03: Intro to Probability

Lisa Yan and Jerry Cain

September 18, 2020

Quick slide reference

3	Defining Probability	03a_definitions
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20	Equally likely outcomes	03c_elo
30	Exercises	LIVE
37	Corollaries	LIVE

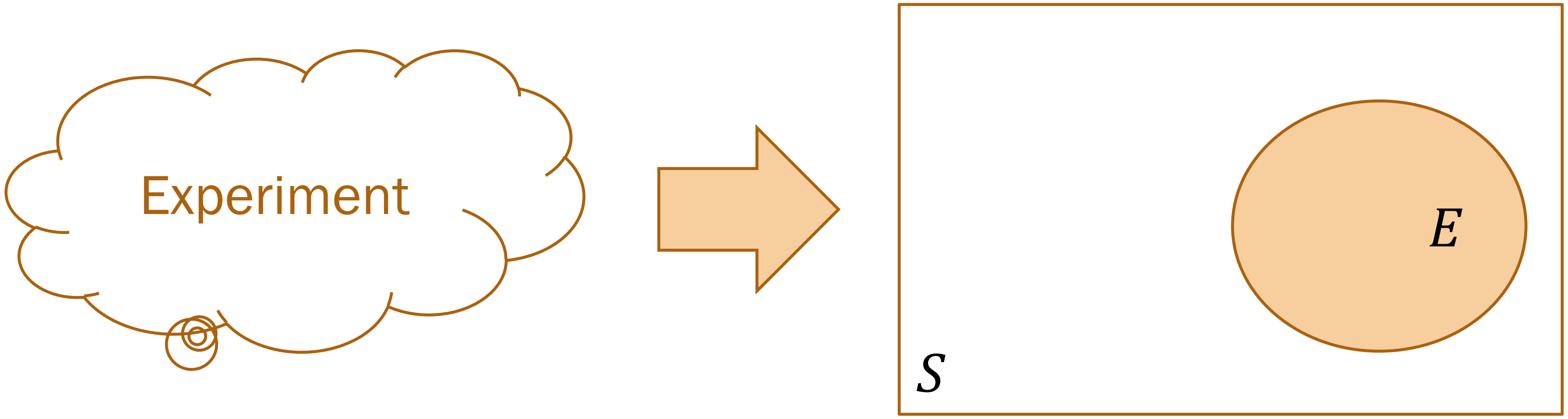
Today's discussion thread: https://us.edstem.org/courses/2678/discussion/124598

Defining Probability

Gradescope quiz, blank slide deck, etc.
http://cs109.stanford.edu/

Key definitions

An experiment in probability:



Sample Space, S: The set of all possible outcomes of an experiment

Event, E: Some subset of S ($E \subseteq S$).

Key definitions

Sample Space, S

- Coin flip
  $S = \{\text{Heads}, \text{Tails}\}$
- Flipping two coins
  $S = \{(\text{H,H}), (\text{H,T}), (\text{T,H}), (\text{T,T})\}$
- Roll of 6-sided die
  $S = \{1, 2, 3, 4, 5, 6\}$
- # emails in a day
  $S = \{x \mid x \in \mathbb{Z},\ x \geq 0\}$
- TikTok hours in a day
  $S = \{x \mid x \in \mathbb{R}, 0 \leq x \leq 24\}$

Event, E

- Flip lands heads
  $E = \{\text{Heads}\}$
- $\geq$ 1 head on 2 coin flips
  $E = \{(\text{H,H}), (\text{H,T}), (\text{T,H})\}$
- Roll is 3 or less:
  $E = \{1, 2, 3\}$
- Low email day ($\leq$ 20 emails)
  $E = \{x \mid x \in \mathbb{Z},\ 0 \leq x \leq 20\}$
- Wasted day ($\geq$ 5 TT hours):
  $E = \{x \mid x \in \mathbb{R}, 5 \leq x \leq 24\}$

What is a probability?

A number between 0 and 1
to which we ascribe meaning.*

*our belief that an event E occurs.

What is a probability?

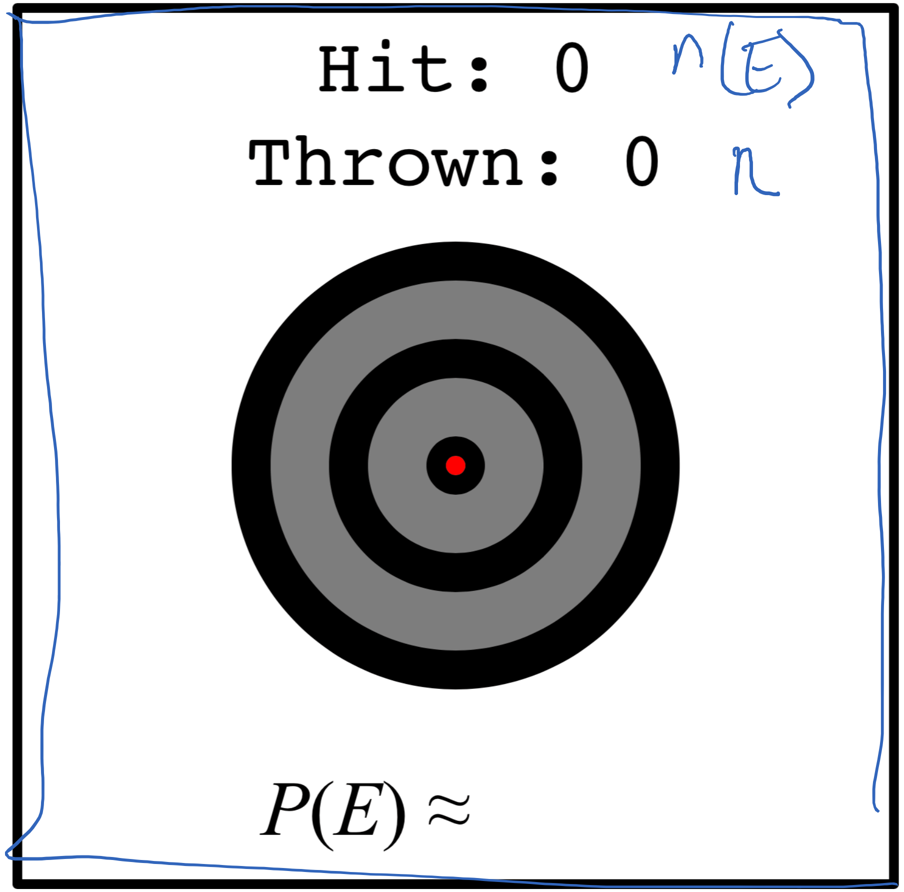
frequentist

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

n = # of total trials

$n(E)$ = # trials where E occurs

Let E = the set of outcomes where you hit the target.



