

02: Combinatorics

Lisa Yan and Jerry Cain
September 16, 2020

Quick slide reference

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Today's discussion thread: <https://us.edstem.org/courses/2678/discussion/124109>

Permutations II

Summary of Combinatorics

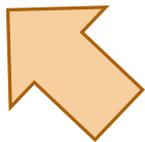
Counting tasks on n objects

Sort objects
(permutations)

Choose k objects
(combinations)

Put objects in r
buckets

Distinct
(distinguishable)



Sort n distinct objects



Ayesha



Tim



Irina



Joey



Waddie

of permutations =

Summary of Combinatorics

Counting tasks on n objects

Sort objects
(permutations)

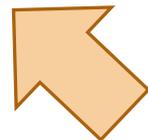
Choose k objects
(combinations)

Put objects in r
buckets

Distinct
(distinguishable)

Some
distinct

$n!$

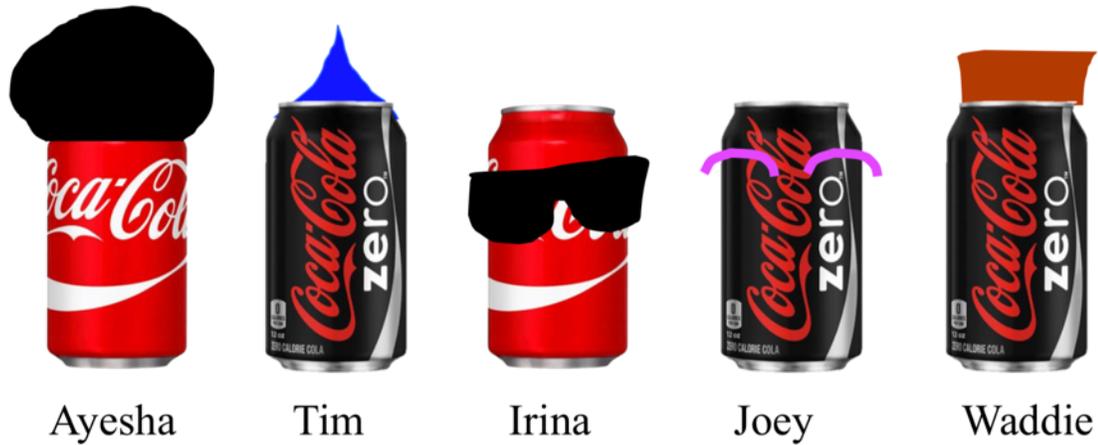


Sort semi-distinct objects

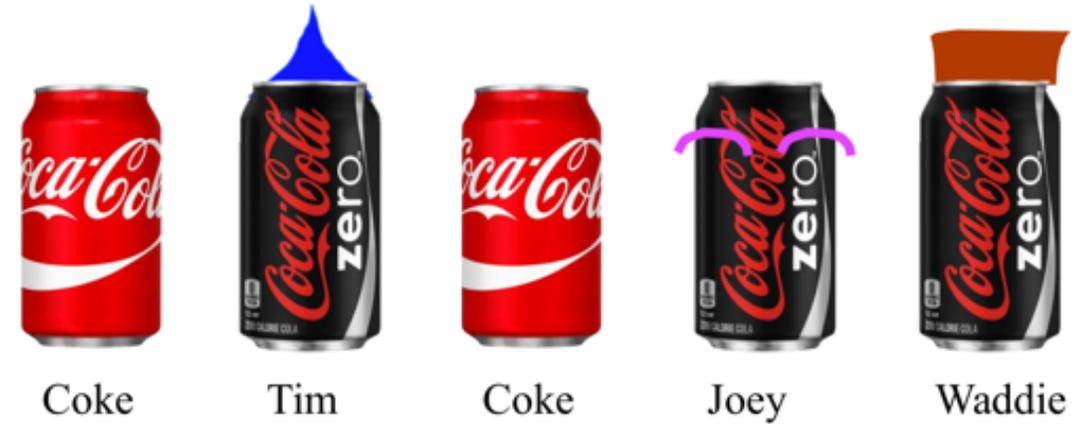
Order n
distinct objects

$n!$

All distinct



Some indistinct



Sort semi-distinct objects

How do we find **the number of permutations considering some objects are indistinct?**

By the product rule, permutations of distinct objects is a two-step process:

$$\begin{array}{ccccc} \text{permutations} & & \text{permutations} & & \text{Permutations} \\ \text{of distinct objects} & = & \text{considering some} & \times & \text{of just the} \\ & & \text{objects are indistinct} & & \text{indistinct objects} \end{array}$$

Sort semi-distinct objects

How do we find **the number of permutations considering some objects are indistinct?**

By the product rule, permutations of distinct objects is a two-step process:

$$\frac{\text{permutations of distinct objects}}{\text{Permutations of just the indistinct objects}} = \text{permutations considering some objects are indistinct}$$

General approach to counting permutations

When there are n objects such that

n_1 are the same (indistinguishable or **indistinct**), and

n_2 are the same, and

...

n_r are the same,

The number of unique orderings (**permutations**) is

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

For each group of indistinct objects,
Divide by the overcounted permutations.

Sort semi-distinct objects

Order n semi-
distinct objects $\frac{n!}{n_1! n_2! \cdots n_r!}$

How many permutations?



Coke



Coke0



Coke



Coke0



Coke0

Summary of Combinatorics

Counting tasks on n objects

Sort objects
(permutations)

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(combinations)

Put objects in r
buckets

Distinct
(distinguishable)

Some
distinct

$$n!$$

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

Strings

Order n semi-
distinct objects $\frac{n!}{n_1! n_2! \cdots n_r!}$

How many orderings of letters are possible for the following strings?

1. BOBA

2. MISSISSIPPI



How many orderings of letters are possible for the following strings?

