Outline

- **Review**
- **Practice Problems!**

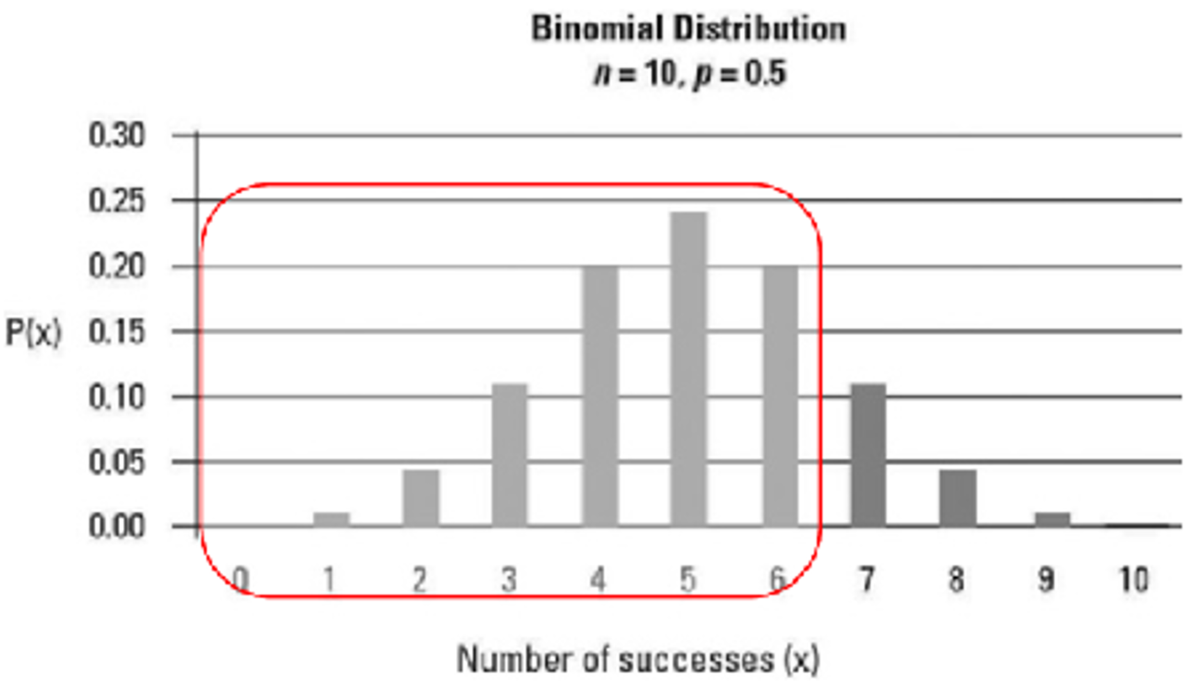
Review Time!

- Random Variables
- Joint Distributions
- Joint RV Statistics
- Conditional Distribution
- General Inference
- **Practice Problems!**

