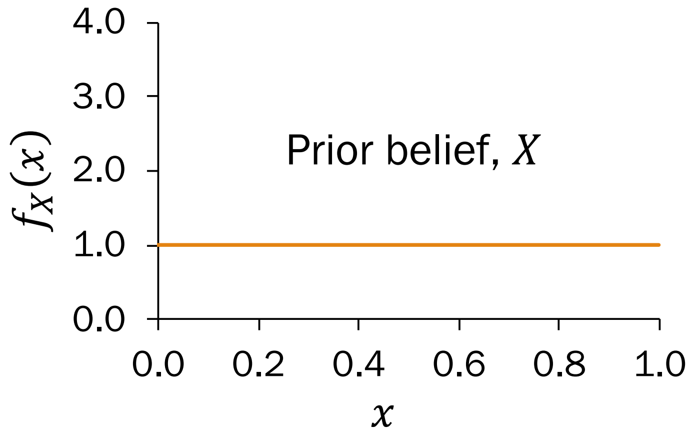
is the PDF for $\text{Beta}(\overset{a}{\overset{\shortparallel}{8}}, \overset{b}{\overset{\shortparallel}{2}})$!

c normalizes to valid PDF

$$\frac{1}{B(8,1)} x^{8-1}(1-x)^{2-1}$$

Let's try it out

1. Start with a $\theta \sim \text{Uni}(0,1)$ over probability that a coin lands heads.



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okay

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c normalizes to valid PDF

Beta(8,2)

3. What is our posterior belief of the probability θ?

- Start with a $\theta \sim \text{Uni}(0,1)$ over probability
- Observe $n = 7$ successes and $m = 1$ failures
- Your new belief about the probability of θ is:

$$f_{\theta|N}(x|n) = \frac{1}{c}\, x^{7}(1-x)^{1}, \text{where } c = \int_0^1 x^{7}(1-x)^{1} dx$$

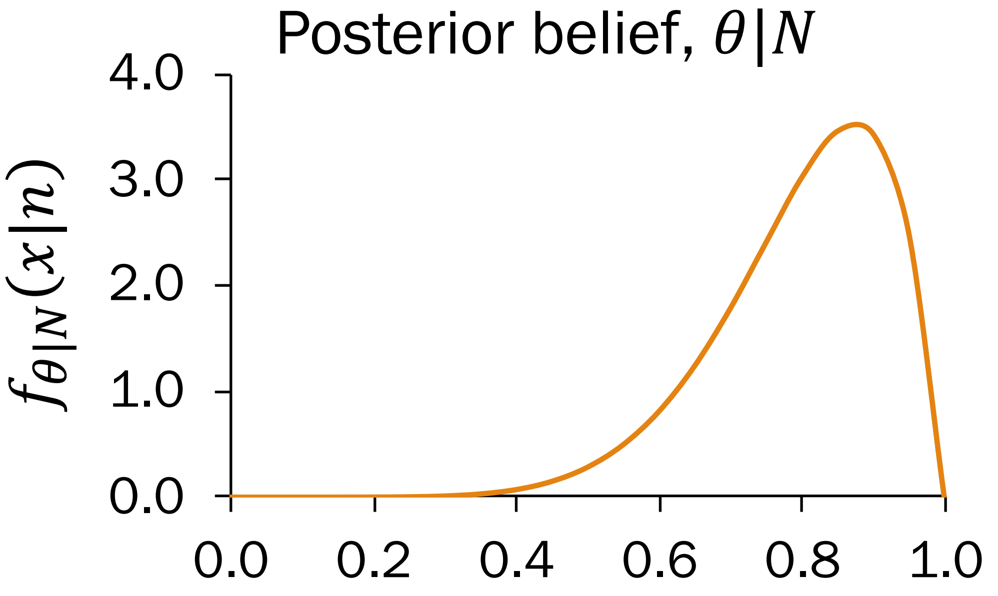
Posterior belief, $\theta|N$:

Beta$(a = 8, b = 2)$

$$f_{\theta|N}(x|n) = \frac{1}{c}\, x^{8-1}(1-x)^{2-1}$$

Beta$(a = n + 1, b = m + 1)$

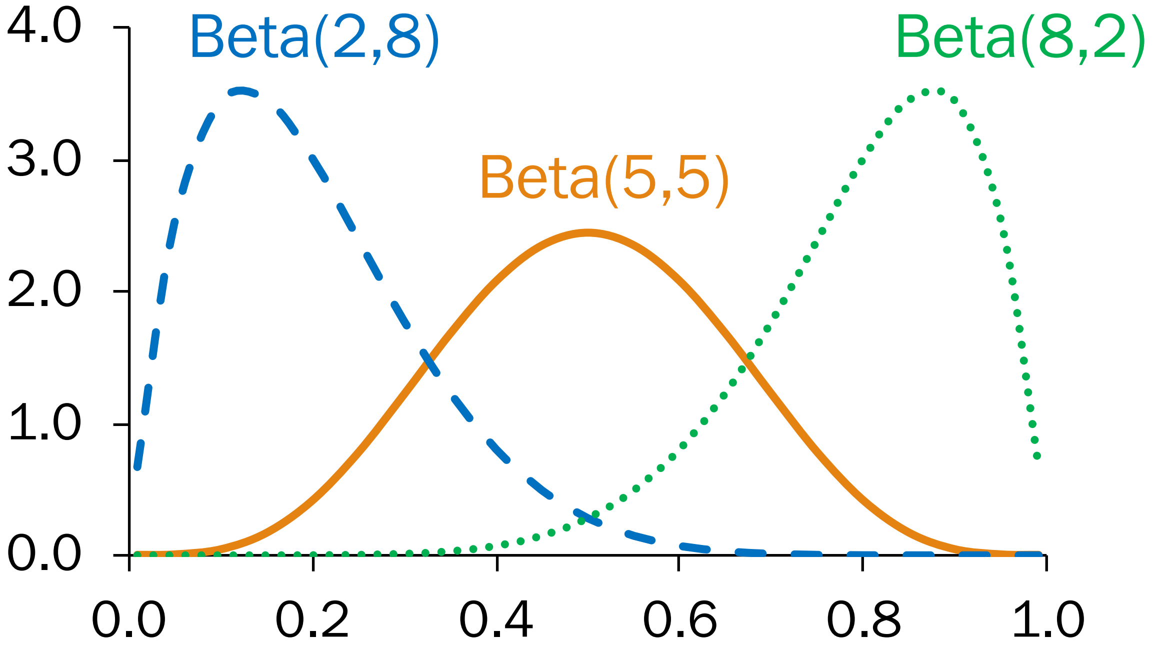
successes failures



x

CS109 focus: Beta where a, b both positive integers

$X \sim \text{Beta}(a, b)$

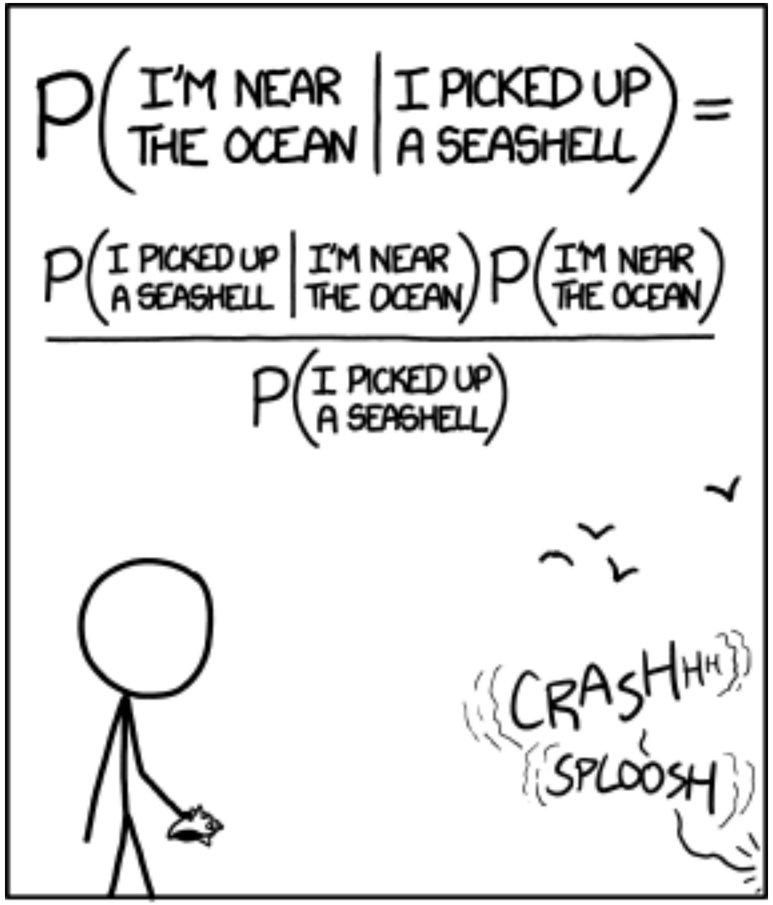


If a, b are positive integers, Beta parameters a, b *could* come from an experiment:

$$a = \text{"successes"} + 1$$
$$b = \text{"failures"} + 1$$

- Beta (in CS109) models the randomness of the probability of experiment success.
- Beta parameters depend our data and our prior.