1. BOBA

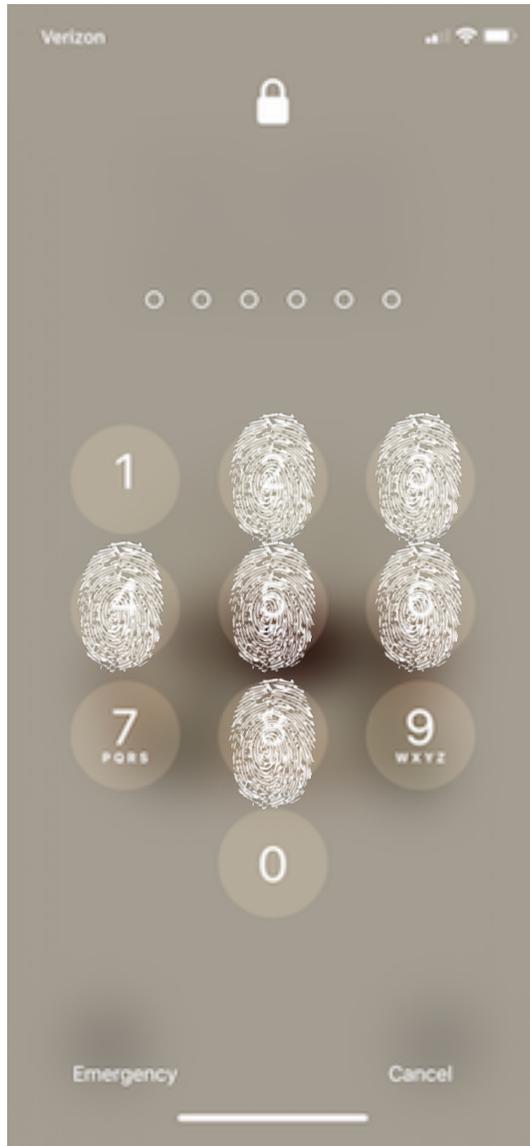
$$= \frac{4!}{2!} = 12$$

2. MISSISSIPPI

$$= \frac{11!}{1!4!4!2!} = 34,650$$

Unique 6-digit passcodes with **six** smudges

Order n semi-
distinct objects $\frac{n!}{n_1! n_2! \cdots n_r!}$

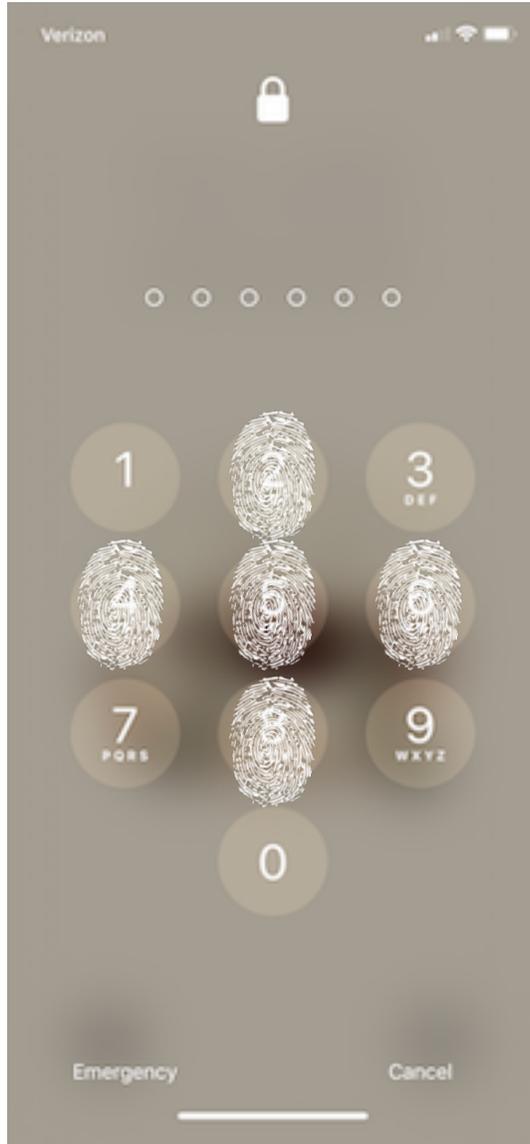


How many unique 6-digit passcodes are possible if a phone password uses each of **six** distinct numbers?

Total = $6!$
= 720 passcodes

Unique 6-digit passcodes with **five** smudges

Order n semi-
distinct objects $\frac{n!}{n_1! n_2! \cdots n_r!}$



How many unique 6-digit passcodes are possible if a phone password uses each of **five** distinct numbers?

Steps:

1. Choose digit to repeat 5 outcomes
2. Create passcode (sort 6 digits:
4 distinct, 2 indistinct)

$$\begin{aligned} \text{Total} &= 5 \times \frac{6!}{2!} \\ &= 1,800 \text{ passcodes} \end{aligned}$$

Combinations I

Summary of Combinatorics

Counting tasks on n objects

Sort objects
(permutations)

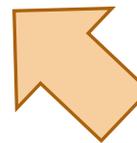
Choose k objects
(combinations)

Put objects in r
buckets

Distinct
(distinguishable)

Some
distinct

Distinct



$$n!$$

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

Combinations with cake

There are $n = 20$ people.

How many ways can we **choose** $k = 5$ people to get cake?

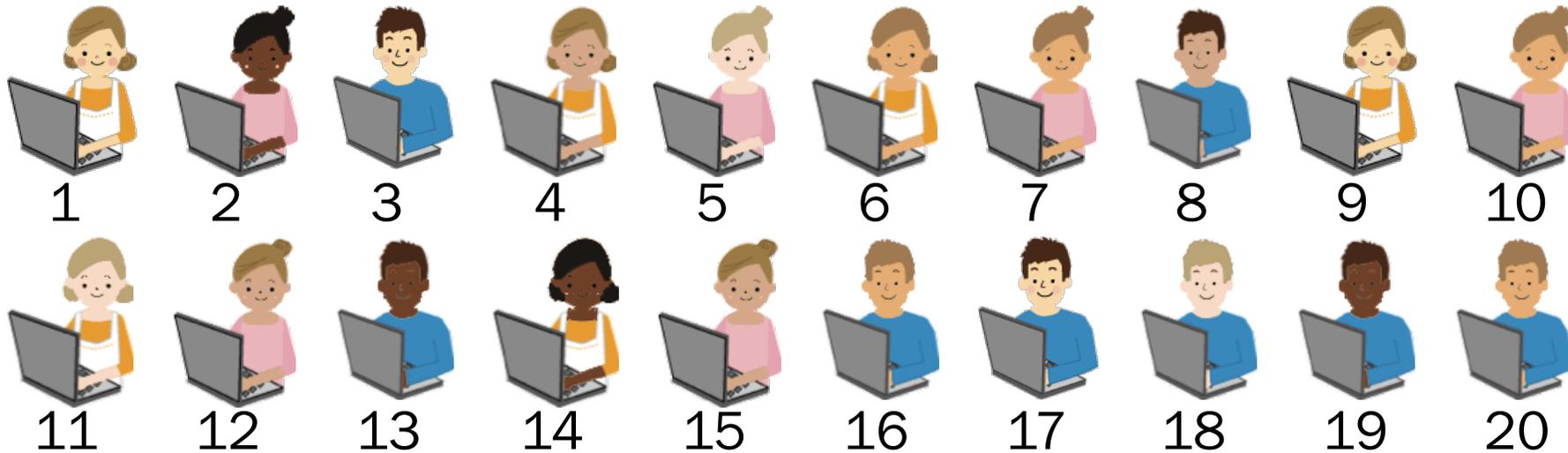


Consider the following generative process...

Combinations with cake

There are $n = 20$ people.

How many ways can we **choose** $k = 5$ people to get cake?



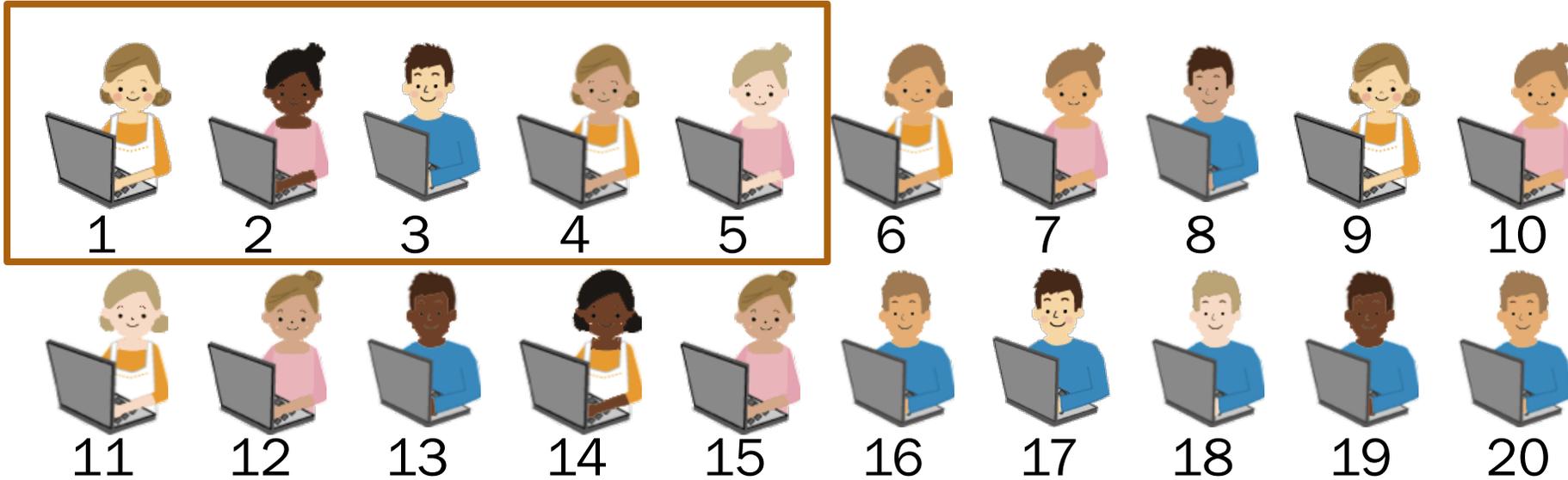
1. n people
get in line

$n!$ ways

Combinations with cake

There are $n = 20$ people.