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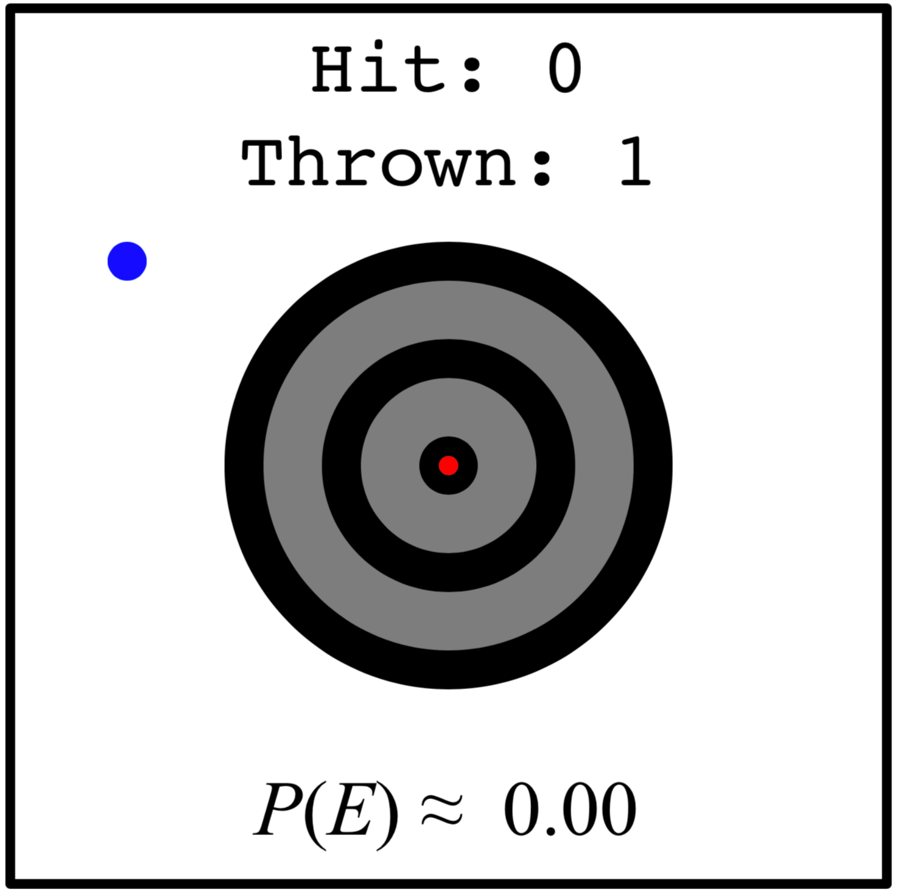

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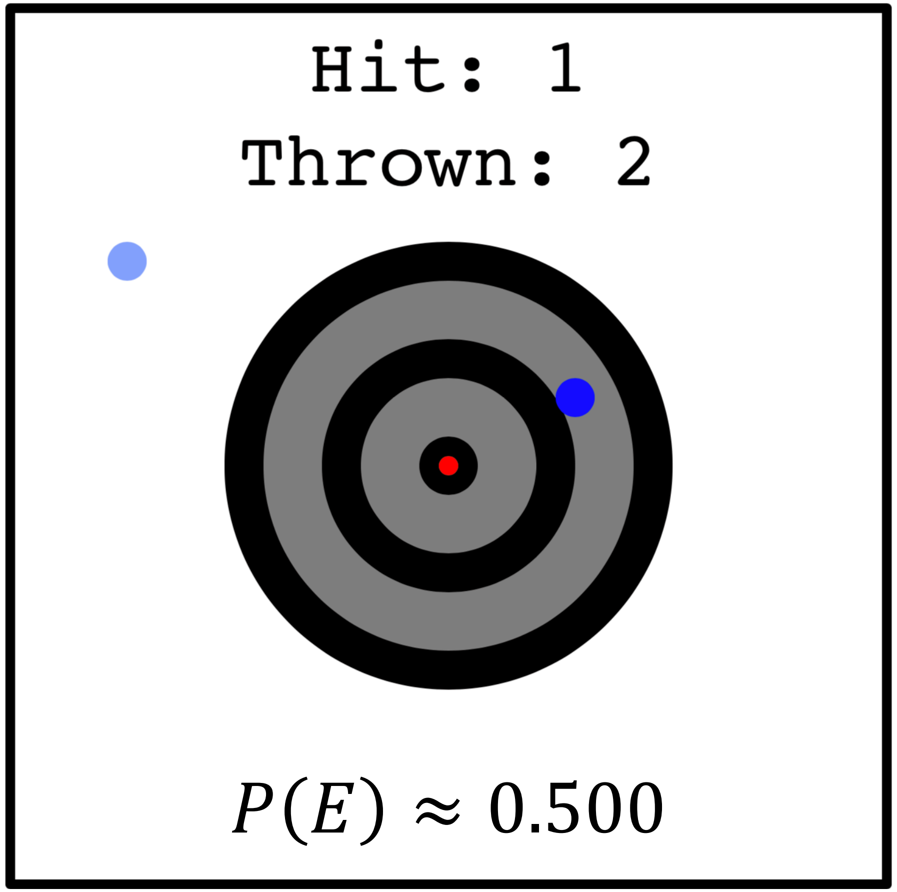

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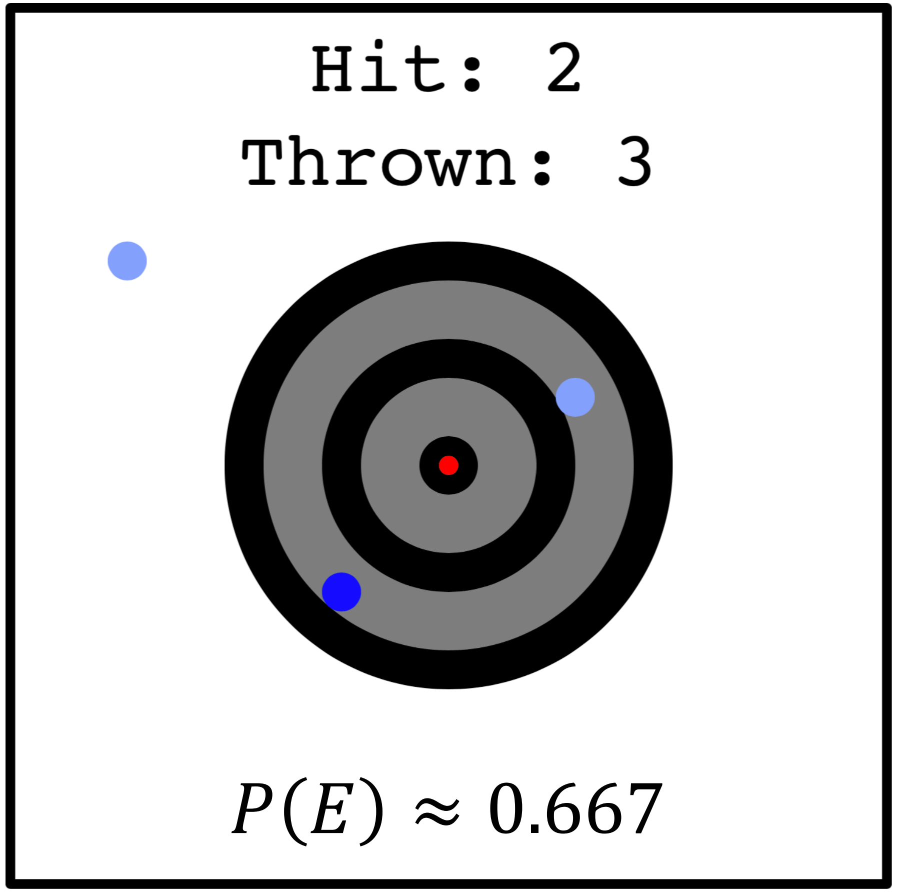