Bayes' on the waves



STATISTICALLY SPEAKING, IF YOU PICK UP A SEASHELL AND *DON'T* HOLD IT TO YOUR EAR, YOU CAN PROBABLY HEAR THE OCEAN.

xkcd.com

2:30

Interlude for jokes/announcements

Announcements

<u>Grading clarification</u>

Two examples

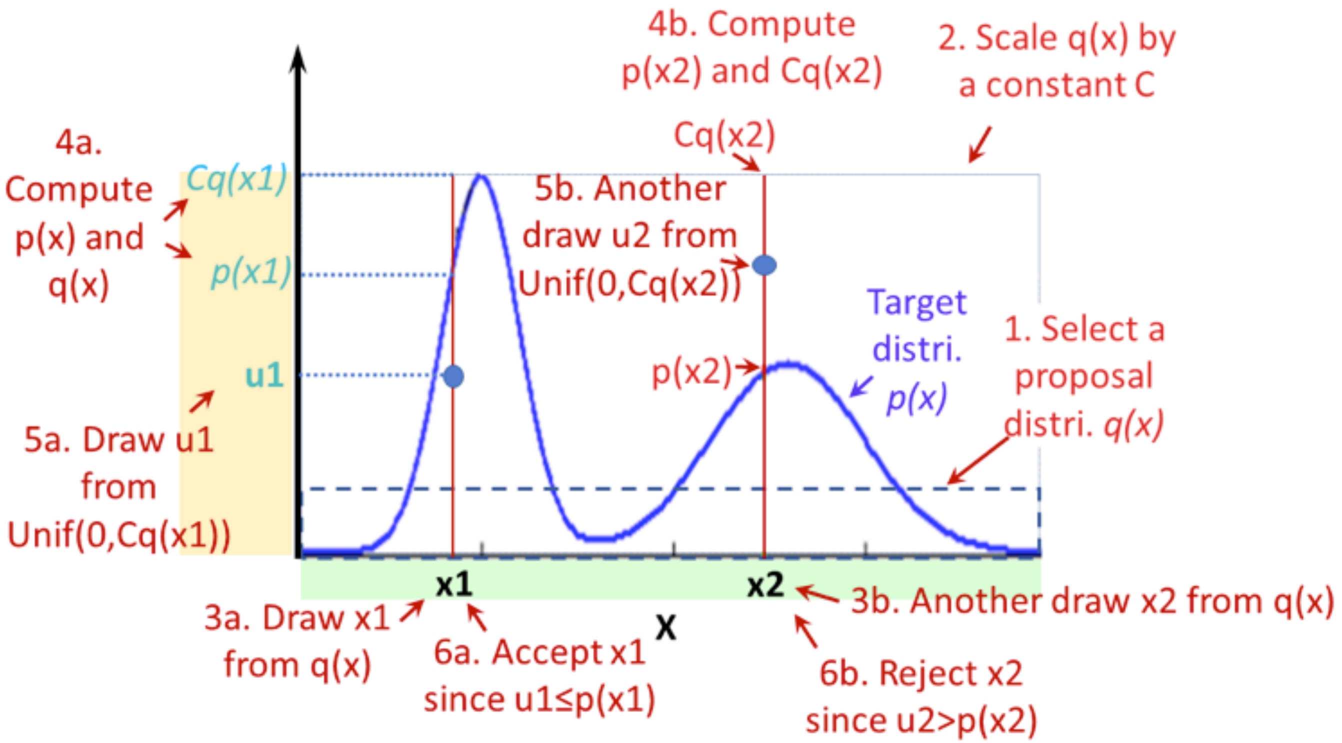
https://us.edstem.org/courses/109/discussion/67686

Problem Set 6: **No late days or on-time bonus**

must submit!

Interesting probability news

Why Rejection Sampling Is Useful in Cat Modeling



Note: Cat Modeling
= Catastrophe Modeling
(e.g., earthquakes, hurricanes, etc.)

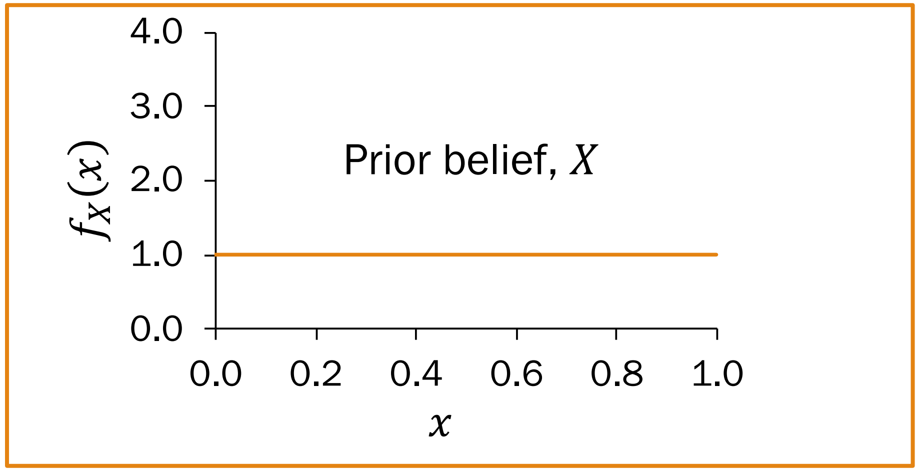
https://www.air-worldwide.com/blog/posts/2018/9/why-rejection-sampling-is-useful-in-cat-modeling/