How many ways can we **choose** $k = 5$ people to get cake?



1. n people
get in line

$n!$ ways

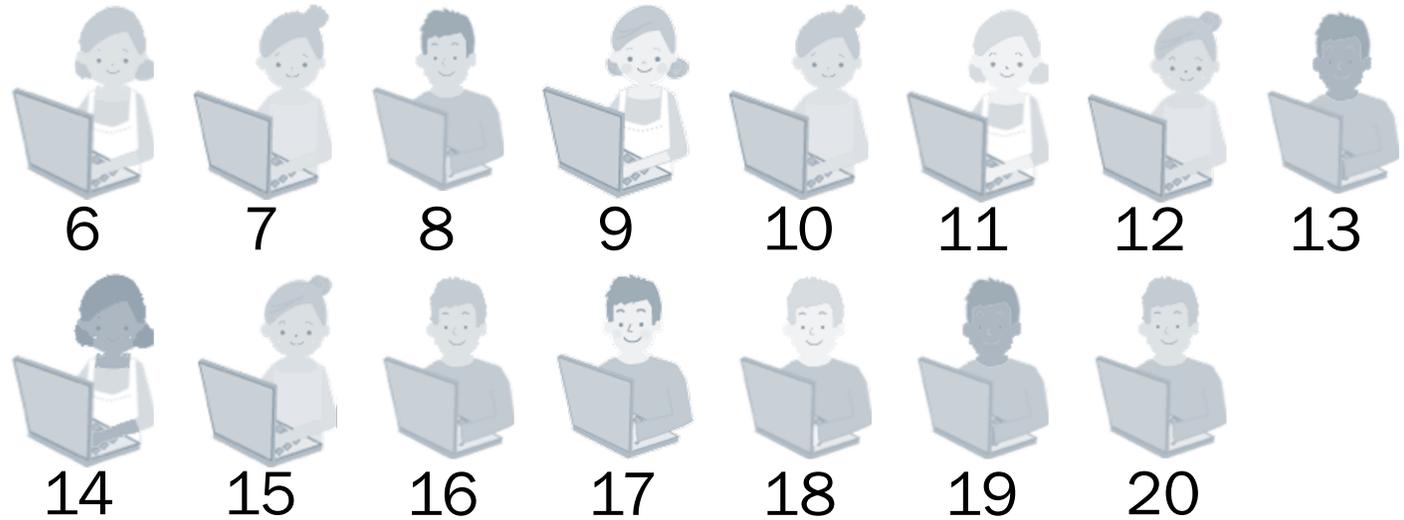
2. Put first k
in cake room

1 way

Combinations with cake

There are $n = 20$ people.

How many ways can we **choose** $k = 5$ people to get cake?



1. n people
get in line

$n!$ ways

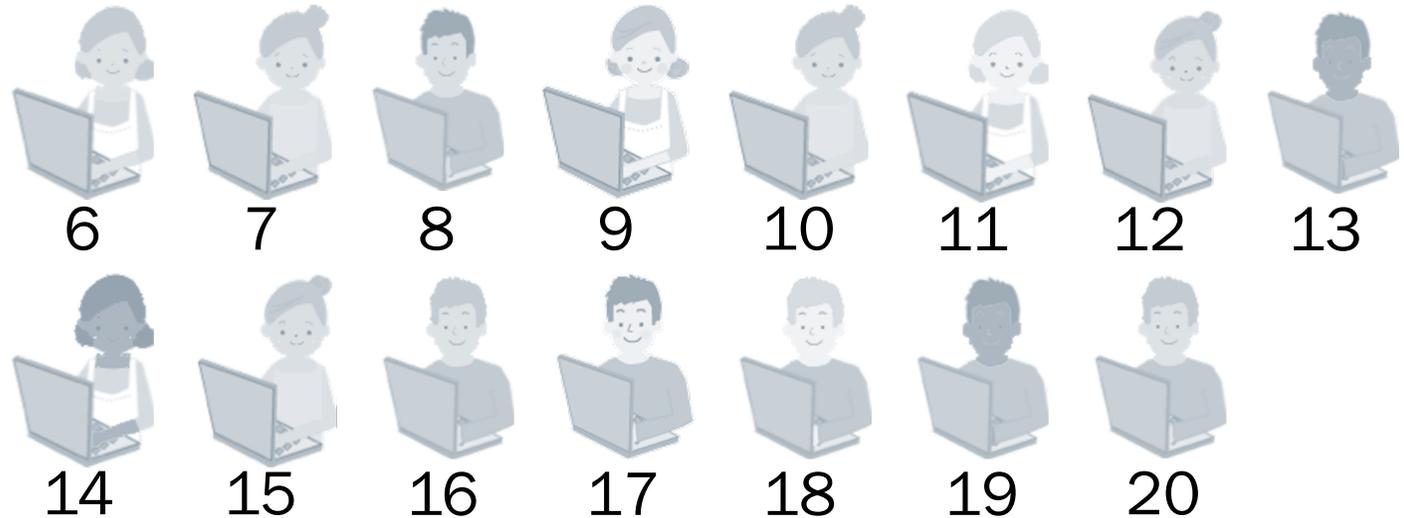
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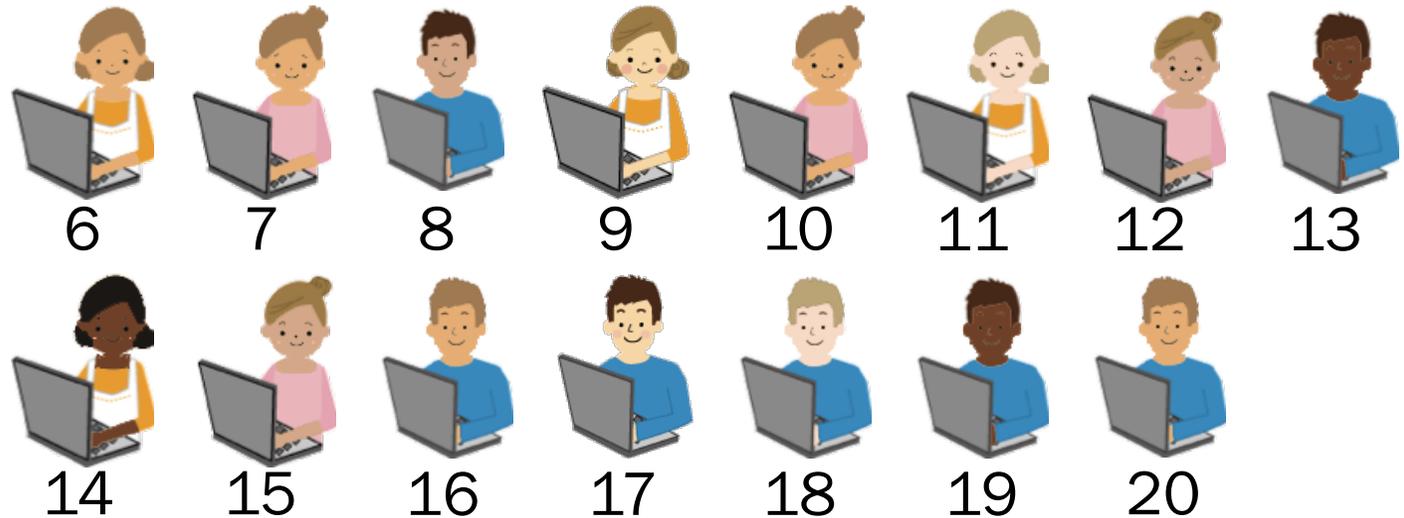
3. **Allow cake
group to mingle**

$k!$ different
permutations lead to
the same mingle

Combinations with cake

There are $n = 20$ people.

How many ways can we **choose** $k = 5$ people to get cake?