Probability Distributions

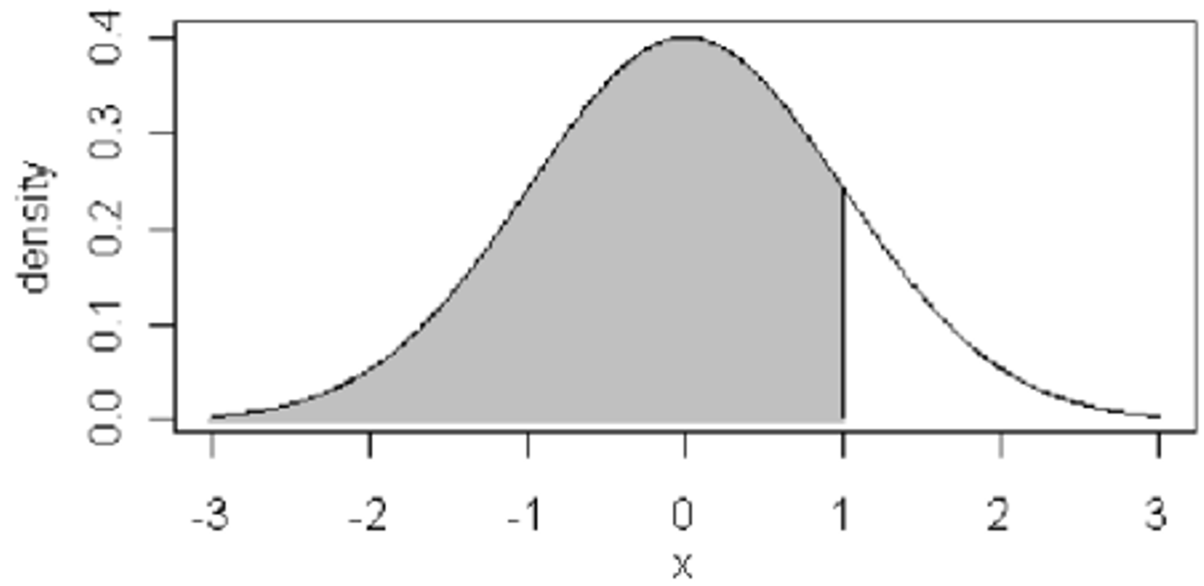
Discrete

CDF:



Continuous

CDF:



Expectation & Variance

Discrete definition

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

Continuous definition

Wait for it…

Expectation & Variance

Discrete definition

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

Continuous definition

Wait for it…

Properties of Expectation

$$E[X + Y] = E[X] + E[Y]$$

$$E[aX + b] = aE[X] + b$$

$$E[g(X)] = \sum_{x} g(x)p(x)$$

Properties of Variance

$$Var(X) = E[(X - \mu)^2]$$

$$Var(X) = E[X^2] - E[X]^2$$

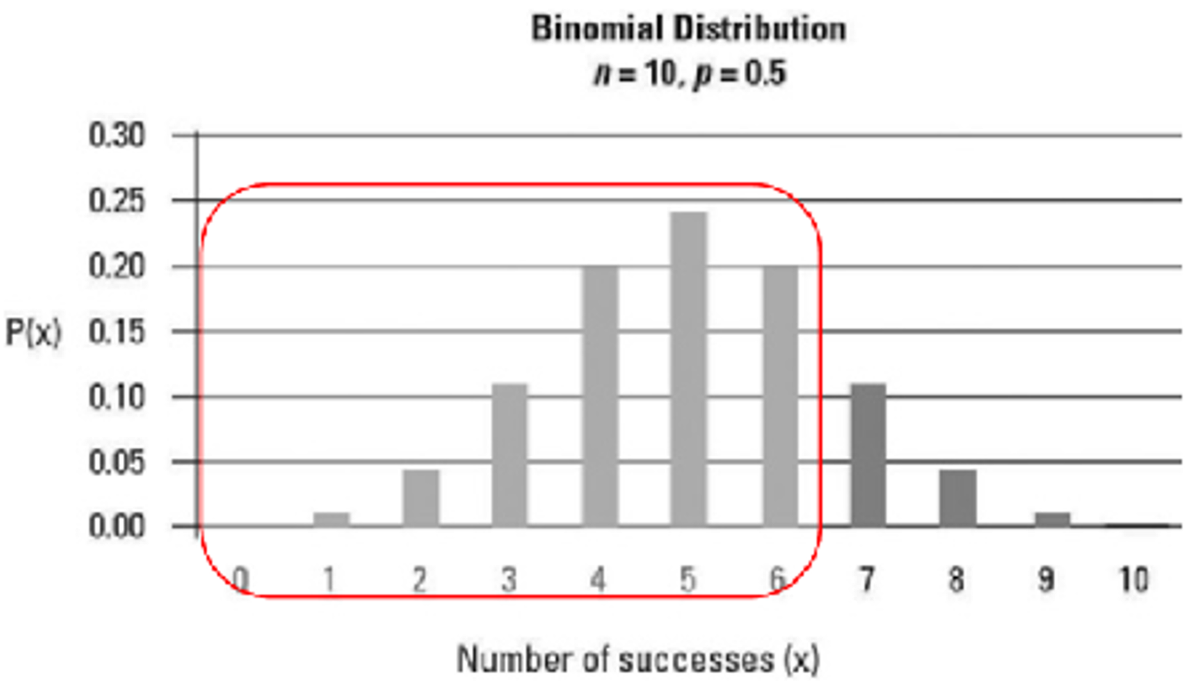
$$Var(aX + b) = a^2 Var(X)$$

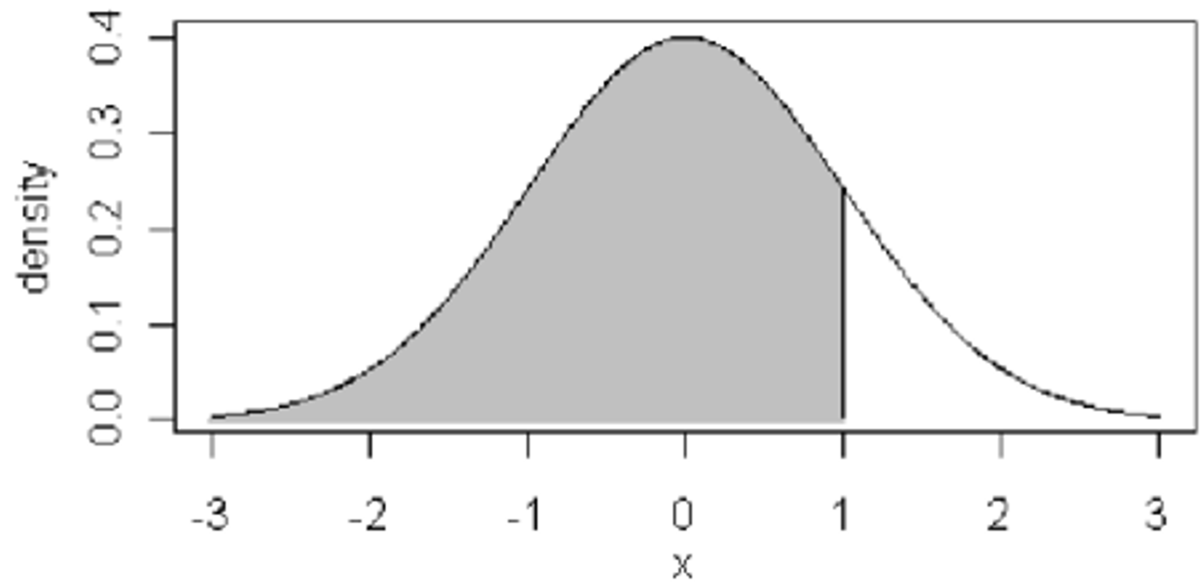
All our (discrete) friends

Ber(p)	Bin(n, p)	Poi(λ)	Geo(p)	NegBin (r, p)
P(X) = p	$\binom{n}{k} p^k (1-p)^{n-k}$	$\frac{\lambda^k e^{-\lambda}}{k!}$	$(1-p)^{k-1}p$	$\binom{k-1}{r-1} p^r (1-p)^{k-r}$
E[X] = p	E[X] = np	E[X] = λ	E[X] = 1 / p	E[X] = r / p
Var(X) = p(1-p)	Var(X) = np(1-p)	Var(X) = λ	$\frac{1-p}{p^2}$	$\frac{r(1-p)}{p^2}$
1 experiment with prob p of success	n independent trials with prob p of success	Number of success over experiment duration, λ rate of success	Number of independent trials until first success	Number of independent trials until r successes

Probability Distributions

Discrete	Continuous
CDF:	CDF:





All our (continuous) friends

For continuous RVs, we need to calculate the PDF, instead of the PMF

PDF for RV X

$f(x) \geq 0$ such that $-\infty < x < \infty$

$$P(a \leq x \leq b) = \int_a^b f(x)dx$$

Expectation & Variance

Discrete definition

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

Continuous definition

$$E[X] = \int_a^b x \cdot f(x)dx$$

Expectation & Variance

Discrete definition

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

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$$E[X] = \int_a^b x \cdot f(x)dx$$

Properties of Expectation

$$E[X + Y] = E[X] + E[Y]$$

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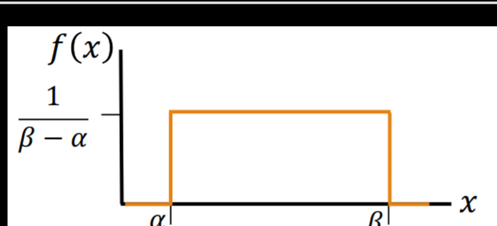
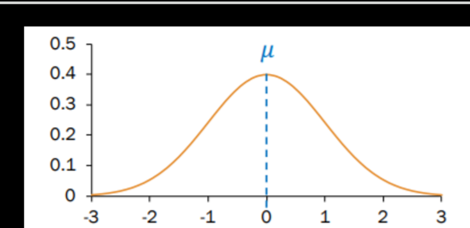
Properties of Variance

