02: Combinatorics

Lisa Yan and Jerry Cain

September 16, 2020

Quick slide reference

Today's discussion thread: https://us.edstem.org/courses/2678/discussion/124109

Permutations II

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable)
- Choose k objects (combinations)
- Put objects in r buckets

Sort n distinct objects

Ayesha Tim Irina Joey Waddie

# of permutations = $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$

Summary of Combinatorics

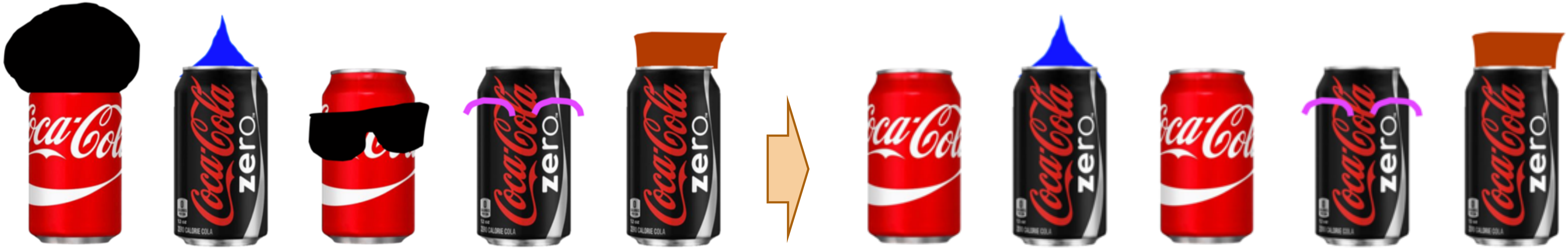
Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct
- Choose k objects (combinations)
- Put objects in r buckets

Sort semi-distinct objects

Order n distinct objects	$n!$

All distinct | Some indistinct



Ayesha Tim Irina Joey Waddie → Coke Tim Coke Joey Waddie

Irina Ayesha

$5! = 120$

$\frac{120}{2}$

Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

$n!$ permutations of distinct objects = permutations considering some objects are indistinct × Permutations of just the indistinct objects

Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

$n!$ permutations of distinct objects = Permutations of just the indistinct objects ($2!$) × permutations considering some objects are indistinct

$$\frac{\text{permutations of distinct objects } (n!)}{\text{Permutations of just the indistinct objects } (2!)} = \text{permutations considering some objects are indistinct}$$

General approach to counting permutations

When there are n objects such that

n_1 are the same (indistinguishable or indistinct), and

n_2 are the same, and

…

n_r are the same,

The number of unique orderings (permutations) is

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}.$$

① permuting assuming that all n objects distinct

② divide by overcounted permutations

For each group of indistinct objects,
Divide by the overcounted permutations.

Sort semi-distinct objects

Order n semi-distinct objects $\frac{n!}{n_1!\, n_2! \cdots n_r!}$

How many permutations?

Coke Coke0 Coke Coke0 Coke0

$$\frac{5!}{2!\,3!} = 10$$

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$
- Choose k objects (combinations)
- Put objects in r buckets

Strings

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many orderings of letters are possible for the following strings?

1. BOBA

2. MISSISSIPPI

Strings

Order n semi-distinct objects $\frac{n!}{n_1!\, n_2! \cdots n_r!}$

How many orderings of letters are possible for the following strings?

1. BOBA

A O B_1 B_2

A O B_2 B_1

$$= \frac{4!}{2!} = 12$$

2. MISSISSIPPI

$$= \frac{11!}{1!4!4!2!} = 34{,}650$$

group 1: 4 I's

" 2: 1 M

3: 4 S's

4: 2 P's

Unique 6-digit passcodes with six smudges

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many unique 6-digit passcodes are possible if a phone password uses each of six distinct numbers?

$$\text{Total} = 6!$$
$$= 720 \text{ passcodes}$$

Unique 6-digit passcodes with five smudges

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many unique 6-digit passcodes are possible if a phone password uses each of five distinct numbers?

<u>Steps</u>:

1. Choose digit to repeat — 5 outcomes
2. Create passcode — (sort 6 digits: 4 distinct, 2 indistinct)

4, 4, 2, 5, 6, 8

$$\text{Total} = 5 \times \frac{6!}{2!}$$

$$= 1{,}800 \text{ passcodes}$$

Combinations I

Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
- **Put objects in r buckets**

Combinations with cake

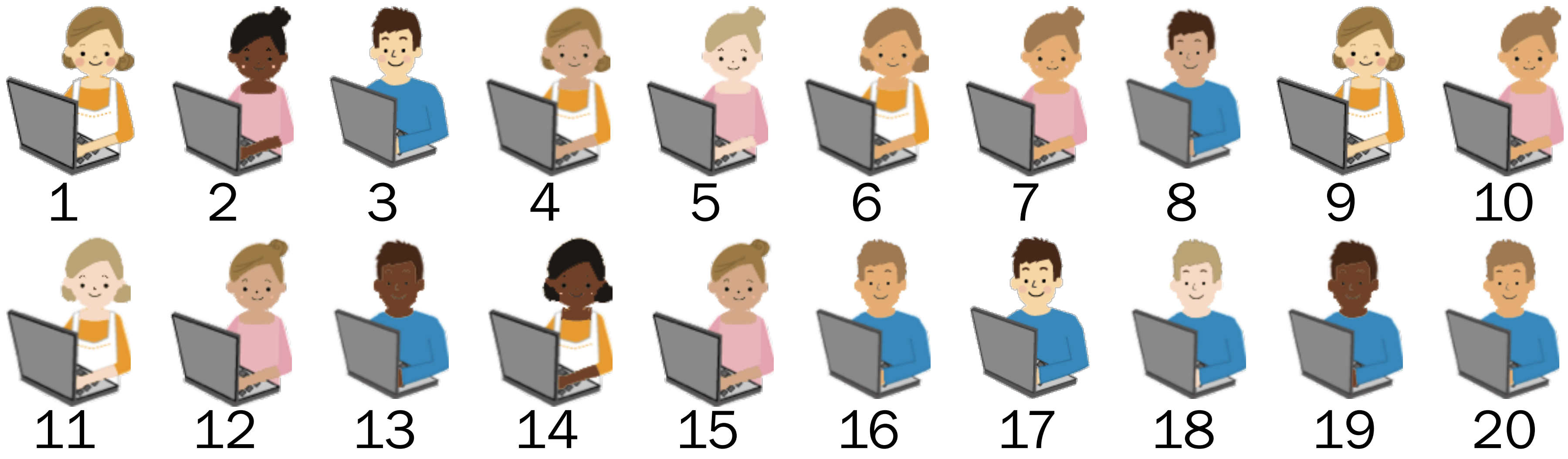
There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

Profiterol

Consider the following generative process...