1. n people get in line

$n!$ ways

2. Put first k in cake room

1 way

3. Allow cake group to mingle

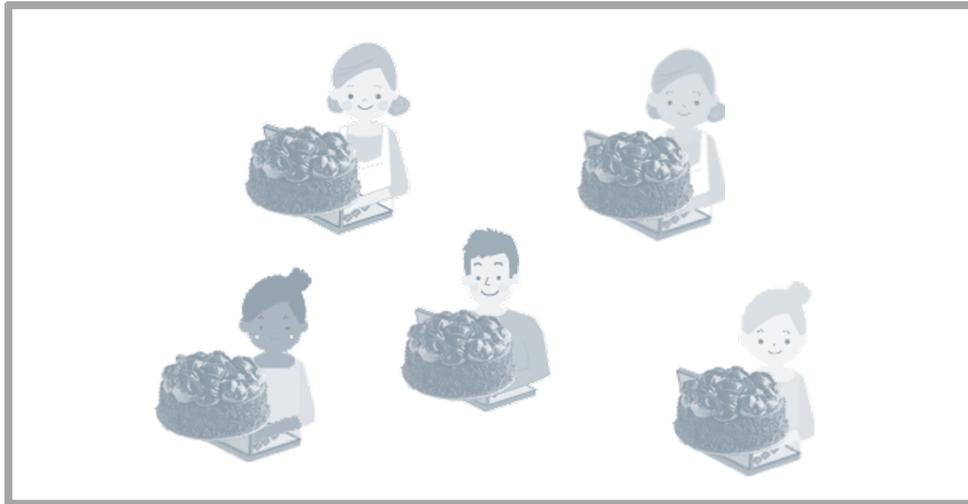
$k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

Combinations with cake

There are $n = 20$ people.

How many ways can we **choose** $k = 5$ people to get cake?



1. n people
get in line

$n!$ ways

2. Put first k
in cake room

1 way

3. Allow cake
group to mingle

$k!$ different
permutations lead to
the same mingle

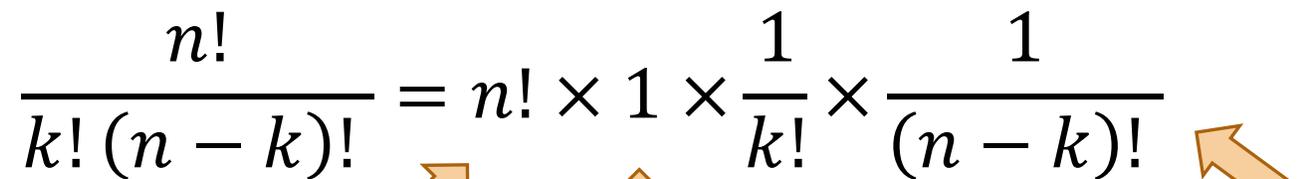
4. Allow non-cake
group to mingle

$(n - k)!$ different
permutations lead to the
same mingle

Combinations

A **combination** is an unordered selection of k objects from a set of n **distinct** objects.

The number of ways of making this selection is

$$\frac{n!}{k!(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!}$$


1. Order n distinct objects

2. Take first k as chosen

3. Overcounted: any ordering of chosen group is same choice

4. Overcounted: any ordering of unchosen group is same choice

Combinations

A **combination** is an unordered selection of k objects from a set of n **distinct** objects.

The number of ways of making this selection is

$$\frac{n!}{k! (n - k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n - k)!} = \binom{n}{k} \text{ Binomial coefficient}$$

Note: $\binom{n}{n-k} = \binom{n}{k}$

Probability textbooks

Choose k of
 n distinct objects $\binom{n}{k}$

How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!3!} = 20 \text{ ways}$$

Combinations II

Summary of Combinatorics

Counting tasks on n objects

Sort objects
(permutations)

Choose k objects
(combinations)

Put objects in r
buckets

Distinct
(distinguishable)

Some
distinct

Distinct

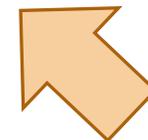
1 group

r groups

$$n!$$

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

$$\binom{n}{k}$$



General approach to combinations

The number of ways to choose r groups of n distinct objects such that

For all $i = 1, \dots, r$, group i has size n_i , and

$\sum_{i=1}^r n_i = n$ (all objects are assigned), is

$$\frac{n!}{n_1! n_2! \cdots n_r!} = \binom{n}{n_1, n_2, \dots, n_r}$$

Multinomial coefficient

Datacenters

Choose k of n distinct objects into r groups of size n_1, \dots, n_r $\binom{n}{n_1, n_2, \dots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	6
B	4
C	3

- A. $\binom{13}{6,4,3} = 60,060$
- B. $\binom{13}{6} \binom{7}{4} \binom{3}{3} = 60,060$
- C. $6 \cdot 1001 \cdot 10 = 60,060$
- D. A and B
- E. All of the above



Datacenters

Choose k of n distinct objects into r groups of size n_1, \dots, n_r $\binom{n}{n_1, n_2, \dots, n_r}$

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A. $\binom{13}{6,4,3} = 60,060$

Strategy: Combinations into 3 groups

Group 1 (datacenter A): $n_1 = 6$

Group 2 (datacenter B): $n_2 = 4$

Group 3 (datacenter C): $n_3 = 3$

Datcenters

Choose k of n distinct objects
into r groups of size n_1, \dots, n_r $\binom{n}{n_1, n_2, \dots, n_r}$

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Group 3 (datacenter C): $n_3 = 3$

B. $\binom{13}{6} \binom{7}{4} \binom{3}{3} = 60,060$

Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

Datacenters

Choose k of n distinct objects
into r groups of size n_1, \dots, n_r $\binom{n}{n_1, n_2, \dots, n_r}$

13 different computers are to be allocated to
3 datacenters as shown in the table:

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Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

Your approach will determine if you use
binomial/multinomial coefficients or factorials.

02: Combinatorics (live)

Lisa Yan

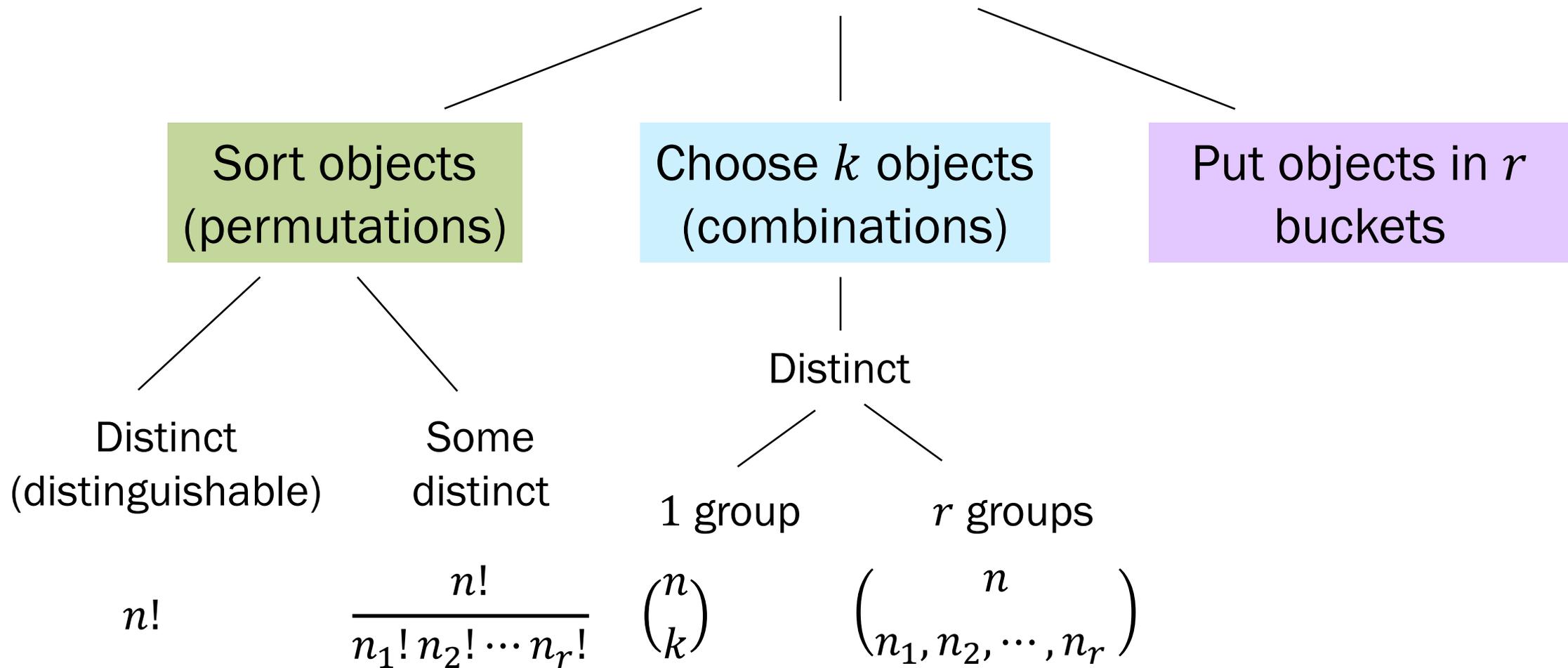
September 16, 2020

Reminders: Lecture with

- Turn on your camera if you are able, mute your mic in the big room
- Virtual backgrounds are encouraged (classroom-appropriate)