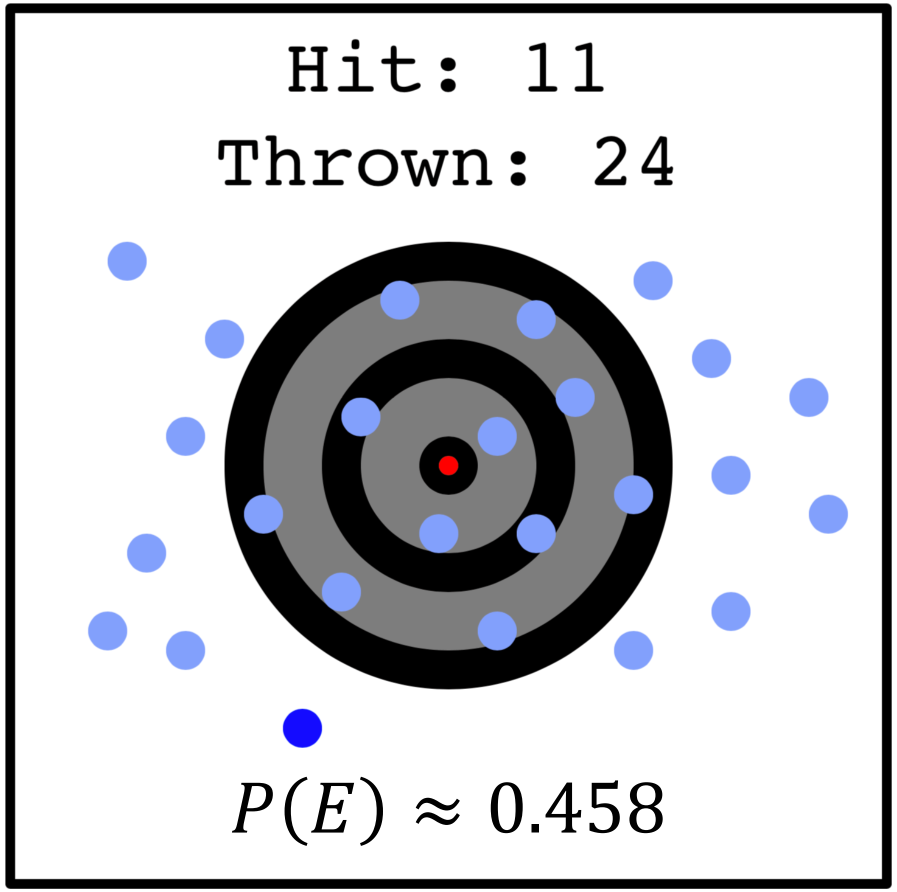
What is a probability?

Let E = the set of outcomes where you hit the target.

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

n = # of total trials

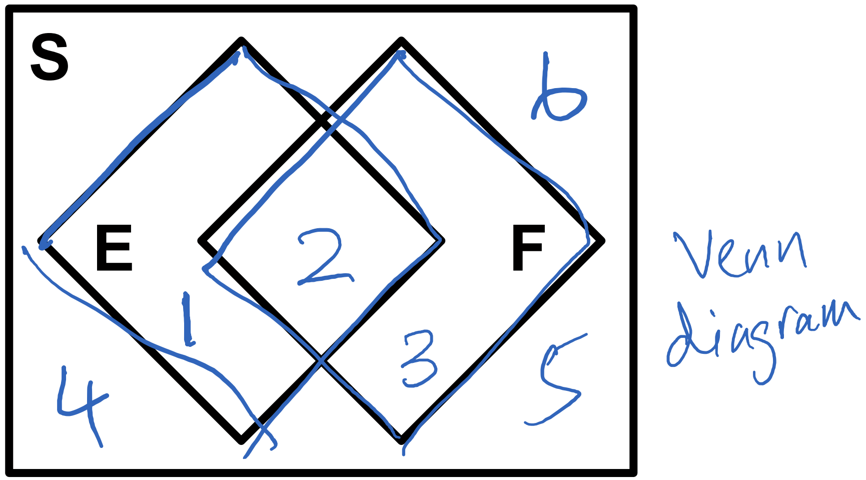
$n(E)$ = # trials where E occurs



Not just yet...

Axioms of Probability

Quick review of sets



E and F are events in S.

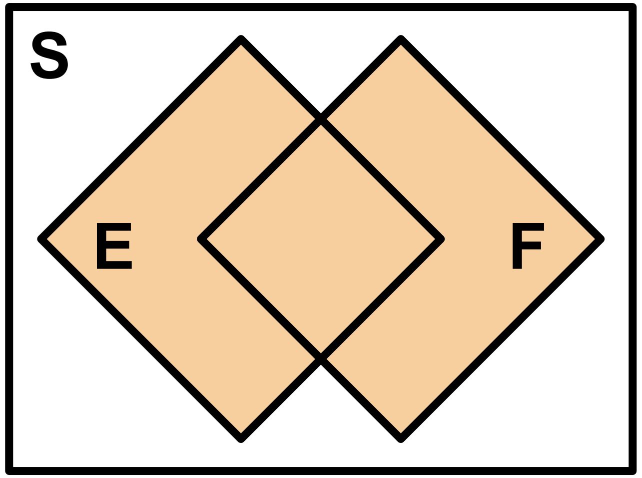
Experiment:

Die roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

Quick review of sets



E and F are events in S.

Experiment:

Die roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

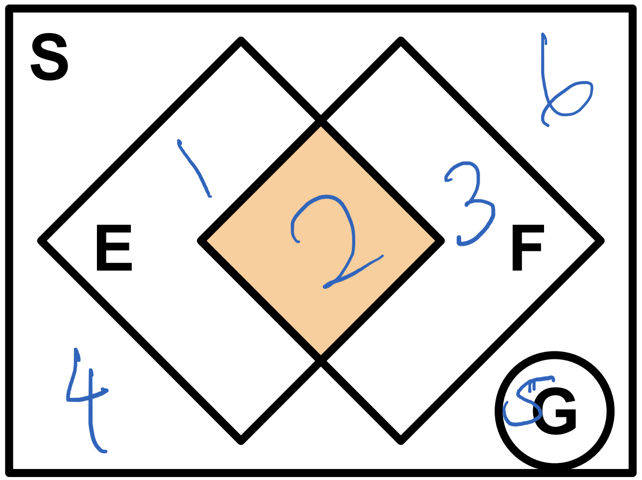
def Union of events, $E \cup F$

\cup

The event containing all outcomes in E or F.

$$E \cup F = \{1,2,3\}$$

Quick review of sets



E and F are events in S.

Experiment:

Die roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

<u>def</u> Intersection of events, $E \cap F$

The event containing all outcomes in E <u>and</u> F.

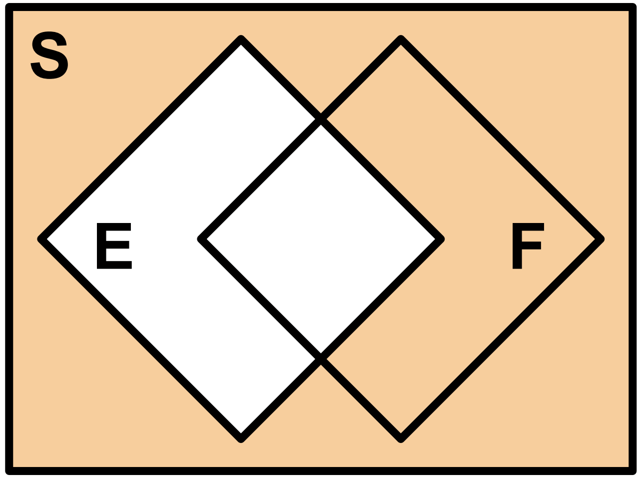
$$E \cap F = EF = \{2\}$$

\cap

<u>def</u> Mutually exclusive events F and G means that $F \cap G = \emptyset$

$G = \{5\}$

Quick review of sets



E and F are events in S.