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

$n!$ ways

Combinations with cake

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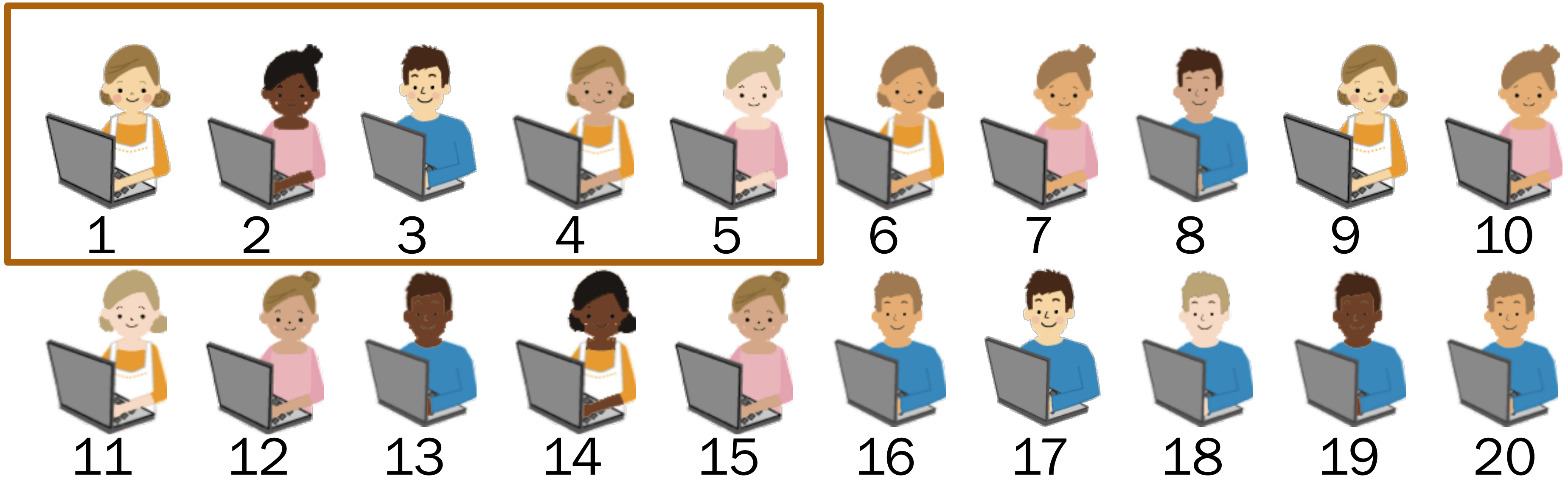

1. n people get in line
2. Put first k in cake room

$n!$ ways

1 way

Combinations with cake

There are $n = 20$ people.
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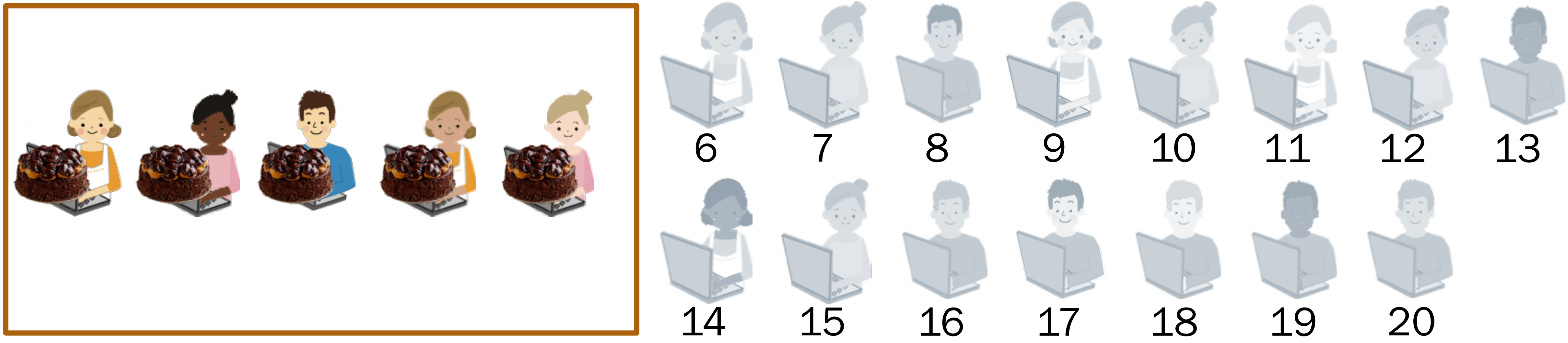