Today's discussion thread: <https://us.edstem.org/courses/2678/discussion/124109>

Counting tasks on n objects



Summary of Combinatorics

Counting tasks on n objects

Sort objects
(permutations)

Choose k objects
(combinations)

Put objects in r
buckets

Distinct
(distinguishable)

Some
distinct

$$n!$$

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

Distinct

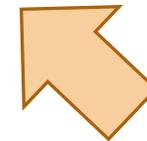
Indistinct?

1 group

r groups

$$\binom{n}{k}$$

$$\binom{n}{n_1, n_2, \dots, n_r}$$



Think

Slide 42 is a question to think over by yourself (~1min).

Post any clarifications here or in chat!

<https://us.edstem.org/courses/2678/discussion/124109>



A trick question

How many **distinct** (distinguishable) ways are there to group 6 **indistinct** (indistinguishable) objects into 3 groups, where groups A, B, and C have sizes 1, 2, and 3, respectively?

A. $\binom{6}{1,2,3}$

B. $\frac{6!}{1!2!3!}$

C. 0

D. 1

E. Both A and B

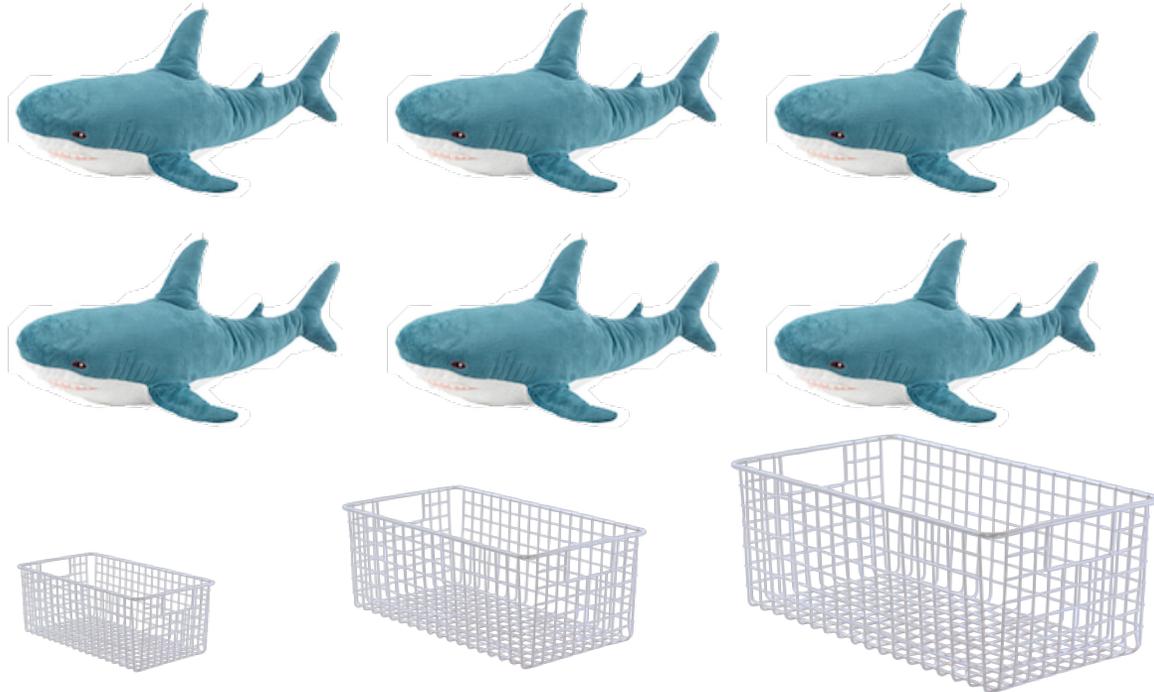
F. Something else



(by yourself)

A trick question

How many **distinct** (distinguishable) ways are there to group 6 **indistinct** (indistinguishable) objects into 3 groups, where groups A, B, and C have sizes 1, 2, and 3, respectively?



A (fits 1)

B (fits 2)

C (fits 3)

- A. $\binom{6}{1,2,3}$
- B. $\frac{6!}{1!2!3!}$
- C. 0
- D. 1
- E. Both A and B
- F. Something else

Probability textbooks

Review

Choose k of
 n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

Think

Slide 46 is a question to think over by yourself (~2min).

Post any clarifications here or in chat!

<https://us.edstem.org/courses/2678/discussion/124109>



Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books? $\binom{6}{3} = \frac{6!}{3!3!} = 20$ ways

2. What if we do not want to read both the 9th and 10th edition of Ross?

A. $\binom{6}{3} - \binom{6}{2} = 5$ ways

D. $\binom{6}{3} - \binom{4}{1} = 16$

B. $\frac{6!}{3!3!2!} = 10$

E. Both C and D

C. $2 \cdot \binom{4}{2} + \binom{4}{3} = 16$

F. Something else

Ask: <https://us.edstem.org/courses/2678/discussion/124109>



Probability textbooks

Choose k of
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1. How many ways are there to choose 3 books from a set of 6 distinct books? $\binom{6}{3} = \frac{6!}{3!3!} = 20$ ways

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Strategy 1: Sum Rule

Probability textbooks

Choose k of
 n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!3!} = 20 \text{ ways}$$

2. What if we do not want to read both the 9th and 10th edition of Ross?

Strategy 2: “Forbidden method” (unofficial name)

Forbidden method: It is sometimes easier to exclude invalid cases than to include cases.

Interlude for jokes/announcements

Announcements

Problem Set #1

Out: today
Due: Friday 9/25, 1:00pm
Covers: through Friday 9/18

Section sign-ups/Acquaintance form

Form released: later today
Form due: Saturday 5:00pm 9/19
Results: latest Sunday

Python tutorial (2 timeslots)

When: Friday 12:00-1:00am PT
Friday 2:00-3:00pm PT
Recorded? yes
Notes: to be posted [online](#)