$$Var(X) = E[(X - \mu)^2]$$

$$Var(X) = E[X^2] - E[X]^2$$

$$Var(aX + b) = a^2 Var(X)$$

All our (continuous) friends

Uni(α, β)	Exp(λ)	N(μ, σ ²)
$f(x) = \frac{1}{\beta - \alpha}$	$f(x) = \lambda e^{-\lambda x}$	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$
$P(a \leq X \leq b) = \frac{b-a}{\beta-\alpha}$	$F(x) = 1 - e^{-\lambda x}$	$F(x) = \phi\left(\frac{x-\mu}{\sigma}\right)$
$E[X] = \frac{\alpha + \beta}{2}$	E[X] = 1 / λ	E[X] = μ
$Var(x) = \frac{(\beta-\alpha)^2}{12}$	$Var(x) = \frac{1}{\lambda^2}$	$Var(x) = \sigma^2$
	Duration of time until success occurs. λ is rate of success	

Approximations

When can we approximate a binomial?

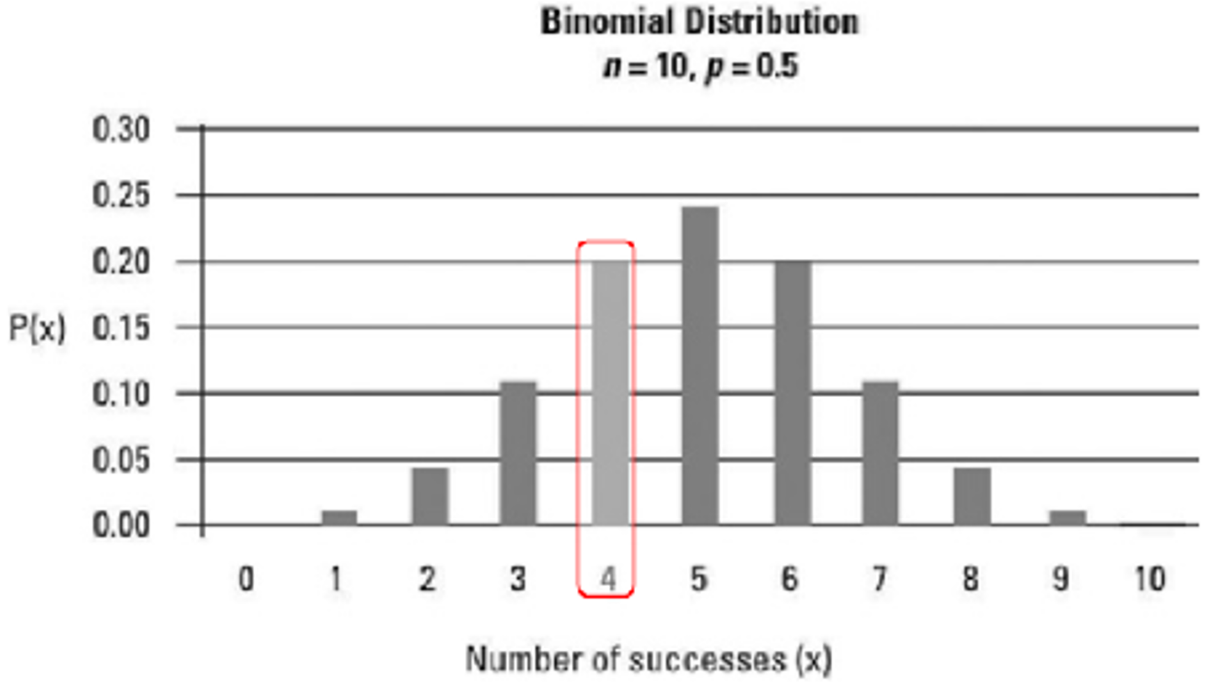
- **Poisson**
  - **$n > 20$**
  - **p is small**
  - **$\lambda = np$ is moderate**
    - **$n > 20$ and $p < 0.05$**
    - **$n > 100$ and $p < 0.1$**
  - **Slight dependence ok**

- **Normal**
  - **$n > 20$**
  - **p is moderate**
    - **$np(1-p) > 10$**
  - **Independent trials**