Experiment:

Die roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

def Complement of event E, E^C

The event containing all outcomes in that are ***not*** in E.

$$E^C = \{3, 4, 5, 6\}$$

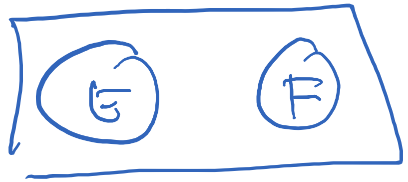
3 Axioms of Probability

Definition of probability: $P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$

Axiom 1: $0 \leq P(E) \leq 1$

Axiom 2: $P(S) = 1$

Axiom 3: If E and F are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$

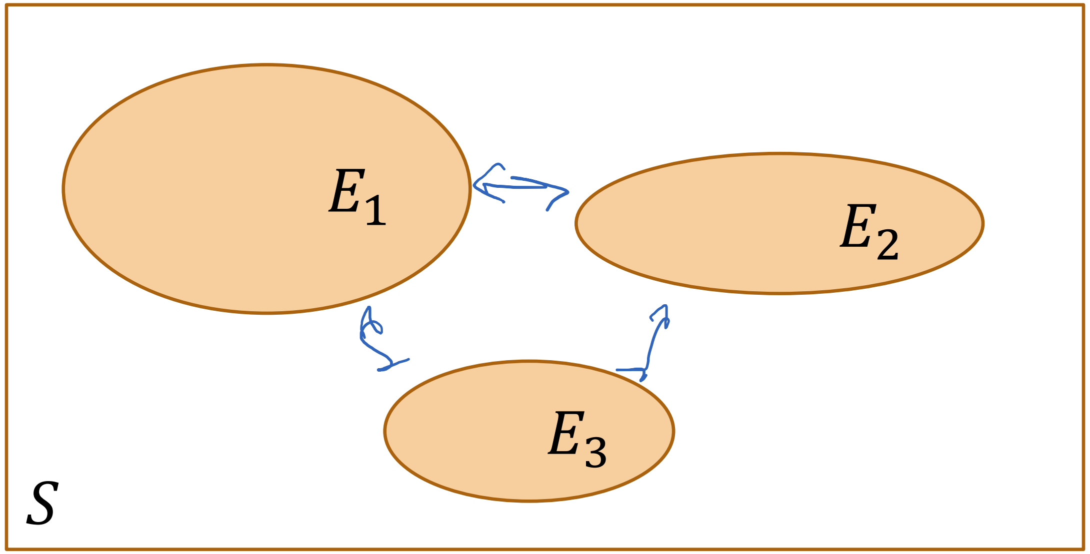


Axiom 3 is the (analytically) useful Axiom

Axiom 3: If E and F are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$

More generally, for any sequence of mutually exclusive events $E_1, E_2, \ldots$:

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$



$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$

(like the Sum Rule of Counting, but for probabilities)

Equally Likely Outcomes

Equally Likely Outcomes

Some sample spaces have **equally likely outcomes**.

- (fair) Coin flip: S = {Head, Tails} — $P(\text{Heads}) = 1/2$
- Flipping two (fair) coins: S = {(H, H), (H, T), (T, H), (T, T)} — $P(\{(H,H)\}) = 1/4$
- Roll of (fair) 6-sided die: S = {1, 2, 3, 4, 5, 6} — $P(\{3\}) = 1/6$

If we have equally likely outcomes, then P(Each outcome) $= \dfrac{1}{|S|}$

Therefore $P(E) = \dfrac{\#\text{ outcomes in E}}{\#\text{ outcomes in S}} = \dfrac{|E|}{|S|}$ (by Axiom 3)

E: 3 or lower

$E = \{1, 2, 3\}$

$E_1 = \{1\}, E_2 = \{2\}, E_3 = \{3\}, P(E_i) = \frac{1}{|S|}$

$P(E) = P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$

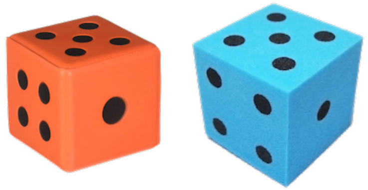
$= 3 \cdot \frac{1}{|S|}$

$= |E| \cdot \frac{1}{|S|} = \frac{|E|}{|S|}$

Roll two dice

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Roll two 6-sided fair dice. What is P(sum = 7)?



S = { (1, 1) , (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1) , (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1) , (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1) , (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1) , (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
(6, 1) , (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

|S| = 36

E = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

Sum is 7

|E| = 6

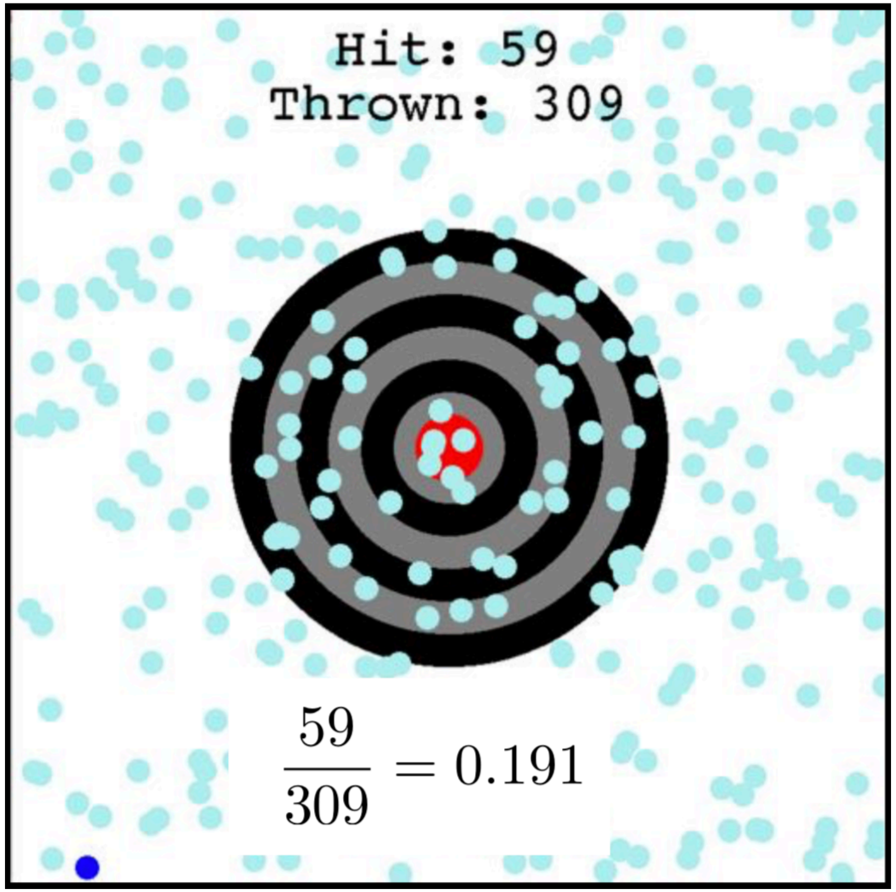
$P(E) = \frac{6}{36} = \frac{1}{6}$

Target revisited

Target revisited

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Let E = the set of outcomes where you hit the target.



Screen size = 800 ×800

Radius of target: 200

The dart is equally likely to land anywhere on the screen. What is $P(E)$, the probability of hitting the target?