CS109 Current Events Spreadsheet

Conjugate distributions

A note about our prior

1. Start with a $\theta \sim \text{Uni}(0,1)$ over probability that a coin lands heads.



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c normalizes to valid PDF

Beta(8,2)

Wait another minute...

Beta RV with different a, b

$X \sim \text{Beta}(a, b)$

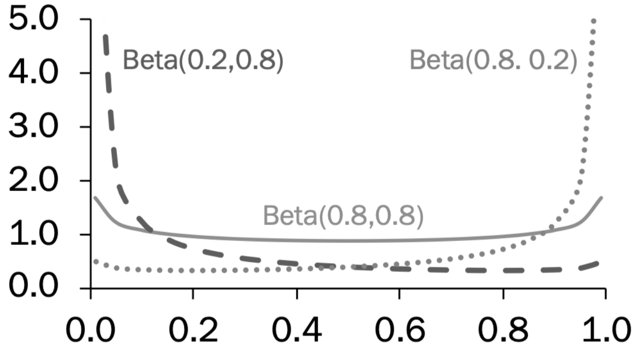
$a > 0, b > 0$

Support of X: $(0, 1)$

PDF $\quad f(x) = \dfrac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}$

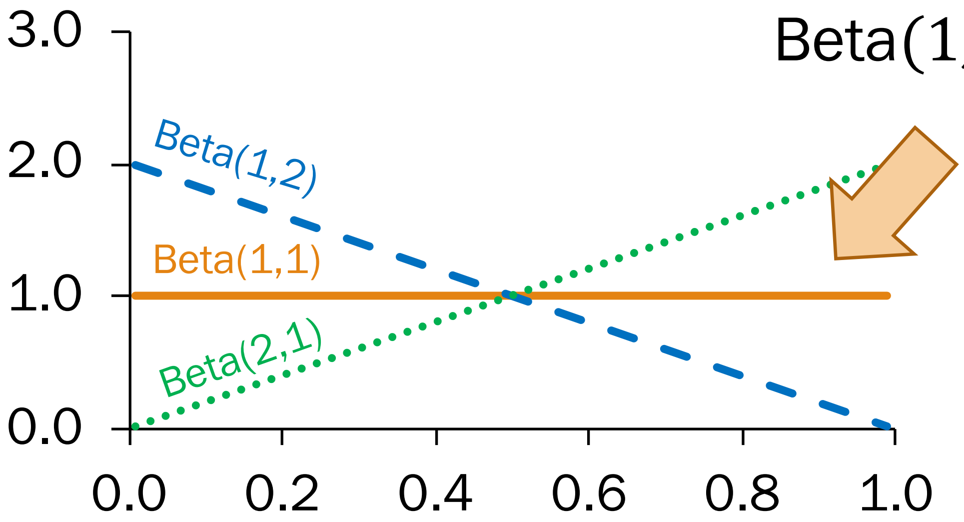
where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$, normalizing constant

$0 < a, b < 1$



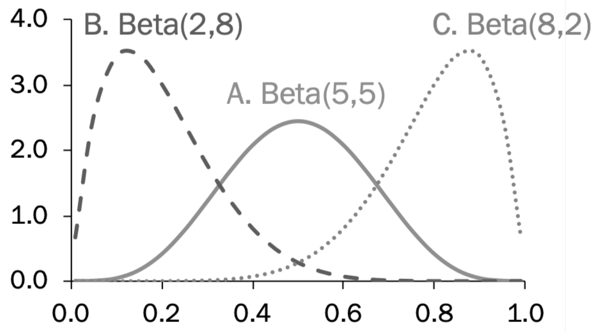
$a = 1$ and/or $b = 1$

Beta(1,1)=Uni(0,1)