Continuity correction

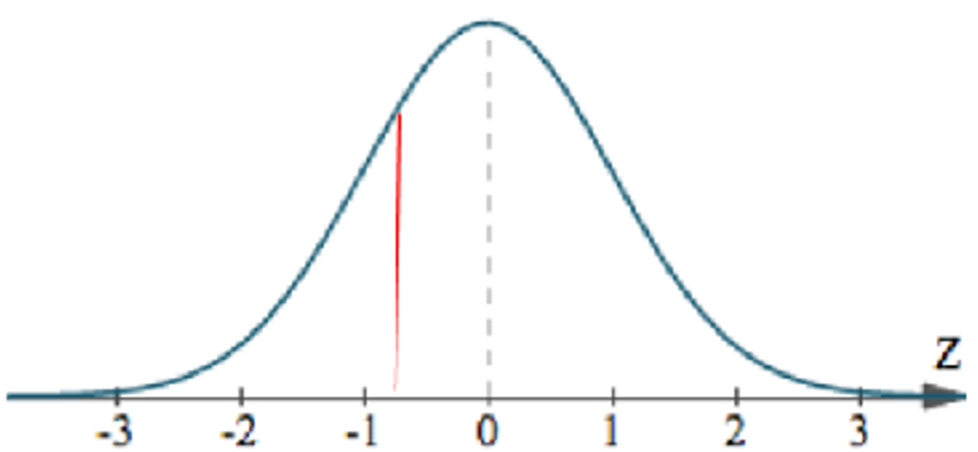
Discrete

PMF:



Continuous

PDF:



Joint Distributions – Discrete

$$p_{x,y}(a, b) = P(X = a, Y = b)$$

$$P_x(a) = \sum_y P_{x,y}(a, y)$$

$$F_{X,Y}(a, b) = \sum_{x \le a} \sum_{y \le b} p_{X,Y}(x, y)$$

Multinomial RVs

Joint PMF

$$P(X_1 = c_1, X_2 = c_2, \dots, X_m = c_m) = \binom{n}{c_1, c_2 \dots, c_m} p_1^{c_1}, p_2^{c_2} \dots p_m^{c_m}$$

Where $\sum_{i=0}^{m} c_i = n$

Generalize to Binomial RVs

Independent Discrete RVs

Two discrete random variables X and Y are independent if for all x,y:

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

Sum of independent Binomials

$$X + Y \sim Bin(n_1 + n_2, p)$$

Sum of independent Poisson RVs

$$X + Y \sim Poi(\lambda_1 + \lambda_2)$$

Covariance

$$
\begin{aligned}
Cov(X,Y) &= E[(X - E[X])(Y - E[Y])] \\
&= E[XY] - E[X]E[Y]
\end{aligned}
$$

Covariance

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

How do you calculate variance of two RVs?

$$Var(X + Y) = Var(X) + 2 \cdot Cov(X,Y) + Var(Y)$$

Covariance

$$Var(X+Y) = Var(X) + 2 \cdot Cov(X,Y) + Var(Y)$$

When X and Y are independent

$$Var(X+Y) = Var(X) + Var(Y)$$

Note when we only know Cov(X,Y)=0 we can’t assume X and Y are independent

Correlation

Correlation of X and Y

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$\sigma_X^2 = \text{Var}(X)$,
$\sigma_Y^2 = \text{Var}(Y)$

Note: $-1 \leq \rho(X,Y) \leq 1$
Measures the linear relationship between X and Y

$\rho(X,Y) = 1 \quad \Rightarrow Y = aX + b$, where $a = \sigma_Y/\sigma_X$
$\rho(X,Y) = -1 \quad \Rightarrow Y = aX + b$, where $a = -\sigma_Y/\sigma_X$
$\rho(X,Y) = 0 \quad \Rightarrow$ "uncorrelated" (absence of linear relationship)

Conditional Distribution

Conditional PMF for discrete X given Y

$$P(X = x|Y = y) = \frac{P(X=x,Y=y)}{P(Y=y)}$$

Conditional Expectation

$$E[X|Y = y] = \sum_{x} xP(X = x|Y = y)$$

Conditional Distribution

Law of Total Expectation

$$E\big[E[X|Y]\big] = \sum_y P(Y = y)E[X|Y = y] = E[X]$$

If we only have a conditional PMF of X on some discrete variable Y, we can compute $E[X]$ as follows:

1. Compute expectation of X given some value of $Y = y$
2. Repeat step 1 for all values of Y
3. Compute a weighted sum (where weights are $P(Y = y)$)