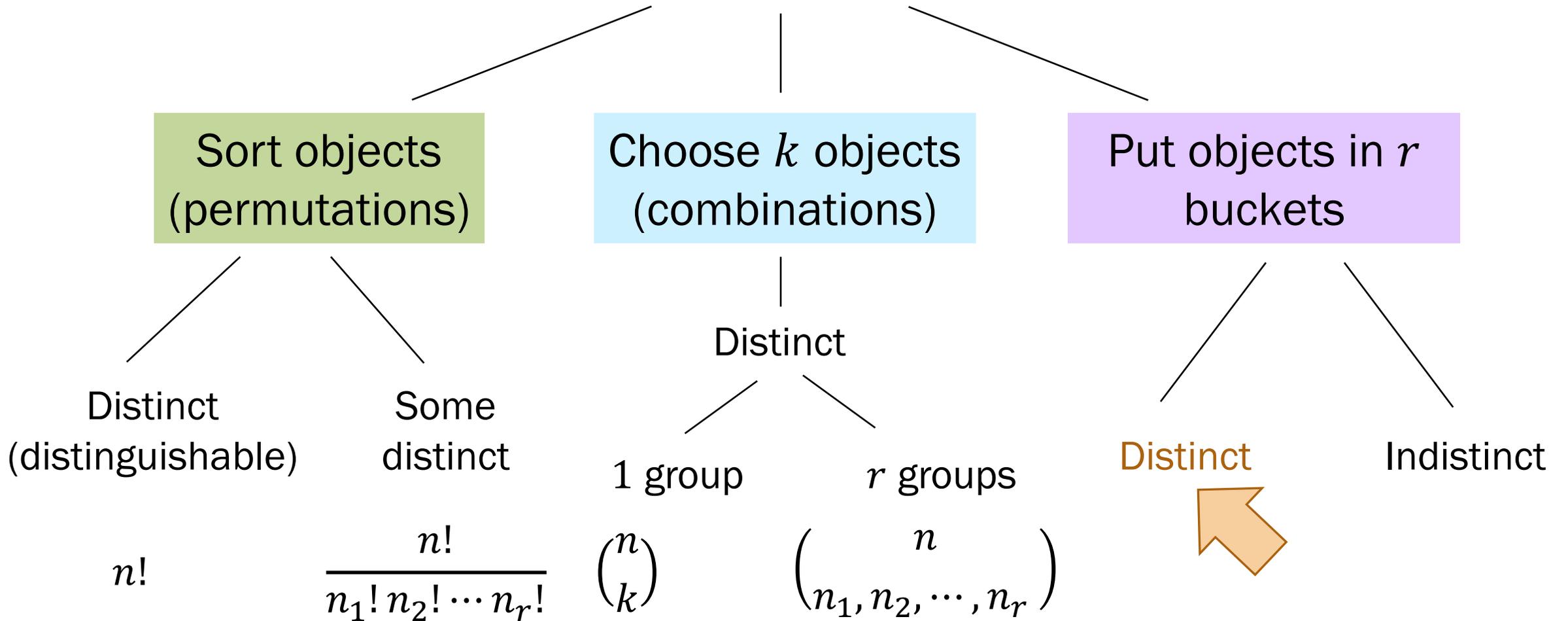
Getting help

Ed discussion: find study buddies!
Office hours: start today
<https://web.stanford.edu/class/cs109/stanford/staff.html>

Buckets and The Divider Method

Summary of Combinatorics

Counting tasks on n objects

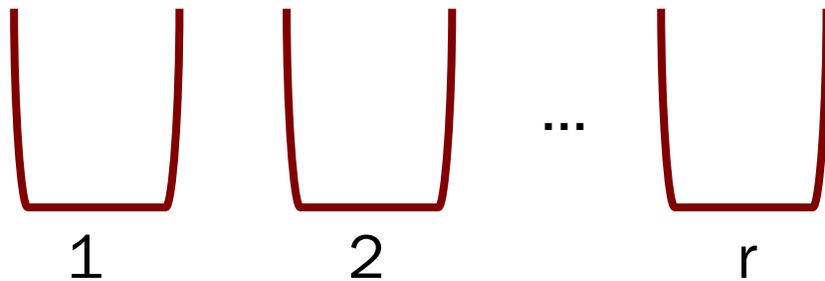
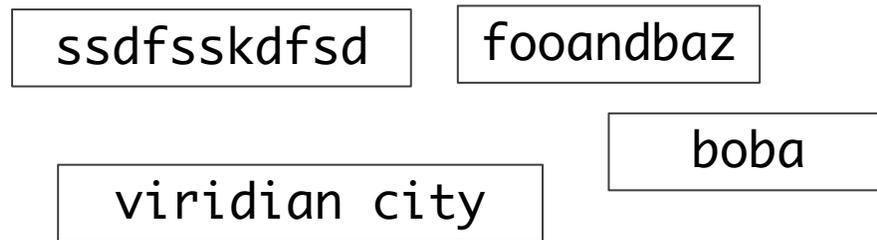


~~Balls and urns~~ Hash tables and **distinct** strings

How many ways are there to hash n **distinct** strings to r buckets?

Steps:

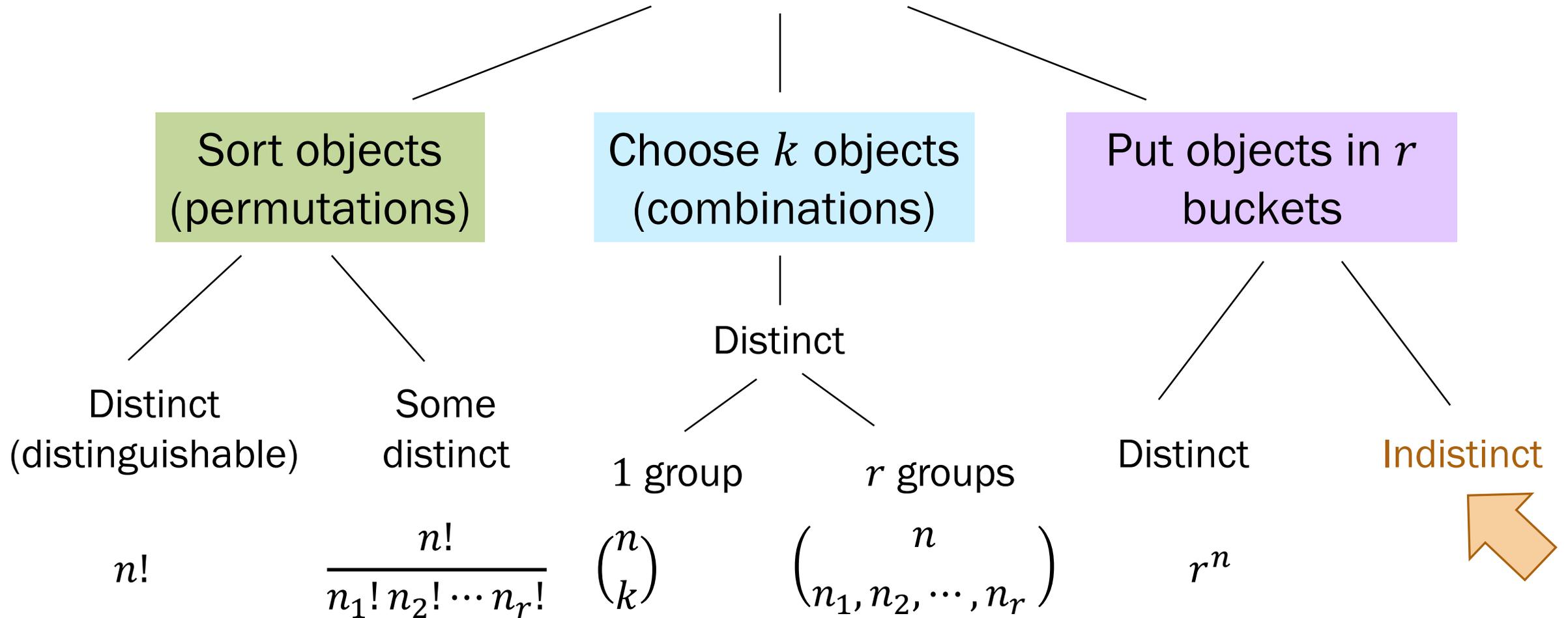
1. Bucket 1st string
2. Bucket 2nd string
- ...
- n . Bucket n^{th} string