$a, b > 1$

$a, b \in \mathbb{Z}^+$

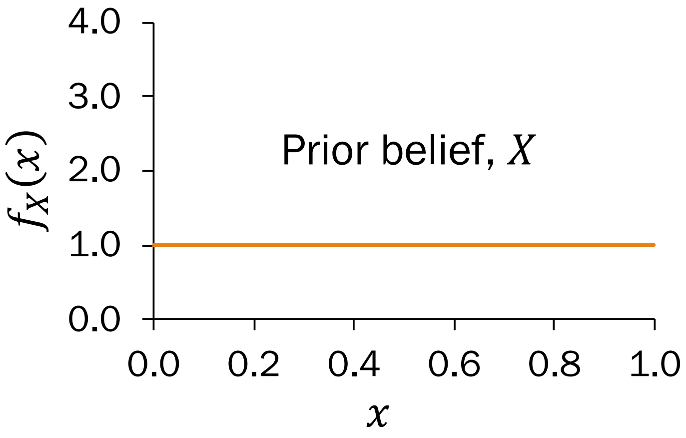


Note: PDF symmetric when $a = b$

A note about our prior

1. Start with a $\theta \sim \text{Uni}(0,1)$ over probability that a coin lands heads.

Beta(1,1)



2. Flip a coin 8 times. Observe $n = 7$ heads and $m = 1$ tail

3. What is our posterior belief of the probability θ?

Beta(8,2)

Check this out. Beta$(a = 1, b = 1)$:

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}$$

$$= \frac{1}{\int_0^1 1dx}$$

$$= 1 \qquad \text{where } 0 < x < 1$$

Beta is a conjugate distribution for Bernoulli

Beta is a conjugate distribution for Bernoulli, meaning:

- Prior and posterior parametric forms are the same

(proof on next slide)

Beta is a conjugate distribution for Bernoulli

Beta is a **conjugate distribution** for Bernoulli, meaning:

1. If our prior belief of the parameter is Beta, and
2. Our experiment is Bernoulli, then (observe n successes, m failures)
3. Our posterior is also Beta.

<u>Proof</u>: $\theta \sim \text{Beta}(a, b)$ $\quad N|\overset{x}{\overset{\shortparallel}{\theta}} \sim \text{Bin}(n + m, x)$

$$f_{\theta|N}(x|n) = \frac{p_{N|\theta}(n|x) f_\theta(x)}{p_N(n)} = \frac{\binom{n+m}{m} x^n (1-x)^m \cdot \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}}{p_N(n)}$$

constants that don't depend on x

$$= C \cdot x^n (1-x)^m \cdot x^{a-1}(1-x)^{b-1}$$

$$= C \cdot x^{(n+a)-1}(1-x)^{(m+b)-1}$$

Beta(n+a, m+b)

Beta is a conjugate distribution for Bernoulli

This is the main takeaway of Beta.

Beta is a conjugate distribution for Bernoulli, meaning:

- Prior and posterior parametric forms are the same
- Practically, conjugate means easy update:
  Add number of "heads" and "tails" seen to Beta parameters.

You can set the prior to reflect how biased you think the coin is apriori:

- $\theta \sim \text{Beta}(a, b)$: have seen $(a + b - 2)$ imaginary trials, where $(a - 1)$ are heads, $(b - 1)$ tails
- Then $\text{Beta}(1, 1) = \text{Uni}(0, 1)$ means we haven't seen any imaginary trials

Prior	$\text{Beta}(a = n_{imag} + 1, b = m_{imag} + 1)$
Experiment	Observe n successes and m failures
Posterior	$\text{Beta}\left(a = n_{imag} + n + 1, b = m_{imag} + m + 1\right)$