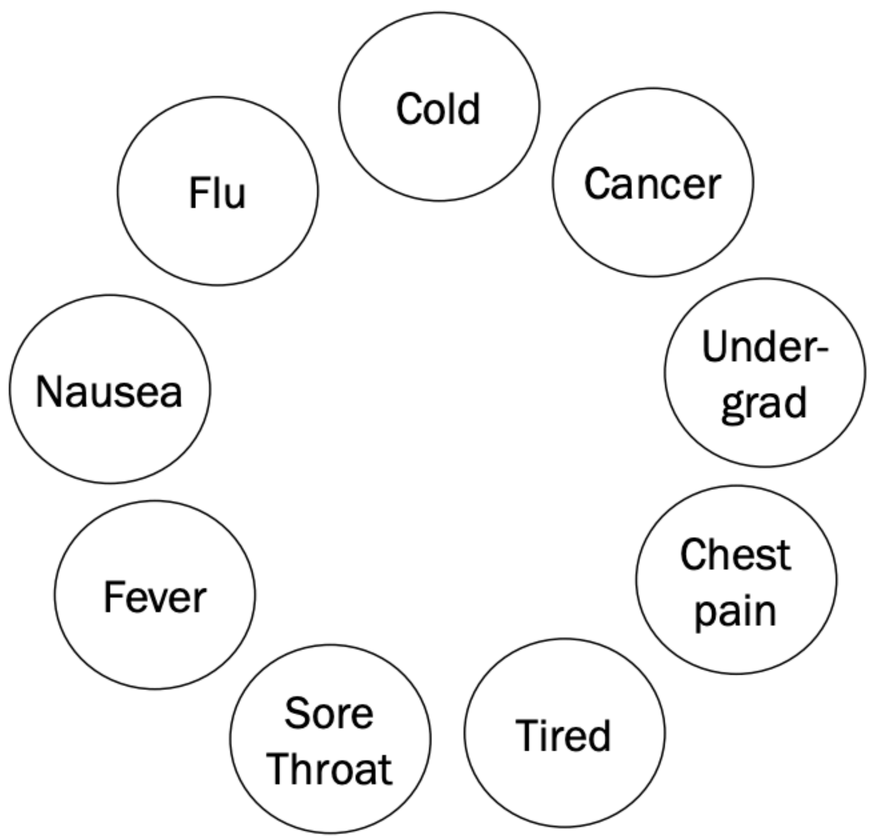
```
def recurse():
  if (random.random() < 0.5):
      return 3
  else: return (2 + recurse())
```

Useful for analyzing recursive code!!

Stay tuned!

General Inference

General Inference

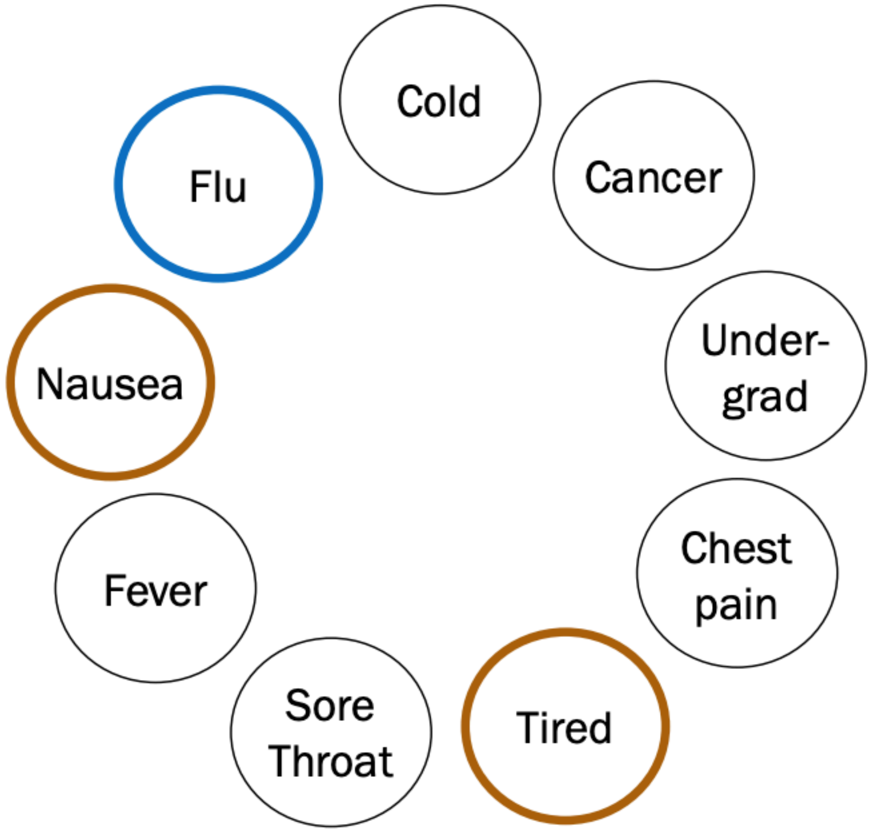


General inference question:

Given the values of some random variables, what is the conditional distribution of some other random variables?