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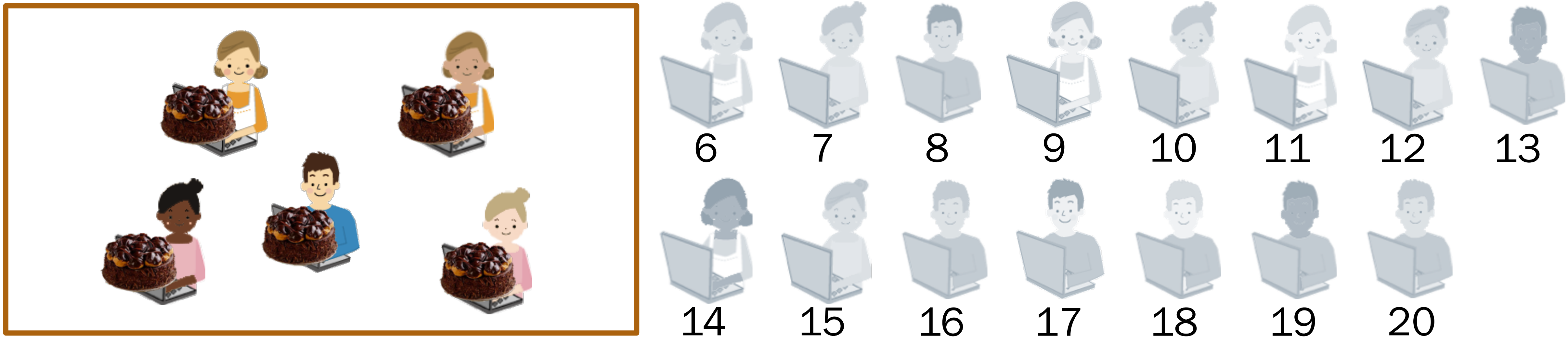

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

Combinations with cake

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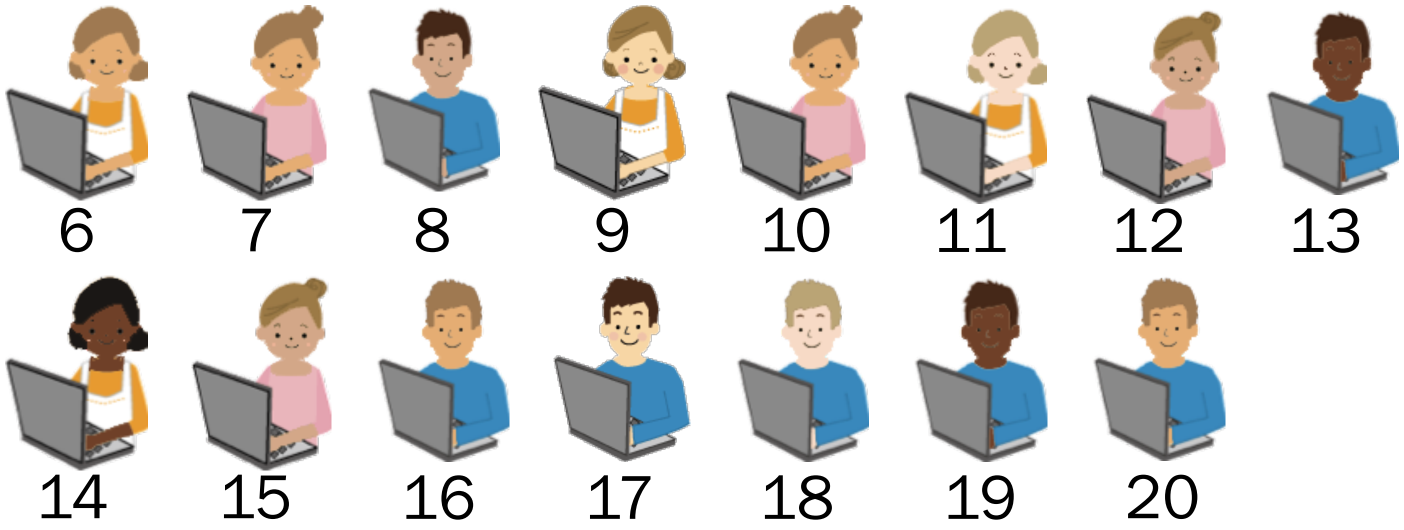
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

   $(n-k)!$ different permutations lead to the same mingle

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!}$$

1. Order n distinct objects
2. Take first k as chosen
3. Overcounted: any ordering of chosen group is same choice
4. Overcounted: any ordering of unchosen group is same choice

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!} = \binom{n}{k}$$

Binomial coefficient

(1)

Note: (2) $\binom{n}{n-k} = \binom{n}{k}$

$\binom{n}{n-k} = \frac{n!}{(n-k)!\,k!}$

$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

Combinations II

Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups
- **Put objects in r buckets**

General approach to combinations

The number of ways to choose r groups of n distinct objects such that

For all $i = 1, \ldots, r$, group i has size n_i, and

$\sum_{i=1}^{r} n_i = n$ (all objects are assigned), is

$$\frac{n!}{n_1!\, n_2! \cdots n_r!} = \binom{n}{n_1, n_2, \cdots, n_r}$$

Multinomial coefficient

n objects → n_1 group 1, n_2 group 2, ⋮ n_r group r

Binomial coefficient

$\binom{n}{k} = \binom{n}{k, n-k}$

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines (goal)
A	6
B	4
C	3

A. $\binom{13}{6,4,3} = 60{,}060$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

C. $6 \cdot 1001 \cdot 10 = 60{,}060$

D. A and B

E. All of the above

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

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13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

A. $\binom{13}{6,4,3} = 60{,}060 = \frac{13!}{6!\,4!\,3!}$

Strategy: Combinations into 3 groups

Group 1 (datacenter A): $n_1 = 6$ ←

Group 2 (datacenter B): $n_2 = 4$ ←

Group 3 (datacenter C): $n_3 = 3$ ←

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \dots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

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Strategy: Combinations into 3 groups

Group 1 (datacenter A): $n_1 = 6$

Group 2 (datacenter B): $n_2 = 4$

Group 3 (datacenter C): $n_3 = 3$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

$$\binom{13}{6}\binom{7}{4}\binom{3}{3} = \frac{13!}{6!\,7!} \times \frac{7!}{4!\,3!} \times \frac{3!}{3!\,0!} = \frac{13!}{6!\,4!\,3!} = \binom{13}{6,4,3}$$

($0! = 1$)

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \dots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

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Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

Your approach will determine if you use binomial/multinomial coefficients or factorials.