The enchanted die

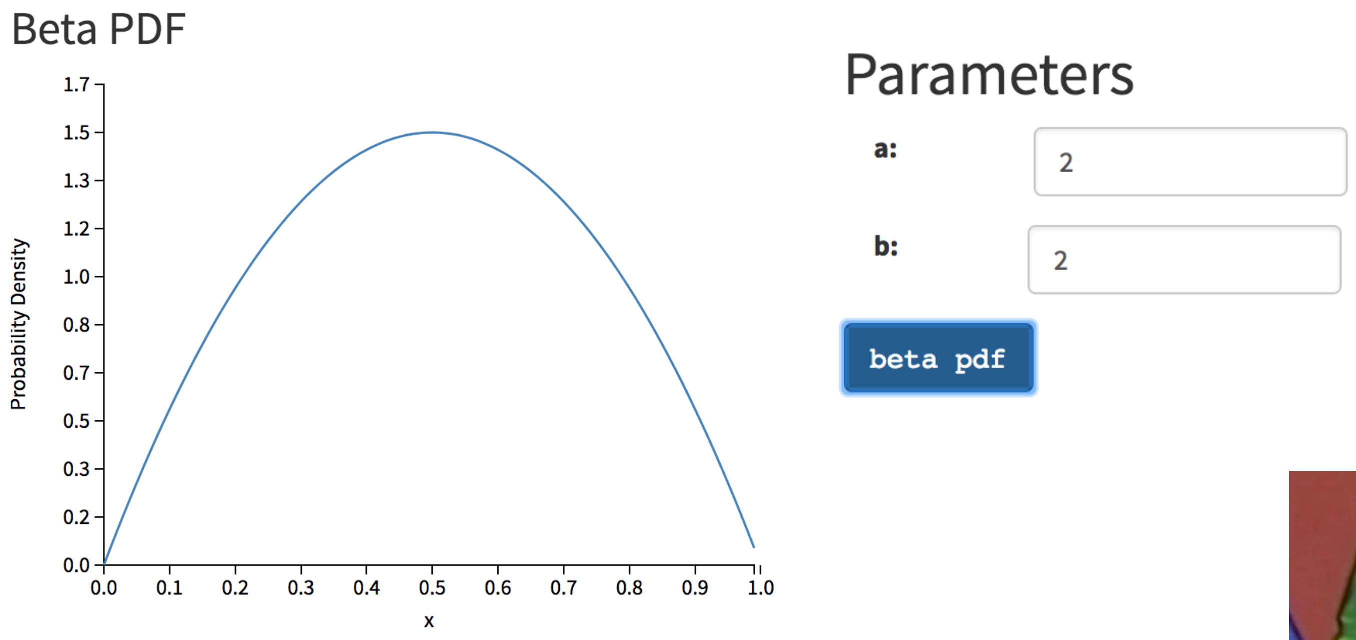
Prior	$\text{Beta}(a = n_{imag} + 1, b = m_{imag} + 1)$
Posterior	$\text{Beta}(a = n_{imag} + n + 1, b = m_{imag} + m + 1)$

Let θ be the probability of rolling a 6 on Lisa's die.

- Prior: Imagine 5 die rolls where only 6 showed up
- Observation: roll it a few times…

What is the updated distribution of θ after our observation?

Check out the demo!



Medicinal Beta

- Before being tested, a medicine is believed to “work” 80% of the time.
- The medicine is tried on 20 patients.
- It “works” for 14, “doesn’t work” for 6.

What is your new belief that the drug “works”?

Frequentist

Let θ be the probability your drug works.

$$\theta \approx \frac{14}{20} = 0.7$$

Bayesian

A frequentist view will not incorporate prior/expert belief about probability.

Medicinal Beta

- Before being tested, a medicine is believed to "work" 80% of the time.
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What is your new belief that the drug "works"?

Frequentist

Let θ be the probability your drug works.

$$\theta \approx \frac{14}{20} = 0.7$$

Bayesian

Let θ be the probability your drug works.

θ is a random variable.

Medicinal Beta

Prior	$\text{Beta}(a = n_{imag} + 1, b = m_{imag} + 1)$
Posterior	$\text{Beta}(a = n_{imag} + n + 1, b = m_{imag} + m + 1)$

- Before being tested, a medicine is believed to "work" 80% of the time.
- The medicine is tried on 20 patients.
- It "works" for 14, "doesn't work" for 6.

What is your new belief that the drug "works"? (Bayesian interpretation)

What is the prior distribution of θ? (select all that apply)

A. $\theta \sim \text{Beta}(1, 1) = \text{Uni}(0, 1)$

B. $\theta \sim \text{Beta}(81, 101)$

C. $\theta \sim \text{Beta}(80, 20)$

D. $\theta \sim \text{Beta}(81, 21)$

E. $\theta \sim \text{Beta}(5, 2)$

Medicinal Beta

Prior	$\text{Beta}(a = n_{imag} + 1, b = m_{imag} + 1)$
Posterior	$\text{Beta}(a = n_{imag} + n + 1, b = m_{imag} + m + 1)$

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What is the prior distribution of θ? (select all that apply)

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B. $\theta \sim \text{Beta}(81,101)$

C. $\theta \sim \text{Beta}(80,20)$

D. $\theta \sim \text{Beta}(\underset{4+1}{\overset{80+1}{81}}, \underset{1+1}{\overset{20+1}{21}})$ Interpretation: 80 successes / 100 imaginary trials

E. $\theta \sim \text{Beta}(5,2)$ 4 ~~4~~ successes 15 imaginary trials

(you can choose either based on how strong your belief is (an engineering choice). We choose E on next slide)

Medicinal Beta

Prior	$\text{Beta}(a = n_{imag} + 1, b = m_{imag} + 1)$
Posterior	$\text{Beta}(a = n_{imag} + n + 1, b = m_{imag} + m + 1)$

- Before being tested, a medicine is believed to "work" 80% of the time.
- The medicine is tried on 20 patients.
- It "works" for 14, "doesn't work" for 6.