General Inference

General Inference



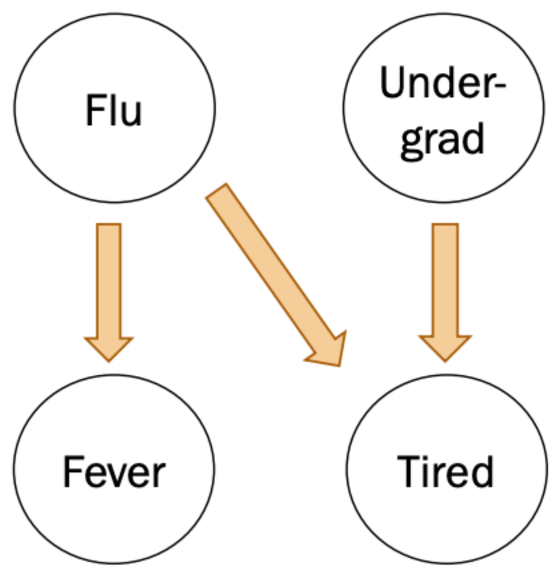
One inference question:

$$P(F = 1 | N = 1, T = 1)$$

$$= \frac{P(F = 1, N = 1, T = 1)}{P(N = 1, T = 1)}$$

General Inference

Bayesian Networks



In a Bayesian Network,

Each random variable is conditionally independent of its non-descendants, given its parents.

- Node: random variable
- Directed edge: conditional dependency

Examples:

- $P(F_{ev} = 1 | T = 0, F_{lu} = 1) = P(F_{ev} = 1 | F_{lu} = 1)$
- $P(F_{lu} = 1, U = 0) = P(F_{lu} = 1) P(U = 0)$

Practice Time

- **Quiz Logistics and Coverage**
- **Random Variables**
- **Joint Distributions**
- **Joint RV Statistics**
- **Conditional Distribution**
- **General Inference**
- **Practice Problems!**

Practice Problems

- 500 year flood planes (“a previous exam” on website)
  - The Huffmeister floodplane in Houston has historically been estimated to flood at an average rate of 1 flood for every 500 years.
- What is the probability of observing at least 3 floods in 500 years?
- What is the probability that a flood will occur within the next 100 years?
- What is the expected number of years until the next flood?

Practice Problems

- What is the probability of observing at least 3 floods in 500 years?
  - Poisson with lambda = 1 (flood per 500 years)
  - P(X >= 3) = 1 – P(X < 3) = 1 – (sum of P(X=i) from 0 to 2)
  - 1 – 5/2e
- What is the probability that a flood will occur within the next 100 years?
  - Exponential with lambda = 1/500
  - F(100) = 1 – e^(-0.2)
- What is the expected number of years until the next flood?
  - Expectation for an exponential RV is 1/lambda = 500

Practice Problems

- Recursive Code Problem

Consider the following recursive function

```
int Near() {
   int b = randomInteger(1, 4); // equally likely to be 1, 2, 3 or 4
   if (b == 1) return 2;
   else if (b == 2) return 4;
   else if (b == 3) return (6 + Near());
   else return (8 + Near());
}

int Far() {
   int a = randomInteger(1, 3); // equally likely to be 1, 2 or 3
   if (a == 1) return 2;
   else if (a == 2) return (2 + Near());
   else return (4 + Far());
}
```

Let Y = the value returned by `Far()`.

Let X = value returned by `Near()`.

Practice Problems

What is E[Y]?

```
int Near() {
   int b = randomInteger(1, 4); // equally likely to be 1, 2, 3 or 4
   if (b == 1) return 2;
   else if (b == 2) return 4;
   else if (b == 3) return (6 + Near());
   else return (8 + Near());
}

int Far() {
   int a = randomInteger(1, 3); // equally likely to be 1, 2 or 3
   if (a == 1) return 2;
   else if (a == 2) return (2 + Near());
   else return (4 + Far());
}
```

Let Y = the value returned by **`Far()`**.