02: Combinatorics (live)

Lisa Yan

September 16, 2020

Reminders: Lecture with zoom

- Turn on your camera if you are able, mute your mic in the big room
- Virtual backgrounds are encouraged (classroom-appropriate)

Today's discussion thread: https://us.edstem.org/courses/2678/discussion/124109

Summary of Combinatorics

Review

Counting tasks on n objects

ordered

unordered groups

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\,n_2!\cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**

Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\,n_2!\cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
  - Indistinct?
- **Put objects in r buckets**

Think

Slide 42 is a question to think over by yourself (~1min).

Post any clarifications here or in chat!

https://us.edstem.org/courses/2678/discussion/124109

🤔 (by yourself)

A trick question

How many distinct (distinguishable) ways are there to group 6 indistinct (indistinguishable) objects into 3 groups, where groups A, B, and C have sizes 1, 2, and 3, respectively?

$\frac{6!}{1!2!3!}$

A. $\binom{6}{1,2,3}$

B. $\frac{6!}{1!2!3!}$

C. 0

D. 1

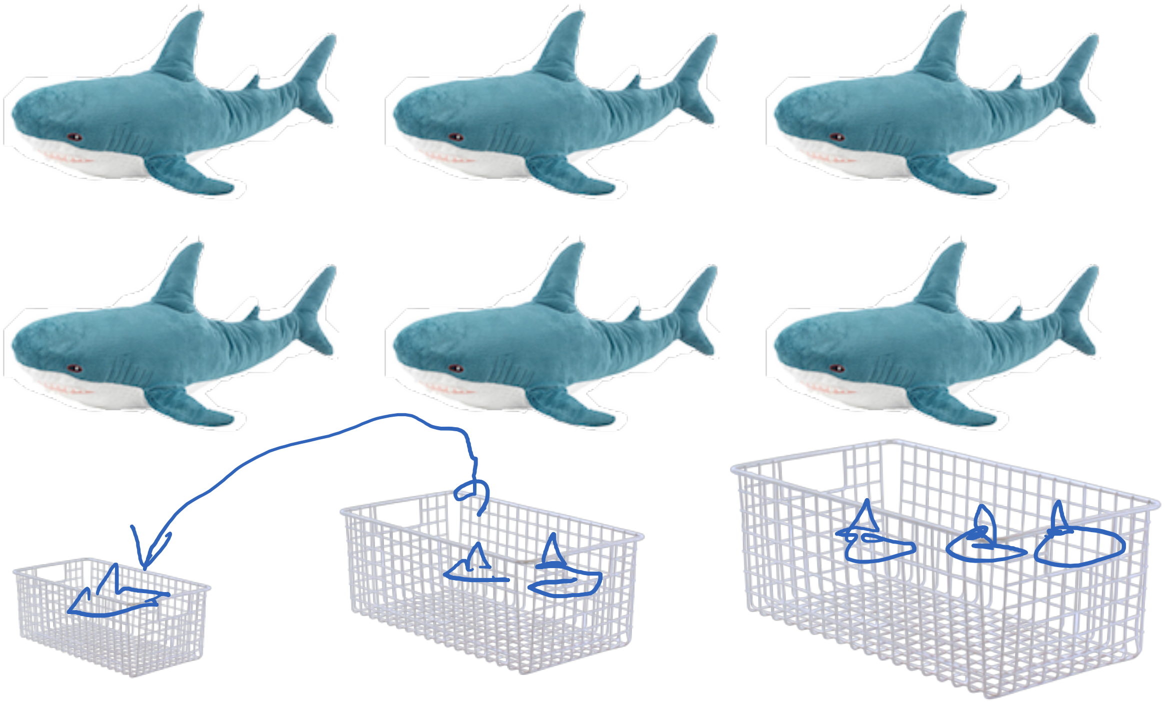
E. Both A and B

F. Something else

(by yourself)

A trick question

How many distinct (distinguishable) ways are there to group 6 indistinct (indistinguishable) objects into 3 groups, where groups A, B, and C have sizes 1, 2, and 3, respectively?



A (fits 1) B (fits 2) C (fits 3)

A. $\binom{6}{1,2,3}$

B. $\frac{6!}{1!2!3!}$

C. 0

D. 1

E. Both A and B

F. Something else

Probability textbooks

Review

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$\binom{6}{3} = \frac{6!}{3!3!} = 20$ ways

Think

Slide 46 is a question to think over by yourself (~2min).

Post any clarifications here or in chat!

https://us.edstem.org/courses/2678/discussion/124109

(by yourself)

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books? $\binom{6}{3} = \frac{6!}{3!\,3!} = 20$ ways

2. What if we do not want to read both the 9th and 10th edition of Ross?

A. $\binom{6}{3} - \binom{6}{2} = 5$ ways

B. $\frac{6!}{3!3!2!} = 10$

C. $2 \cdot \binom{4}{2} + \binom{4}{3} = 16$

D. $\binom{6}{3} - \binom{4}{1} = 16$

E. Both C and D

F. Something else

Ask: https://us.edstem.org/courses/2678/discussion/124109

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

2. What if we do not want to read both the 9th and 10th edition of Ross?

Strategy 1: Sum Rule

3 cases — choose:
- 9th & 2 other books: $1 \cdot \binom{4}{2}$
- 10th & 2 others: $1 \cdot \binom{4}{2}$
- 3 of neither 9th nor 10th books: $+ \binom{4}{3}$

Sum Rule: $2\binom{4}{2} + \binom{4}{3} = 16$ [C]

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

2. What if we do not want to read both the 9th and 10th edition of Ross?

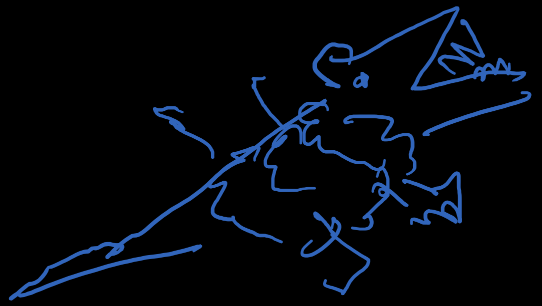
Strategy 2: "Forbidden method" (unofficial name)

DO NOT WANT: choose both 9th & 10th, & one other book $\quad 1 \cdot 1 \cdot \binom{4}{1}$

$$20 - \binom{4}{1} = 16 \text{ ways}$$

Forbidden method: It is sometimes easier to exclude invalid cases than to include cases.