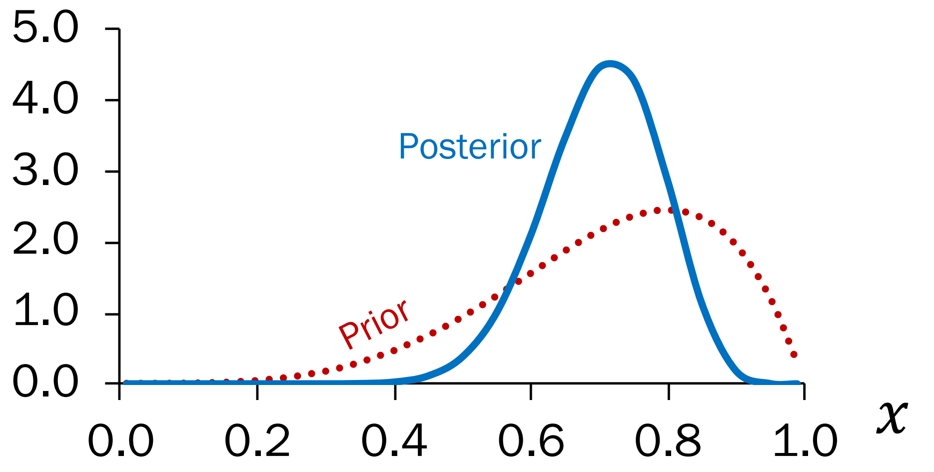
What is your new belief that the drug "works"?

(Bayesian interpretation)

Prior: $\theta \sim \text{Beta}(a = 5, b = 2)$

Posterior: $\theta \sim \text{Beta}(a = 5 + 14, b = 2 + 6)$

$\sim \text{Beta}(a = 19, b = 8)$



Medicinal Beta

Prior	$\text{Beta}(a = n_{imag} + 1, b = m_{imag} + 1)$
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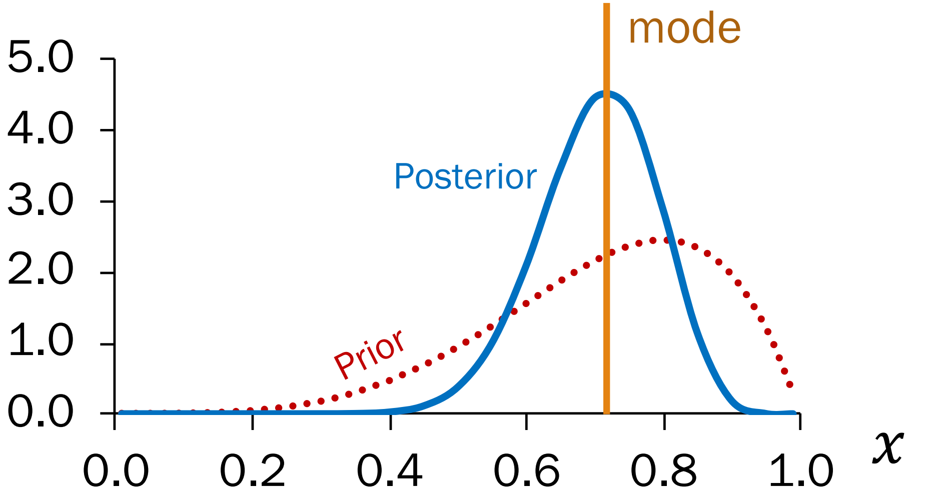

What do you report to pharmacists?

A. Expectation of posterior
B. Mode of posterior
C. Distribution of posterior
D. Nothing

Medicinal Beta

Prior	$\text{Beta}(a = n_{imag} + 1, b = m_{imag} + 1)$
Posterior	$\text{Beta}(a = n_{imag} + n + 1, b = m_{imag} + m + 1)$

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(Bayesian interpretation)

Prior: $\theta \sim \text{Beta}(a = 5, b = 2)$

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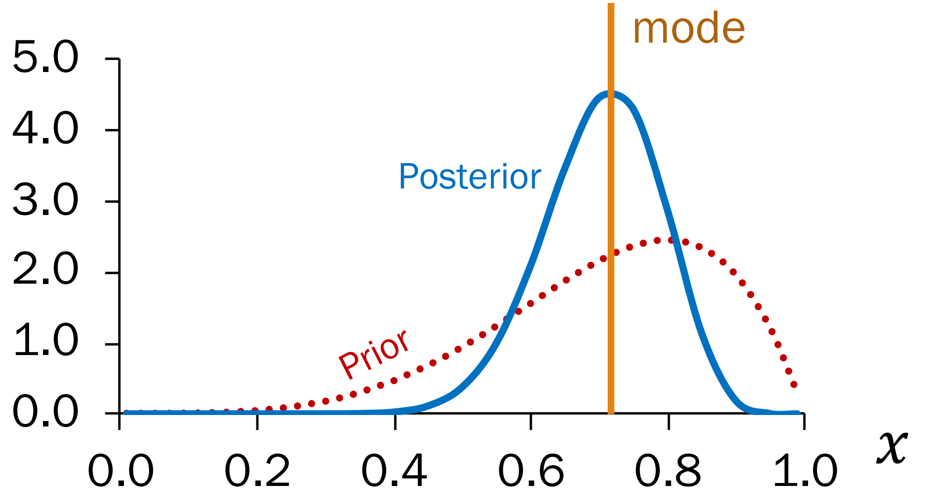