r^n outcomes

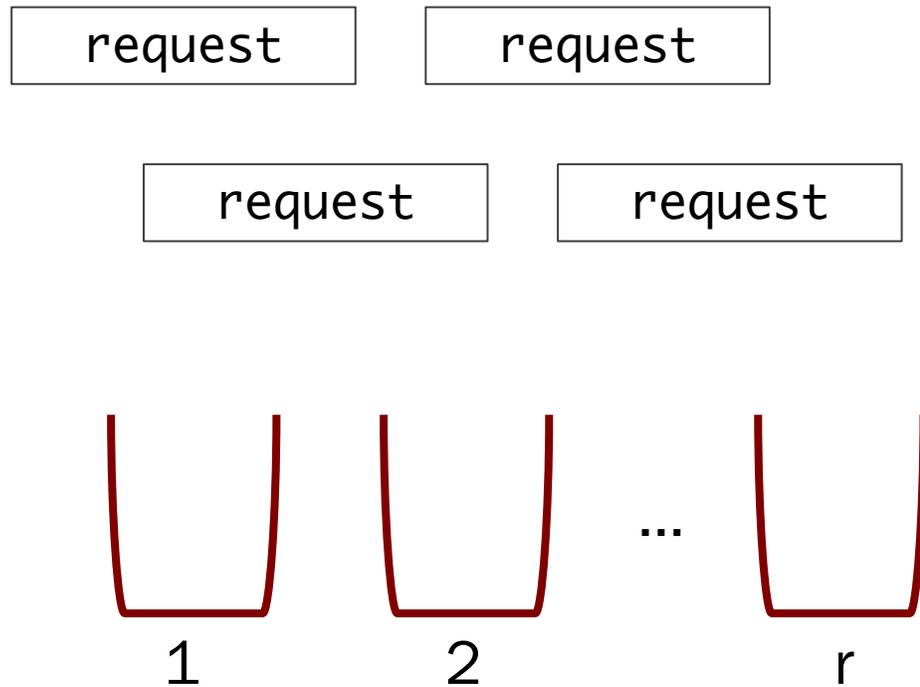
Summary of Combinatorics

Counting tasks on n objects



Servers and **indistinct** requests

How many ways are there to distribute n **indistinct** web requests to r servers?



Goal

Server 1 has x_1 requests,

Server 2 has x_2 requests,

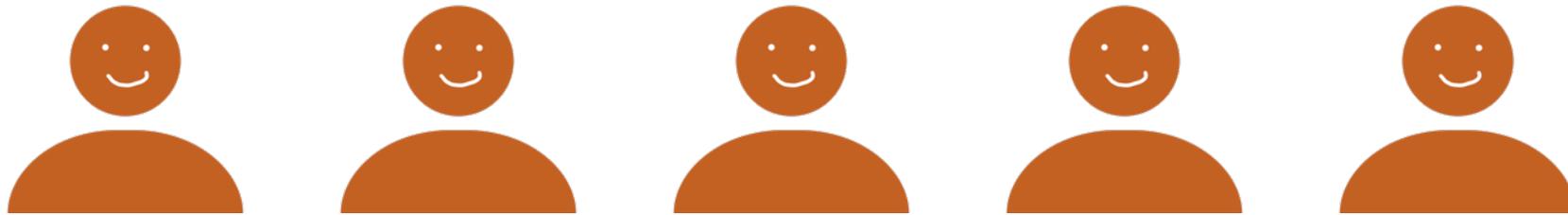
...

Server r has x_r requests (the rest)

Simple example: $n = 3$ requests and $r = 2$ servers

Bicycle helmet sales

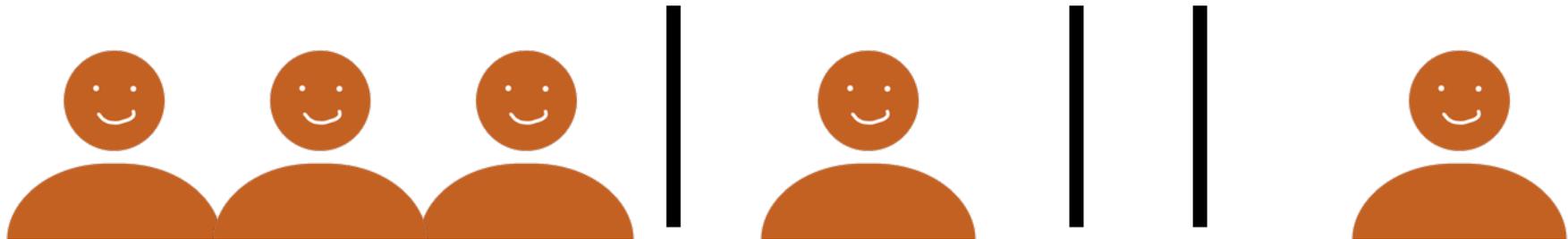
How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?



Bicycle helmet sales

1 possible assignment outcome:

Goal Order n indistinct objects and $r - 1$ indistinct dividers.



Consider the following generative process...

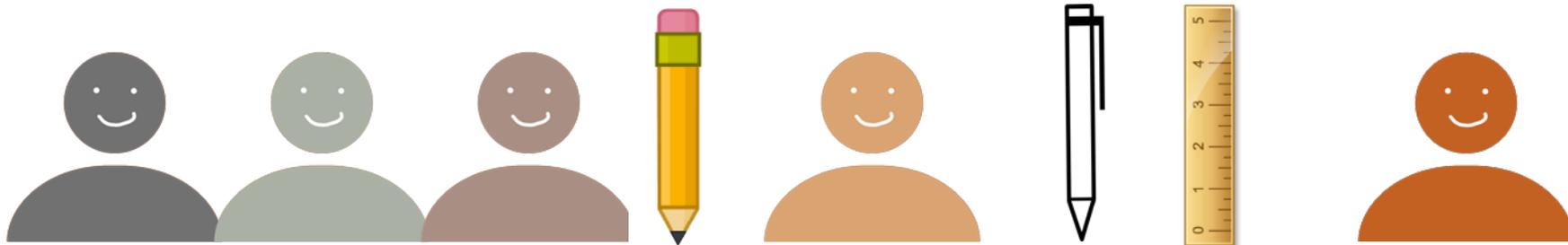


The divider method: A generative proof

How many ways can we **assign** $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

Goal Order n **indistinct** objects and $r - 1$ **indistinct** dividers.

0. Make objects and dividers distinct

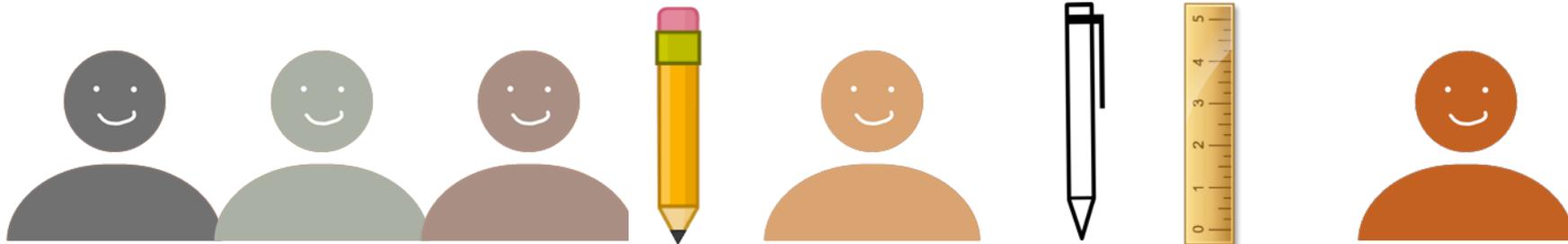


The divider method: A generative proof

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1. Order n distinct objects and $r - 1$ distinct dividers

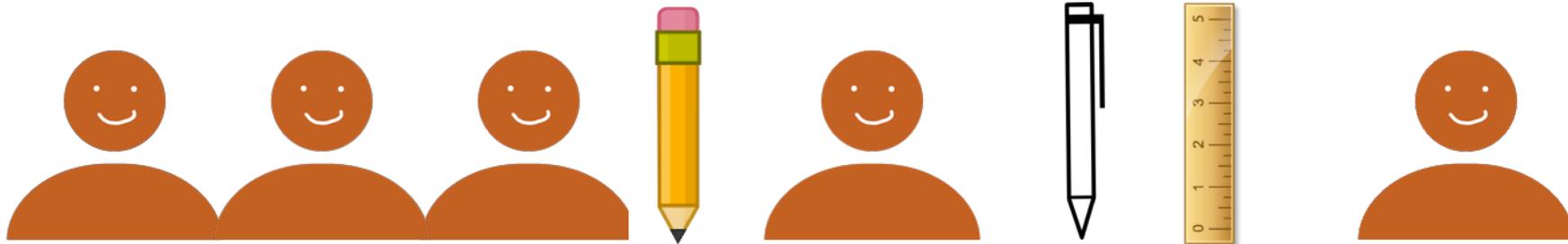
$$(n + r - 1)!$$

The divider method: A generative proof

How many ways can we **assign** $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

Goal Order n **indistinct** objects and $r - 1$ **indistinct** dividers.