Interlude for jokes/announcements



investigator

Announcements

Problem Set #1

Out: today
Due: Friday 9/25, 1:00pm
Covers: through Friday 9/18

Python tutorial (2 timeslots)

#1

When: Friday 12:00-1:00**am** PT
Friday 2:00-3:00**pm** PT
Recorded? yes
Notes: to be posted online Zoom

Section sign-ups/Acquaintance form

Form released: later today
Form due: Saturday 5:00pm 9/19 Pacific
Results: latest Sunday

Getting help

Ed discussion: find study buddies!
Office hours: start today
https://web.stanford.edu/class/cs109/stanford/staff.html

Buckets and The Divider Method

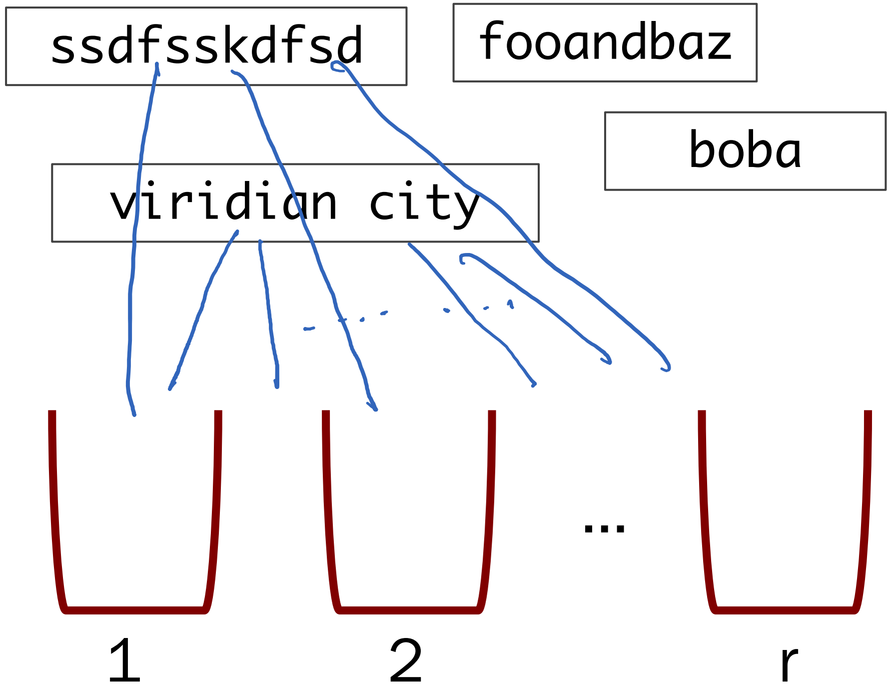
Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\,n_2!\cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct
  - Indistinct

~~Balls and urns~~ Hash tables and distinct strings

How many ways are there to hash n distinct strings to r buckets?



Steps:

1. Bucket 1st string r
2. Bucket 2nd string r

...

n. Bucket nth string r

$\underbrace{r \cdot r \cdot r \cdots r}_{n \text{ steps}}$

r^n outcomes

Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
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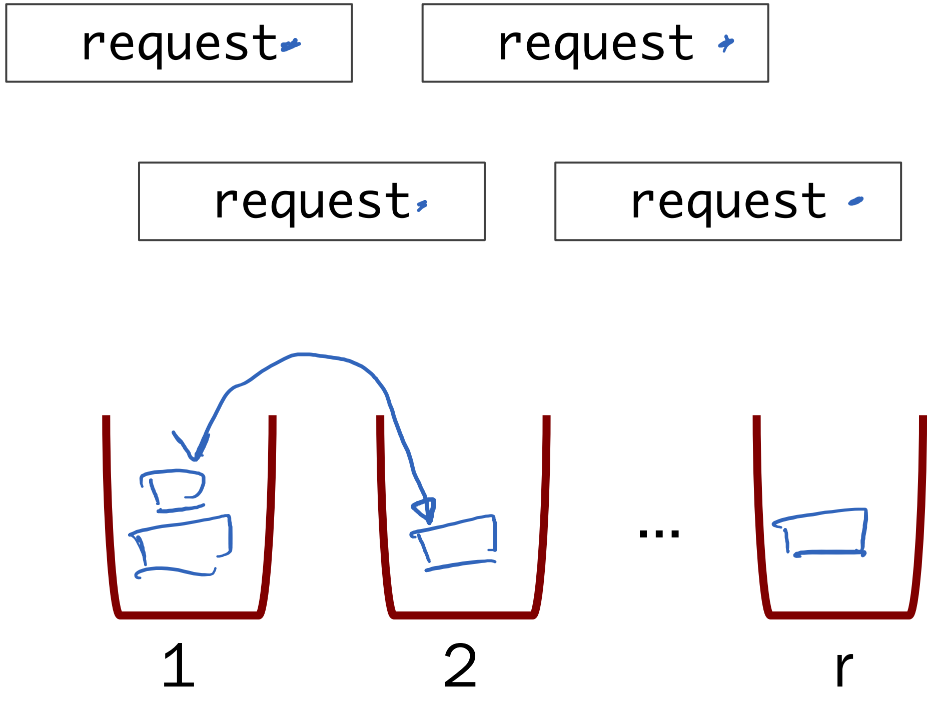
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct: r^n
  - Indistinct

Servers and indistinct requests

How many ways are there to distribute n indistinct web requests to r servers?