What do you report to pharmacists?

$$E[\theta] = \frac{a}{a+b} = \frac{19}{19+8} \approx 0.70$$

$$\text{mode}(\theta) = \frac{a-1}{a+b-2} = \frac{18}{18+7} \approx 0.72$$

(we'll derive the formula for Beta's mode next lecture)

In CS109, we report the **mode**: The "most likely" parameter given the data.

Food for thought

In this lecture:

$$X \sim \mathrm{Ber}(p)$$

If we don't know the **parameter** p,
Bayesian statisticians will:

- Treat the parameter p as a random variable θ with a Beta prior distribution
- Perform an experiment
- Based on experiment outcomes, update the posterior distribution of θ

Food for thought:

Any parameter for a "parameterized" random variable can be thought of as a random variable.

$$Y \sim \mathcal{N}(\mu, \sigma^2)$$

Estimating our parameter directly

(our focus so far)

Maximum Likelihood Estimator (MLE)

What is the parameter θ that **maximizes the likelihood** of our observed data $(x_1, x_2, \dots, x_n)$?

$$L(\theta) = f(X_1, X_2, \dots, X_n|\theta)$$
$$= \prod_{i=1}^{n} f(X_i|\theta)$$

$$\theta_{MLE} = \arg\max_{\theta} \underbrace{f(X_1, X_2, \dots, X_n|\theta)}_{\text{likelihood of data}}$$

Observations:

- MLE maximizes probability of observing data given a parameter θ.
- If we are estimating θ, shouldn't we maximize the probability of θ directly?

See you next time!

Extra: MLE: Multinomial derivation

Okay, just one more MLE with the Multinomial

Consider a sample of n i.i.d. random variables where

- Each element is drawn from one of m outcomes. $P(\text{outcome } i) = p_i$, where $\sum_{i=1}^{m} p_i = 1$
- X_i = # of trials with outcome i, where $\sum_{i=1}^{m} X_i = n$

1. What is the likelihood of observing the sample $(X_1, X_2, \dots, X_m)$, given the probabilities $p_1, p_2, \dots, p_m$?

$$L(\theta) = \frac{n!}{X_1!\, X_2! \cdots X_m!} p_1^{X_1} p_2^{X_2} \cdots p_m^{X_m}$$

2. What is θ_{MLE}?

$$LL(\theta) = \log(n!) - \sum_{i=1}^{m} \log(X_i!) + \sum_{i}^{m} X_i \log(p_i)\,, \quad \text{such that } \sum_{i=1}^{m} p_i = 1$$

Optimize with Lagrange multipliers in extra slides

$$\theta_{MLE}\colon\ p_i = \frac{X_i}{n}$$

Intuitively, probability p_i = proportion of outcomes

Optimizing MLE for Multinomial

$$\theta = (p_1, p_2, \dots, p_m)$$

$$\theta_{MLE} = \arg\max_{\theta} LL(\theta), \text{ where } \sum_{i=1}^{m} p_i = 1$$

Use Lagrange multipliers to account for constraint

Lagrange multipliers:

$$A(\theta) = LL(\theta) + \lambda\left(\sum_{i=1}^{m} p_i - 1\right) = \sum_{i} X_i \log(p_i) + \lambda\left(\sum_{i=1}^{m} p_i - 1\right)$$

(drop non-p_i terms)

Differentiate w.r.t. each p_i, in turn:

$$\frac{\partial A(\theta)}{\partial p_i} = X_i \frac{1}{p_i} + \lambda = 0 \quad \Rightarrow \quad p_i = -\frac{X_i}{\lambda}$$

Solve for λ, noting $\sum_{i=1}^{m} X_i = n, \sum_{i=1}^{m} p_i = 1$:

$$\sum_{i=1}^{m} p_i = \sum_{i=1}^{m} -\frac{X_i}{\lambda} = 1 \quad \Rightarrow 1 = -\frac{n}{\lambda} \quad \Rightarrow \lambda = -n$$

Substitute λ into p_i

$$p_i = \frac{X_i}{n}$$