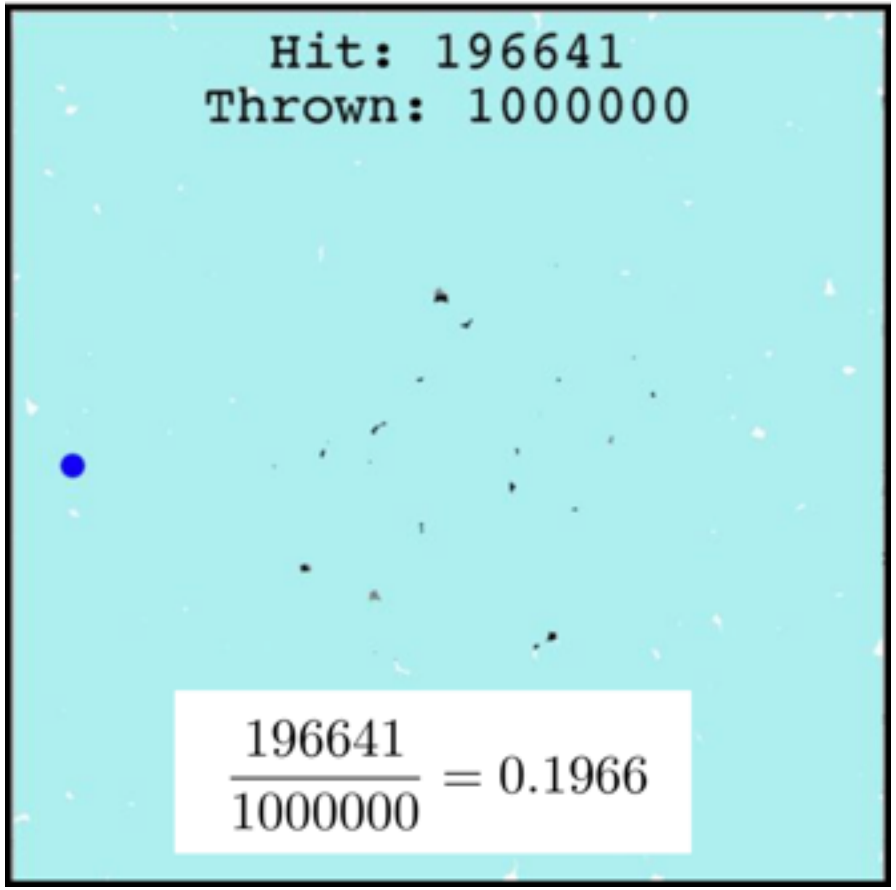
$|S| = 800^2$ $\quad$ $|E| \approx \pi \cdot 200^2$

$$P(E) = \frac{|E|}{|S|} \approx \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$

Target revisited

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Let E = the set of outcomes where you hit the target.



Screen size = 800 ×800

Radius of target: 200

The dart is equally likely to land anywhere on the screen. What is $P(E)$, the probability of hitting the target?

$$|S| = 800^2 \qquad |E| \approx \pi \cdot 200^2$$

$$P(E) = \frac{|E|}{|S|} \approx \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$

Not equally likely outcomes

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Play the lottery.

What is $P(\text{win})$?

calottery

$S = \{\text{Lose}, \text{Win}\}$

$E = \{\text{Win}\}$

$$P(E) = \frac{|E|}{|S|} = \frac{1}{2} = 50\%?$$

41,416,355 tickets sold

1 winning

The hard part: defining outcomes consistently across sample space and events

$P(\text{win}) = 1/41416355$

Cats and sharks

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 sharks in a bag. 3 drawn.

What is P(1 cat and 2 sharks drawn)?

Make indistinct items distinct to get equally likely outcomes.

A. $\frac{3}{7}$

B. $\frac{1}{4} \cdot \frac{2}{3}$

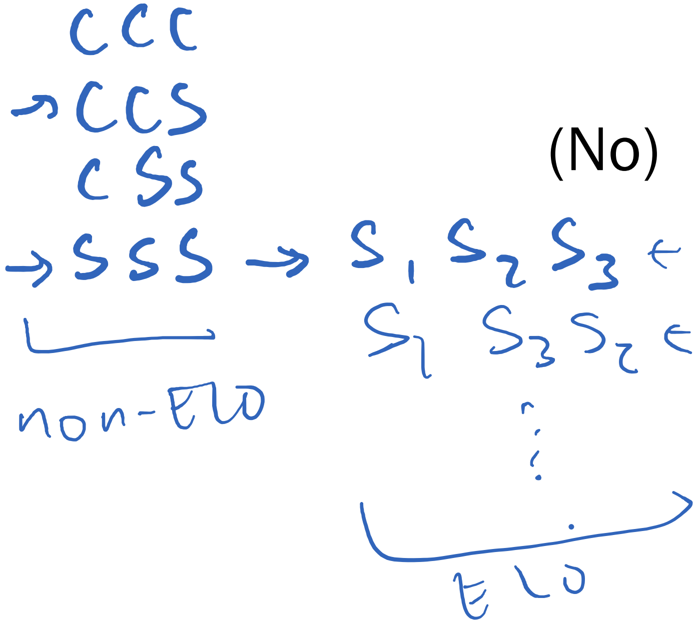
C. $\frac{4}{7} + 2 \cdot \frac{3}{6}$

D. $\frac{12}{35}$

E. Zero/other

Note: Do indistinct objects give you an equally likely sample space?

(No)



Cats and sharks (ordered solution)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 sharks in a bag. 3 drawn.

What is P(1 cat and 2 sharks drawn)?

Make indistinct items distinct to get equally likely outcomes.

Define

- S = Pick 3 distinct items
- E = 1 distinct cat, 2 distinct sharks

$7 \cdot 6 \cdot 5$ $|S| = 210$

pick C first: $4 \cdot 3 \cdot 2$ (c s s)

pick C second $3 \cdot 4 \cdot 2$ (s c s)

pick C third $3 \cdot 2 \cdot 4$ (s s c)

+

$|E| = 72$

$P(E) = \frac{12}{35} = \frac{72}{210}$

Cats and sharks (unordered solution)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 sharks in a bag. 3 drawn.

What is P(1 cat and 2 sharks drawn)?

Make indistinct items distinct to get equally likely outcomes.

Define

- S = Pick 3 distinct items $|S| = \binom{7}{3} = \frac{7!}{3!4!} = 35$
- E = 1 distinct cat, 2 distinct sharks $|E| = \binom{4}{1}\binom{3}{2} = 4 \cdot 3 = 12$ $P(E) = \frac{12}{35}$

(live) 03: Intro to Probability

Lisa Yan and Jerry Cain

September 18, 2020

Holy cow, are all the pre-lecture videos going to be this long??

This course is <u>**front-loaded with material/definitions**</u>.

As we move into the latter half of the course, we will achieve a better balance.

Lecture Notes are a perfectly good substitute for videos if you learn better through reading.

123

SESAME STREET®

The Count

Chance The Rapper

Summary so far

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct: r^n
  - Indistinct: $\dfrac{(n+r-1)!}{n!\,(r-1)!}$

Equally likely outcomes:

$$P(E) = \frac{|E|}{|S|}$$

Combinatorics

Probability

Counting? Probability? Distinctness?

We choose **3 books** from a set of
4 distinct (distinguishable) and **2 indistinct** (indistinguishable) books.
Each set of 3 books is equally likely.

Let event E = our choice does **not** include both indistinct books.