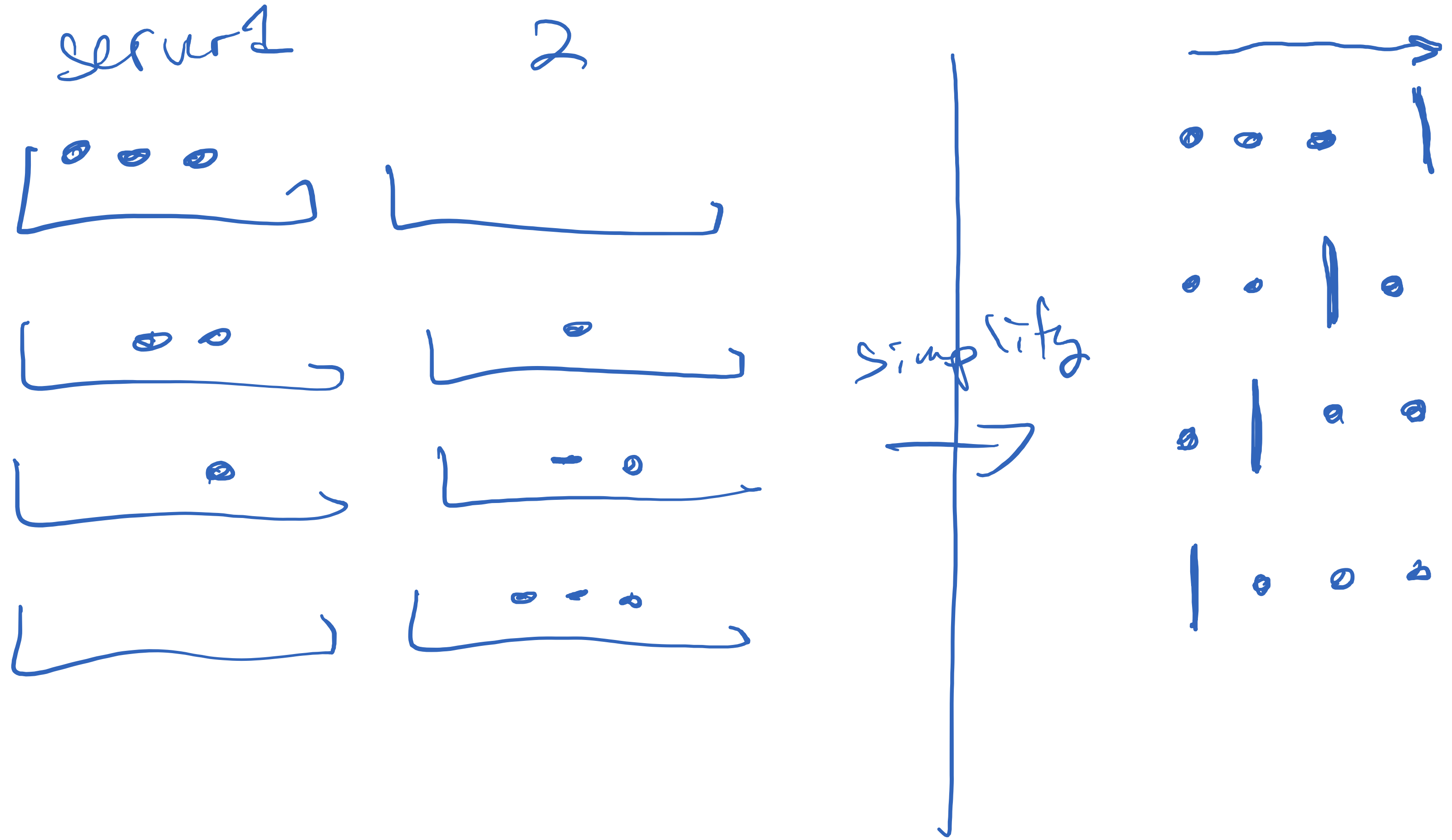


COUNTS IN BUCKETS

Goal

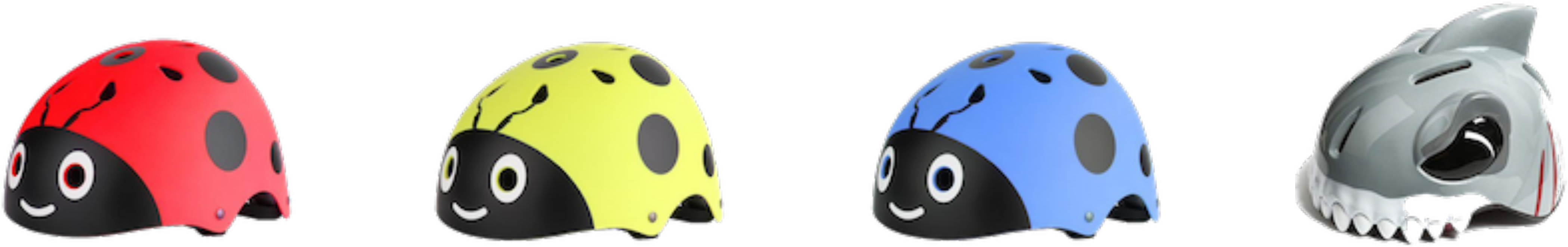
Server 1 has x_1 requests,
Server 2 has x_2 requests,
...
Server r has x_r requests (the rest)