Let X = value returned by **`Near()`**.

Practice Problems

What is E[Y]? First notice Far() calculated based on Near()

```
int Near() {
   int b = randomInteger(1, 4); // equally likely to be 1, 2, 3 or 4
   if (b == 1) return 2;
   else if (b == 2) return 4;
   else if (b == 3) return (6 + Near());
   else return (8 + Near());
}

int Far() {
   int a = randomInteger(1, 3); // equally likely to be 1, 2 or 3
   if (a == 1) return 2;
   else if (a == 2) return (2 + Near());
   else return (4 + Far());
}
```

Let Y = the value returned by **Far()**.

Let X = value returned by **Near()**.

Practice Problems

Probability for Far() is based on Near(), so calculate E[X]

```
int Near() {
   int b = randomInteger(1, 4); // equally likely to be 1, 2, 3 or 4
   if (b == 1) return 2;
   else if (b == 2) return 4;
   else if (b == 3) return (6 + Near());
   else return (8 + Near());
}
```

$$
\begin{aligned}
E[X] &= 1/4(2 + 4 + E[6 + X] + E[8 + X]) \\
&= 1/4(2 + 4 + 6 + E[X] + 8 + E[X]) \\
&= 1/4(20 + 2E[X]) \\
&= 5 + 1/2E[X]
\end{aligned}
$$

So, E[X] = 10

Practice Problems

Now we are ready to calculate E[Y]

```
int Far() {
   int a = randomInteger(1, 3); // equally likely to be 1, 2 or 3
   if (a == 1) return 2;
   else if (a == 2) return (2 + Near());
   else return (4 + Far());
}
```

$$
\begin{aligned}
E[Y] &= 1/3(2 + E[2 + X] + E[4 + Y]) \\
&= 1/3(2 + 2 + E[X] + 4 + E[Y]) \\
&= 1/3(8 + E[X] + E[Y]) \\
&= 1/3(8 + 10 + E[Y]) \\
&= 18/3 + 1/3E[Y]
\end{aligned}
$$

So, $E[Y] = 9$

Practice Problems

What is Var[Y]?

```
int Near() {
   int b = randomInteger(1, 4); // equally likely to be 1, 2, 3 or 4
   if (b == 1) return 2;
   else if (b == 2) return 4;
   else if (b == 3) return (6 + Near());
   else return (8 + Near());
}

int Far() {
   int a = randomInteger(1, 3); // equally likely to be 1, 2 or 3
   if (a == 1) return 2;
   else if (a == 2) return (2 + Near());
   else return (4 + Far());
}
```

Let Y = the value returned by **`Far()`**.

Let X = value returned by **`Near()`**.

Practice Problems

Calculate E[X^2]

```
int Near() {
   int b = randomInteger(1, 4); // equally likely to be 1, 2, 3 or 4
   if (b == 1) return 2;
   else if (b == 2) return 4;
   else if (b == 3) return (6 + Near());
   else return (8 + Near());
}
```

E[X^2] = 1/4(2^2 + 4^2 + E[(6 + X)^2] + E[(8 + X)^2]
= 1/4(4 + 16 + 36 + 12E[X] + E[X^2] + 64 + 16E[X] + E[X^2])
= 1/4(120 + 28E[X] + 2E[X^2])
= 1/4(120 + 28(10) + 2E[X^2])
= 1/4(400 + 2E[X^2])
= 100 + 1/2E[X^2]
So, E[X^2] = 2(100) = 200

Practice Problems

Calculate E[Y^2]

```
int Far() {
   int a = randomInteger(1, 3); // equally likely to be 1, 2 or 3
   if (a == 1) return 2;
   else if (a == 2) return (2 + Near());
   else return (4 + Far());
}
```

E[Y^2] = 1/3(2^2 + E[(2 + X)^ 2]+ E[(4 + Y)^2]
= 1/3(4 + 4 + 4E[X] + E[X^2] + 16 + 8E[Y] + E[Y^2])
= 1/3(24 + 40 + E[X^2] + 8(9) + E[Y^2])
= 1/3(136 + 200 + E[Y^2])
= 1/3(336 + E[Y^2])
So, E[Y^2] = 336/2 = 168

Practice Problems

Now that we have $E[X^2]$ and $E[Y^2]$, we are ready to calculate Var(Y)

$Var(Y) = E[Y^2] – E[Y]^2 = 168 – (9)^2 = 168 – 81 = 87$

Good Luck!!!