1. How many distinct outcomes are in E?

{ one indistinct
{ no indistinct

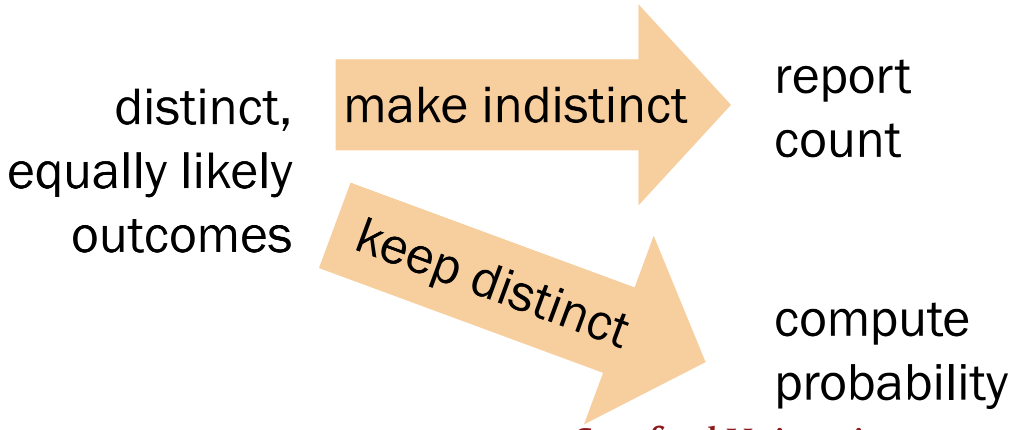
1. $\binom{4}{2} = 6$
$\binom{4}{3} = 4$
$\boxed{10}$

2. What is $P(E)$?

$|S_{dist}| = \binom{6}{3} = 20$

$|E_{dist}| = \binom{2}{1}\binom{4}{2} + \binom{4}{3} = 16$

$P(E) = \frac{|E_{dist}|}{|S_{dist}|} = \frac{16}{20} = 0.8$



Lecture with zoom

Breakout Rooms for working through lecture exercises

- We may incorporate some of these during lecture
- You are <u>always welcome</u> to exit breakout rooms if you are more comfortable staying in the main room
- Turn on your camera if you are comfortable doing so

Think, then Breakout Rooms

Then check out the questions on the next slide (Slide 37). Post any clarifications here!

https://us.edstem.org/courses/2678/discussion/124598

Read both questions: 2 min

Breakout rooms: 5 min. Introduce yourself!

Poker Straights and Computer Chips

1. Consider equally likely 5-card poker hands.
   - Define "poker straight" as 5 consecutive rank cards of any suit   A 2 3 4 5 .... 10 J Q K A

   What is P(Poker straight)?

- What is an example of an equally likely outcome?
- Should objects be ordered or unordered?

2. Computer chips: n chips are manufactured, 1 of which is defective. k chips are randomly selected from n for testing.

   What is P(defective chip is in k selected chips?)

Q1: odd-numbered breakout rooms
Q2: even-numbered breakout rooms
(if time, switch to other question)

1. Any Poker Straight

Consider equally likely 5-card poker hands.

- "straight" is 5 consecutive rank cards of any suit

What is P(Poker straight)?

A 2 3 4 5
⋮
3 4 5 6 7
⋮
9 10 J Q K
10 J Q K A

Define

- S (unordered) $|S| = \binom{52}{5}$
- E (unordered, consistent with S) $10 \cdot \binom{4}{1}^5$

Compute $P(\text{Poker straight}) = \frac{10 \cdot \binom{4}{1}^5}{\binom{52}{5}} \approx 0.00394$

Lecture Notes:
Another example with "official" definition of a Poker straight (straight vs straight flush)

2. Chip defect detection

n chips are manufactured, 1 of which is defective.
k chips are randomly selected from n for testing.

What is P(defective chip is in k selected chips?)

Define

- S (unordered) $\quad |S| = \binom{n}{k}$
- E (unordered, consistent with S) $\quad |E| = \binom{1}{1} \cdot \binom{n-1}{k-1}$

defective $\quad$ not defective

Compute

$$P(E) = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{\frac{(n-1)!}{(k-1)!\,(n-k)!}}{\frac{n!}{k!\,(n-k)!}} = \frac{k}{n}$$

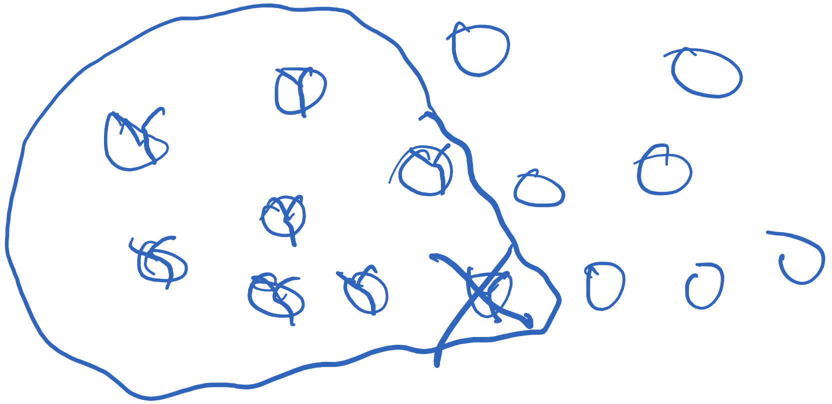
2. Chip defect detection, solution #2

n chips are manufactured, 1 of which is defective.
k chips are randomly selected from n for testing.

What is P(defective chip is in k selected chips?)

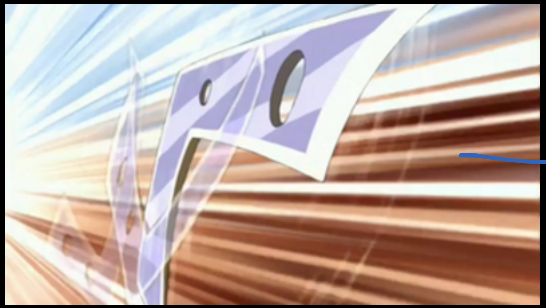
Redefine experiment

1. Choose k indistinct chips (1 way)
2. Throw a dart and make one defective



Define

- S (unordered)
- E (unordered, consistent with S)



forever

boomerang

stick

Interlude for jokes/announcements

Announcements

CS109 Course ▾ Lecture ▾ Problem Sets ▾ Section ▾ Resources/Demos ▾ Section / Office Hours Schedule

Python tutorial

When:

~~earlier today~~

after class

Recorded/posted [online]

Resources on CS109 website

Resources/Demos ▾

- Calculation Ref
- Python for Probability
- LaTeX Guides
- Latex Cheat Sheet
- Full Probability Reference (Overleaf)

Geometric series, Integration by parts…

Section sign-ups/ Acquaintance form

[form link]

Form due:

Sat. 5:00pm 9/19

Results:

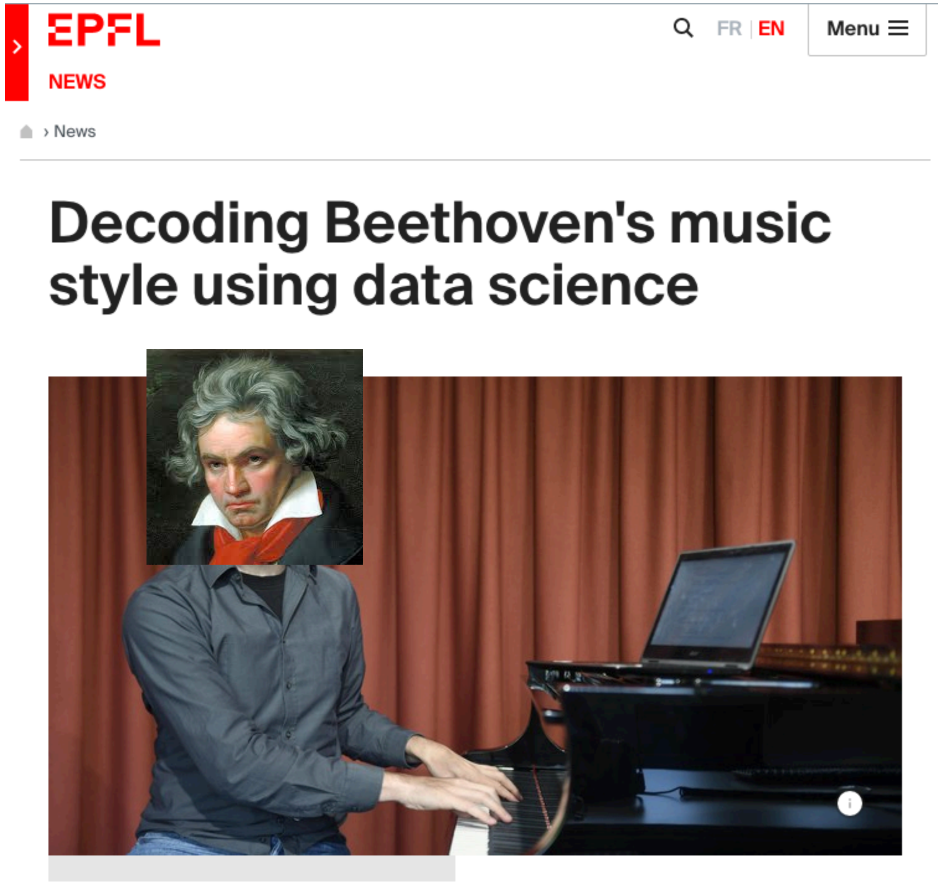
latest Sunday

Office hours

Have started!

nooks

Interesting probability news



"The study finds that **very few chords govern most of the music, a phenomenon that is also known in linguistics**, where very few words dominate language corpora…. It characterizes Beethoven's specific composition style for the String Quartets, through a distribution of all the chords he used, **how often they occur**, and how they commonly transition from one to the other."

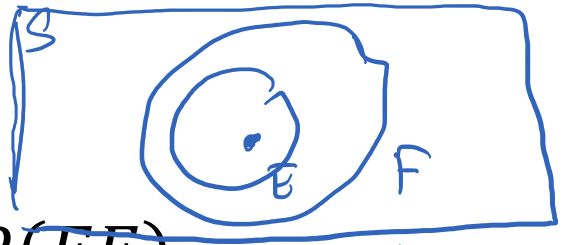
https://actu.epfl.ch/news/decoding-beethoven-s-music-style-using-data-scienc/

Corollaries of Probability

3 Corollaries of Axioms of Probability

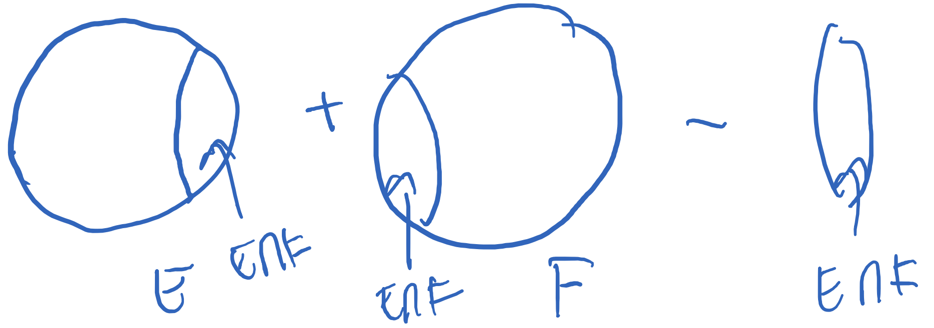
Corollary 1: $P(E^C) = 1 - P(E)$

Corollary 2: If $E \subseteq F$, then $P(E) \leq P(F)$



Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$

(Inclusion-Exclusion Principle for Probability)



Selecting Programmers

- P(student programs in Python) = 0.28 $= P(E)$
- P(student programs in C++) = 0.07 $= P(F)$
- P(student programs in Python and C++) = 0.05. $= P(E \cap F) = P(EF)$

What is P(student does not program in (Python or C++))?

1. Define events & state goal
2. Identify <u>known</u> probabilities
3. Solve

E: Python

F: C++

$P((E \cup F)^C) = 0.7$

Corollary 1: $P((E \cup F)^c) = 1 - P(E \cup F)$

Corollary 3: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$= 0.28 + 0.07 - 0.05$

$= 0.3$

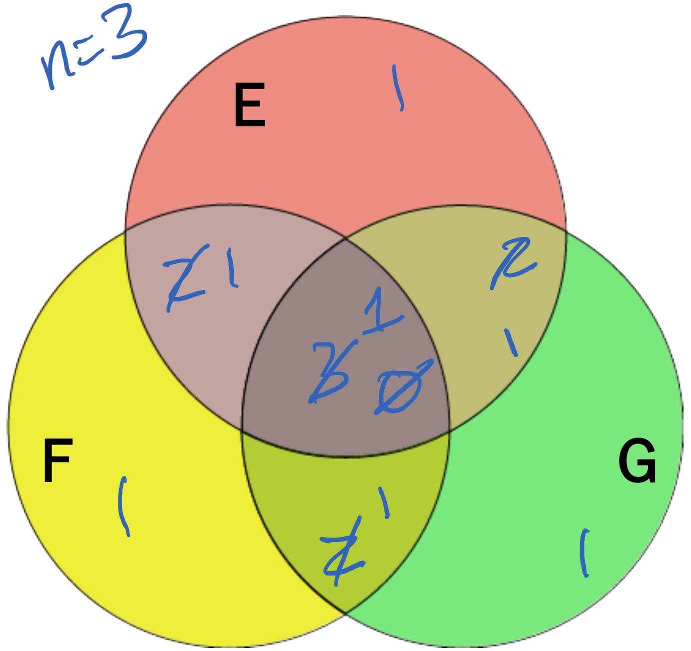
Inclusion-Exclusion Principle (Corollary 3)

Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$

General form: $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{(r+1)} \sum_{i_1 < \cdots < i_r} P\left(\bigcap_{j=1}^{r} E_{i_j}\right)$ (see Lecture Notes)

sets in intersection

n=3



$P(E \cup F \cup G) =$

r = 1: $P(E) + P(F) + P(G)$

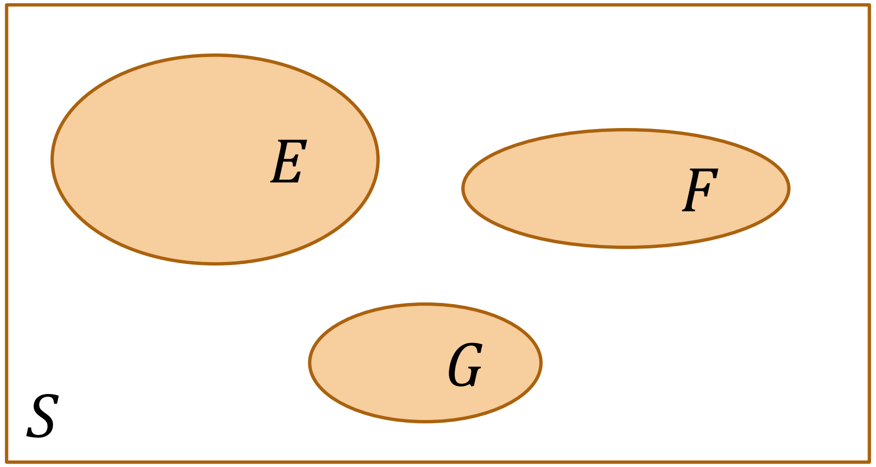
r = 2: $-P(E \cap F) - P(E \cap G) - P(F \cap G)$