Simple example: $n = 3$ requests and $r = 2$ servers



Bicycle helmet sales

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

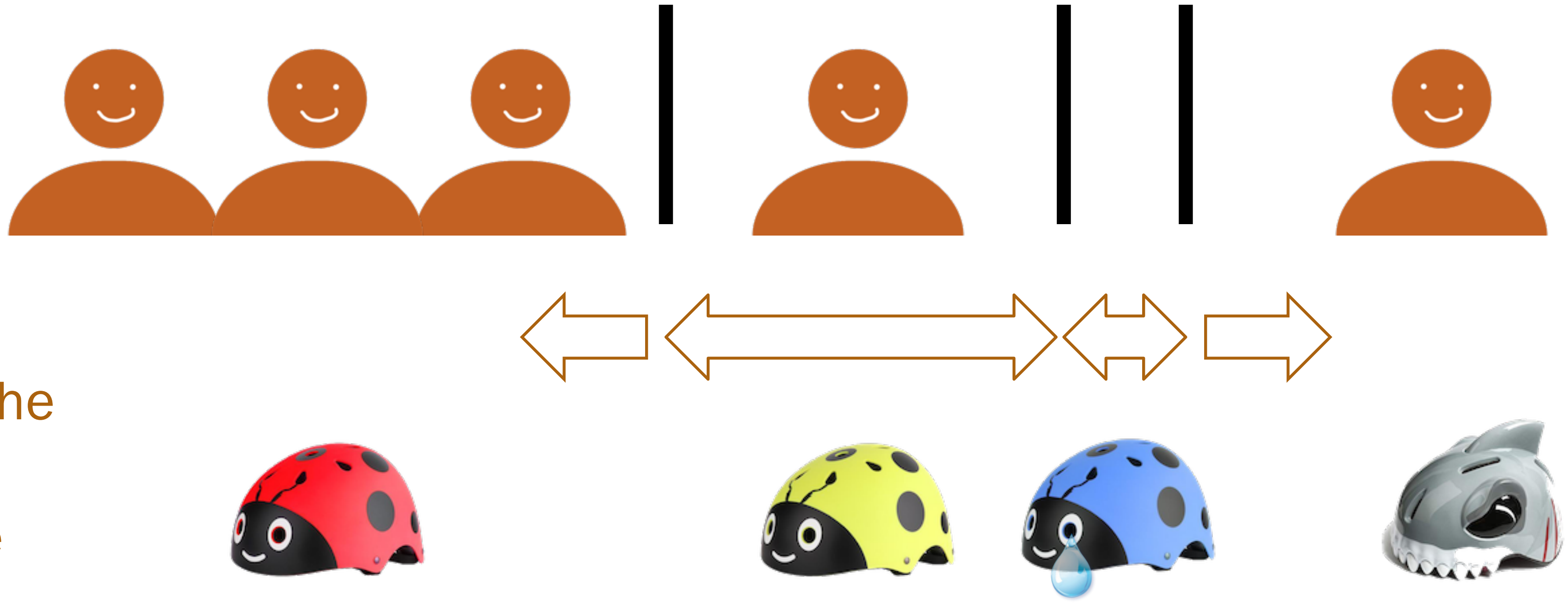


Bicycle helmet sales

1 possible assignment outcome:

$n = 5$ indistinct children
$r = 4$ distinct bicycle helmets
3 dividers

Goal Order n indistinct objects and $r - 1$ indistinct dividers.



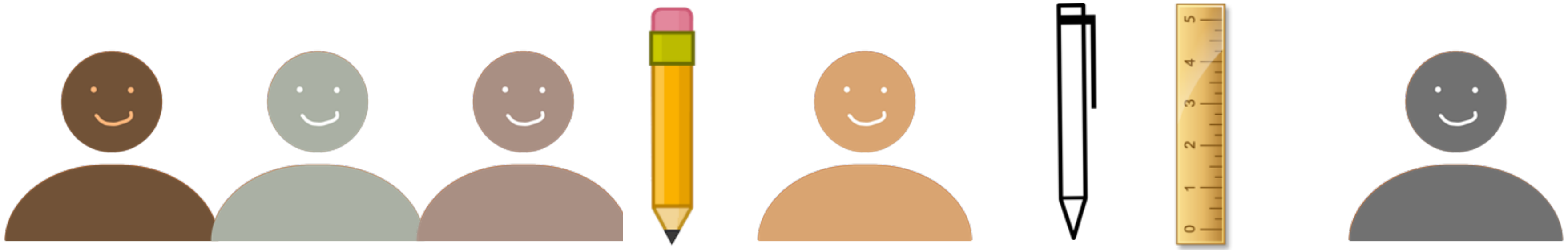
Consider the following generative process…

The divider method: A generative proof

$n = 5$ indistinct children
$r = 4$ distinct bicycle helmets

Goal Order n indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct



The divider method: A generative proof

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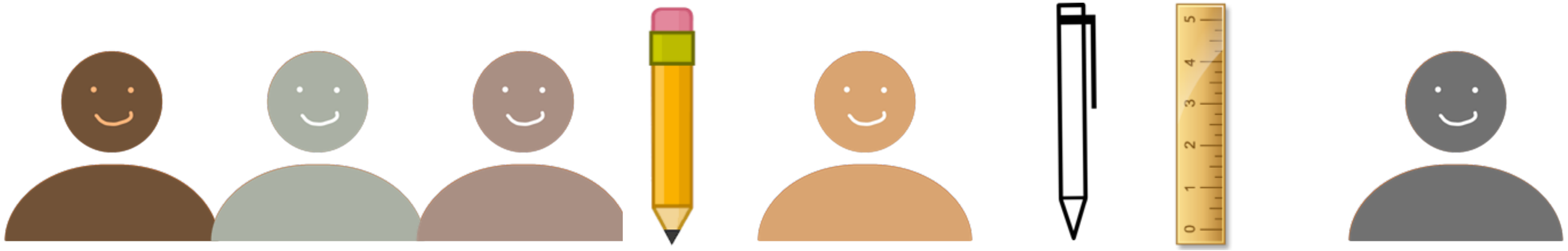

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

The divider method: A generative proof

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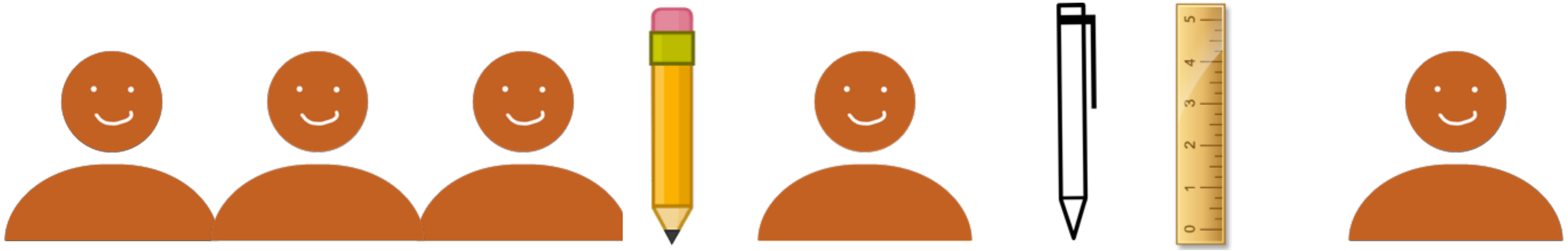
$r = 4$ distinct bicycle helmets