0. Make objects and dividers distinct



1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

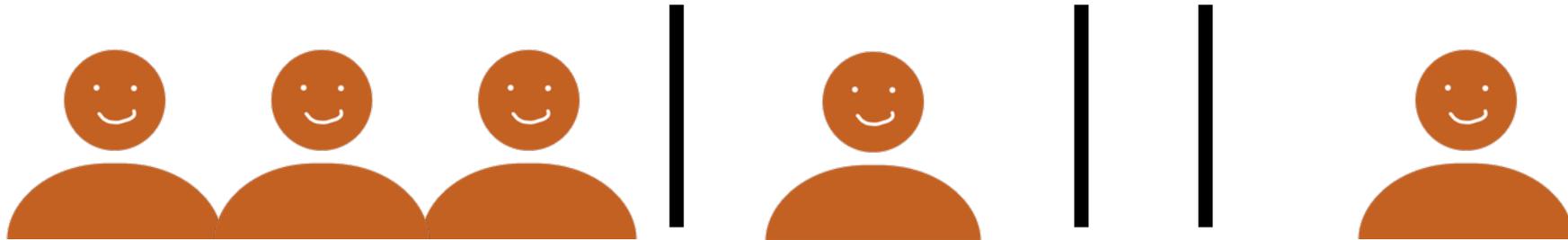
$$\frac{1}{n!}$$

The divider method: A generative proof

How many ways can we **assign** $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

Goal Order n **indistinct** objects and $r - 1$ **indistinct** dividers.

0. Make objects and dividers distinct



1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

3. Make $r - 1$ dividers indistinct

$$\frac{1}{(r - 1)!}$$

The divider method

The number of ways to distribute n indistinct objects into r buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that n are indistinct objects, and $r - 1$ are indistinct dividers:

$$\text{Total} = (n + r - 1)! \times \frac{1}{n!} \times \frac{1}{(r-1)!}$$

$$= \binom{n + r - 1}{r - 1} \text{ outcomes}$$

Integer solutions to equations

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

How many integer solutions are there to the following equation:

$$x_1 + x_2 + \cdots + x_r = n,$$

where for all i , x_i is an integer such that $0 \leq x_i \leq n$?

Positive integer equations can be solved with the divider method.

Breakout Rooms for working through lecture exercises

- We may incorporate some of these during lecture
- You are always welcome to exit breakout rooms if you are more comfortable staying in the main room

Breakout Rooms

Introduce yourself!

Then check out the three questions on the next slide (Slide 67).

Post any clarifications here:

<https://us.edstem.org/courses/2678/discussion/124109>

Breakout Room time: 5 minutes

We'll then all come back as a big group to go over our approach.



Venture capitalists

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?
3. What if you don't have to invest all your money?



Venture capitalists. #1

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in \$1 million increments).

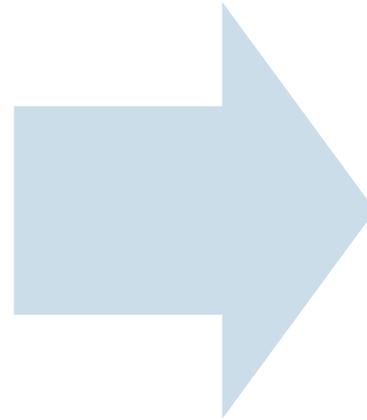
1. How many ways can you fully allocate your \$10 million?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i : amount invested in company i

$$x_i \geq 0$$



Solve

Venture capitalists. #2

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

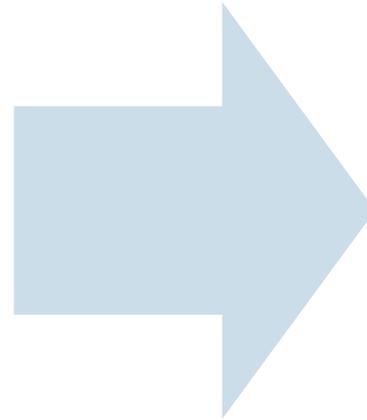
You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i : amount invested in company i



Solve

Venture capitalists. #3

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?
3. What if you don't have to invest all your money?

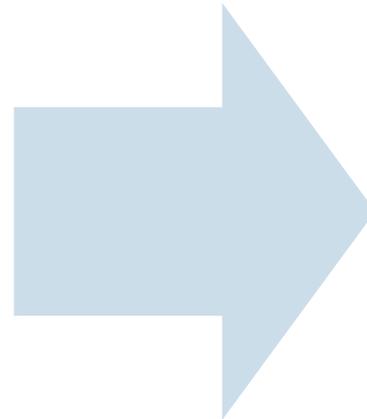
Set up



$$x_1 + x_2 + x_3 + x_4 \leq 10$$

x_i : amount invested in company i

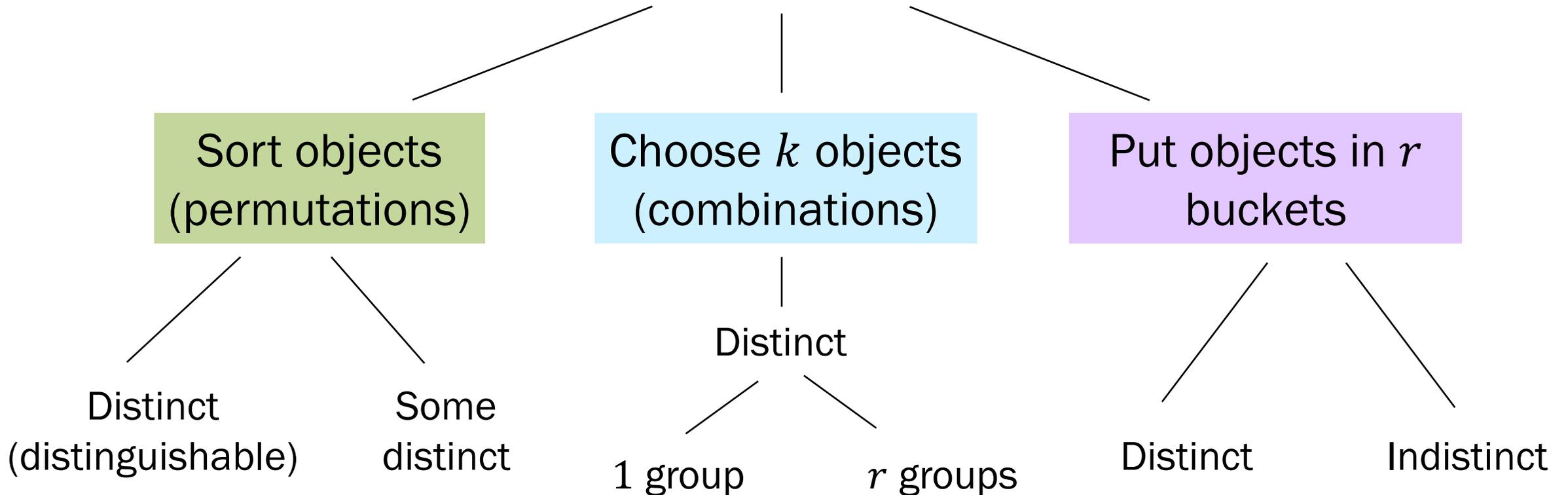
$$x_i \geq 0$$



Solve

Summary of Combinatorics

Counting tasks on n objects



- Determine if objects are distinct
- Use Product Rule if several steps
- Use Inclusion-Exclusion if different cases

See you next time...