r = 3: $+P(E \cap F \cap G)$

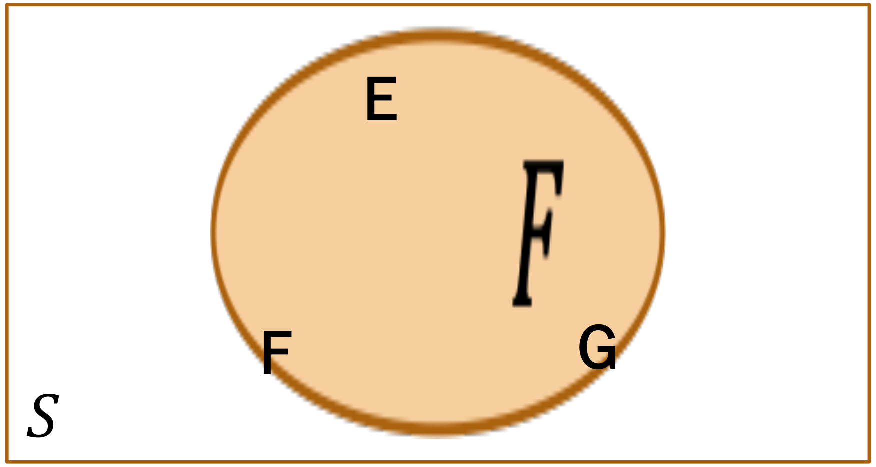
Takeaway: Union of events

$P(E \cup F \cup G)$

Axiom 3,
Mutually exclusive events



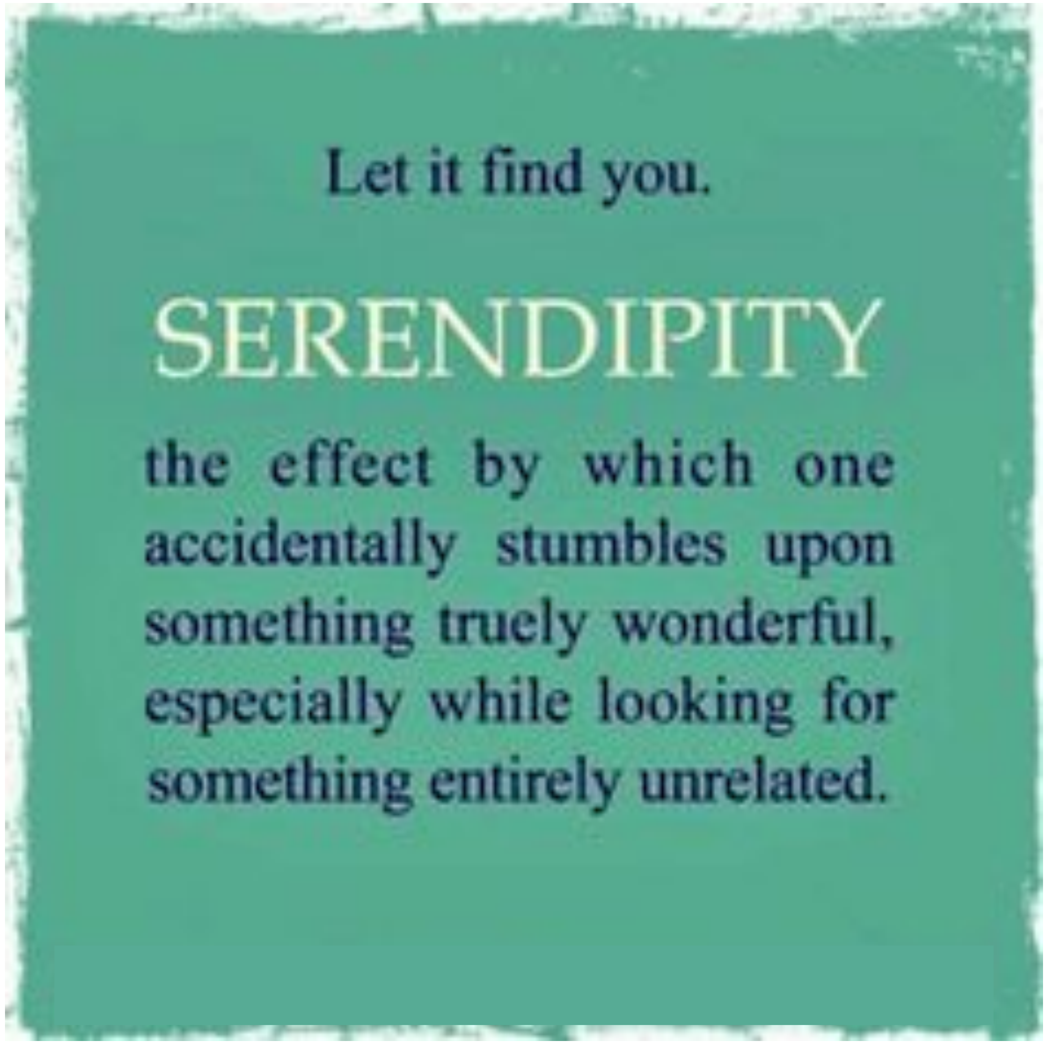
Corollary 3,
Inclusion-Exclusion Principle



The challenge of probability is in defining events.
Some event probabilities are easier to compute than others.

Serendipity

$P(E) = 1 - P(E^c)$



Serendipity

- The population of Stanford is $n = 17{,}000$ people.
- You are friends with $r = 100$ people.
- Walk into a room, see $k = 200$ random people. 223
- Assume each group of k Stanford people is equally likely to be in room.

What is the probability that you see someone you know in the room?

at least one friend?

http://web.stanford.edu/class/cs109/demos/serendipity.html

~ 70%

Think

Slide 52 is a question to think over by yourself (~2min).

Post any clarifications here or in chat!

https://us.edstem.org/courses/2678/discussion/124109

(by yourself)

Serendipity

- The population of Stanford is $n = 17{,}000$ people.
- You are friends with $r = 100$ people.
- Walk into a room, see $k = 223$ random people.
- Assume each group of k Stanford people is equally likely to be in room.

What is the probability that you see at least one friend in the room?

Define

- S (unordered)
- E: ≥ 1 friend in the room

What strategy would you use?

A. $P(\text{exactly } 1) + P(\text{exactly } 2)$ $P(\text{exactly } 3) + \cdots$

$$\binom{r}{1}\binom{n-r}{k-1} \Big/ \binom{n}{k} + \binom{r}{2}\binom{n-r}{k-2} \Big/ \binom{n}{k} + \ldots$$

B. $1 - P(\text{see no friends})$

(by yourself)

Serendipity

E^c ← complement

- The population of Stanford is $n = 17{,}000$ people.
- You are friends with $r = 100$ people.
- Walk into a room, see $k = 223$ random people.
- Assume each group of k Stanford people is equally likely to be in room.

What is the probability that you see at least one friend in the room?

Define

- S (unordered)
- E: ≥ 1 friend in the room

E^c: no friends in room :(

$$P(E^c) = \frac{\binom{n-r}{k}}{\binom{n}{k}} = \frac{\binom{16900}{223}}{\binom{17000}{223}}$$

$$P(E) = 1 - P(E^c) = 1 - \frac{\binom{16900}{223}}{\binom{17000}{223}} \approx 0.7$$

It is often much easier to compute $P(E^c)$.

The Birthday ~~Paradox~~ Problem

What is the probability that in a set of *n* people, <u>at least one</u> pair of them will share the same birthday?

For you to think about (and discuss in section!)

Card Flipping

In a 52-card deck, cards are flipped one at a time.
After the first ace (of any suit) appears, consider the next card.

Is P(next card = Ace Spades) < P(next card = 2 Clubs)?

Once you think you have an answer, vote on our Zoom poll:

https://us.edstem.org/courses/2678/discussion/124598

Check out the Lecture Notes!

(by yourself)

Have a good weekend!