Goal Order n indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct



1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

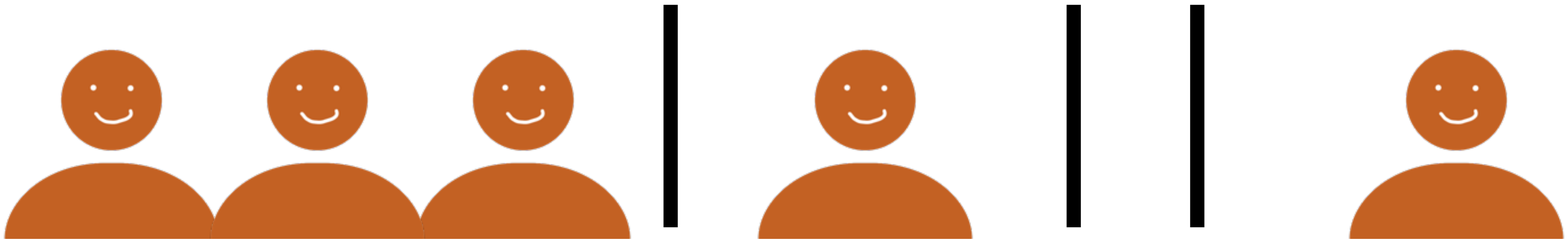
The divider method: A generative proof

$n = 5$ indistinct children
$r = 4$ distinct bicycle helmets

Goal Order n indistinct objects and $r-1$ indistinct dividers.

$$\frac{(n+r-1)!}{n!(r-1)!}$$

0. Make objects and dividers distinct



1. Order n distinct objects and $r-1$ distinct dividers

$$(n+r-1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

3. Make $r-1$ dividers indistinct

$$\frac{1}{(r-1)!}$$

The divider method

The number of ways to distribute n indistinct objects into r buckets is equivalent to the number of ways to permute $n+r-1$ objects such that

n are indistinct objects, and

$r-1$ are indistinct dividers:

$$\text{Total} = (n+r-1)! \times \frac{1}{n!} \times \frac{1}{(r-1)!}$$

$$= \binom{n+r-1}{r-1} \text{ outcomes}$$

Integer solutions to equations

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

How many integer solutions are there to the following equation:

$$x_1 + x_2 + \cdots + x_r = n,$$

where for all i, x_i is an integer such that $0 \leq x_i \leq n$? $i = 1, \ldots, r$

objects: n units of 1

buckets: r buckets, each of them has x_i units

$$\binom{n+r-1}{r-1}$$

non-negative

~~Positive~~ integer equations can be solved with the divider method.

Lecture with zoom

Breakout Rooms for working through lecture exercises

- We may incorporate some of these during lecture
- You are <u>always welcome</u> to exit breakout rooms if you are more comfortable staying in the main room

Breakout Rooms

Introduce yourself!

Then check out the three questions on the next slide (Slide 67).

Post any clarifications here:

https://us.edstem.org/courses/2678/discussion/124109

Breakout Room time: 5 minutes

We'll then all come back as a big group to go over our approach.

Venture capitalists

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have $10 million to invest in 4 companies (in $1 million increments).

1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1?
3. What if you don't have to invest all your money?

Venture capitalists. #1

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

$$x_i \geq 0$$

[0, 0, 5, 5]

Solve

$n = 10$ units

$r = 4$ buckets

$$\binom{10+4-1}{4-1} = \binom{13}{3} = 286$$

Venture capitalists. #2

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

$x_1 \geq 3$

$x_2, x_3, x_4 \geq 0$

fix x_1 bound

Solve

$x_1 + x_2 + x_3 + x_4 = 7$

$x_i \geq 0$

$n = 7$

$r = 4$

$\binom{7+4-1}{4-1} = \binom{10}{3} = 120$

Venture capitalists. #3

Divider method (*n* indistinct objects, *r* buckets) $\binom{n+r-1}{r-1}$

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?
3. What if you don't have to invest all your money?

Set up

⚠️

$$x_1 + x_2 + x_3 + x_4 \le 10$$

x_i: amount invested in company i

$$x_i \ge 0$$

add another bucket

Solve

you

$$x_1 + x_2 + x_3 + x_4 + x_5 = 10$$

$$x_i \ge 0$$

$n = 10$

$r = 5$

$$\binom{10+5-1}{5-1} = \binom{14}{4} = 1001$$

Summary of Combinatorics

Counting tasks on n objects

ordered

unordered | fixed # in each group

unlimited # in each bucket

Sort objects (permutations)

- Distinct (distinguishable)
- Some distinct

Choose k objects (combinations)

- Distinct
  - 1 group
  - r groups

Put objects in r buckets

- Distinct
- Indistinct

- Determine if objects are distinct
- Use Product Rule if several steps
- Use Inclusion-Exclusion if different cases

See you next time...

Venture capitalists. #2

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have $10 million to invest in 4 companies (in $1 million increments).

1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1?

$x_1 + x_2 + \dots + x_r = n$, $x_i \geq 0$, $\binom{n+r-1}{r-1}$

[3, 1, 2, 4] ← [0, 1, 2, 4]

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

$[x_1 \geq 3]$

$x_2, x_3, x_4 \geq 0$

fix x_1 bound

Solve

$x_1 + x_2 + x_3 + x_4 = 7$

$x_i \geq 0$

$n = 7$, $r = 4$

$\binom{7+4-1}{4-1} = \binom{10}{3} = 120$

"indistinct buckets"

[1, 3, 4, 2]

[1, 2, 3, 4]

[0, 0, 0, 10]

[0, 1, 2, 9]

$\binom{10}{3}$ vs $\binom{10}{7}$