04: Conditional Probability and Bayes

Lisa Yan and Jerry Cain

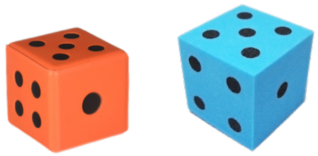
September 21, 2020

Quick slide reference

Conditional Probability

Dice, our misunderstood friends

Roll two 6-sided dice, yielding values D_1 and D_2.



Let E be event: $D_1 + D_2 = 4$.

What is $P(E)$?

$|S| = 36$

$E = \{(1,3), (2,2), (3,1)\}$

$P(E) = 3/36 = 1/12$

Let F be event: $D_1 = 2$.

What is $P(E,$ given F already observed$)$?

Conditional Probability

The conditional probability of E given F is the probability that E occurs given that F has already occurred. This is known as conditioning on F.

Written as: $P(E|F)$

Means: "$P(E,$ given F already observed)"

Sample space → all possible outcomes consistent with F (i.e. $S \cap F$)

Event → all outcomes in E consistent with F (i.e. $E \cap F$)

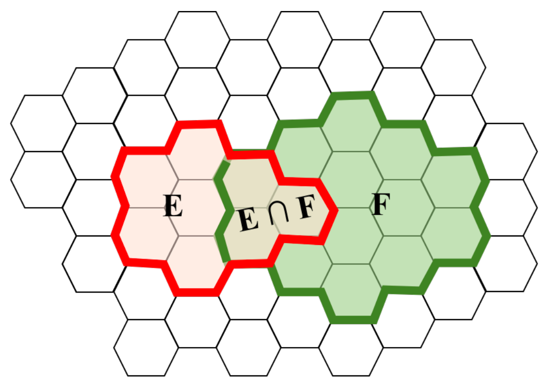
Conditional Probability, equally likely outcomes

The conditional probability of E given F is the probability that E occurs given that F has already occurred. This is known as conditioning on F.

With equally likely outcomes:

$$P(E|F) = \frac{\text{\# of outcomes in E consistent with F}}{\text{\# of outcomes in S consistent with F}} = \frac{|E \cap F|}{|S \cap F|}$$

$$P(E|F) = \frac{|EF|}{|F|}$$



$$P(E) = \frac{8}{50} \approx 0.16$$

$$P(E|F) = \frac{3}{14} \approx 0.21$$

Slicing up the spam

$P(E|F) = \frac{|EF|}{|F|}$ Equally likely outcomes

24 emails are sent, 6 each to 4 users.

- 10 of the 24 emails are spam.
- All possible outcomes are equally likely.

Let E = user 1 receives 3 spam emails.

What is $P(E)$?

Let F = user 2 receives 6 spam emails.

What is $P(E|F)$?

Let G = user 3 receives 5 spam emails.

What is $P(G|F)$?

Slicing up the spam

$P(E|F) = \frac{|EF|}{|F|}$ Equally likely outcomes

24 emails are sent, 6 each to 4 users.

- 10 of the 24 emails are spam.
- All possible outcomes are equally likely.

Let E = user 1 receives 3 spam emails.

What is $P(E)$?

$$P(E) = \frac{\binom{10}{3}\binom{14}{3}}{\binom{24}{6}}$$

$$\approx 0.3245$$

Let F = user 2 receives 6 spam emails.

What is $P(E|F)$?

$$P(E|F) = \frac{\binom{4}{3}\binom{14}{3}}{\binom{18}{6}}$$

$$\approx 0.0784$$

Let G = user 3 receives 5 spam emails.

What is $P(G|F)$?

⚠

$$P(G|F) = \frac{\binom{4}{5}\binom{14}{1}}{\binom{18}{6}}$$

$$= 0$$

No way to choose 5 spam from 4 remaining spam emails!

Conditional probability in general

General definition of conditional probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

The Chain Rule (aka Product rule):

$$P(EF) = P(F)P(E|F)$$

These properties hold even when outcomes are not equally likely.

NETFLIX

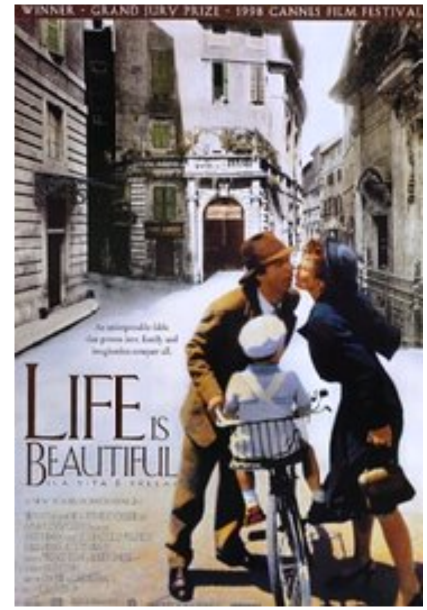
and Learn

Netflix and Learn

$P(E|F) = \frac{P(EF)}{P(F)}$ Definition of Cond. Probability

Let E = a user watches Life is Beautiful.

What is $P(E)$?



❌ Equally likely outcomes?

$S = \{\text{watch, not watch}\}$

$E = \{\text{watch}\}$

$P(E) = 1/2$?

✅ $P(E) = \lim_{n\to\infty} \frac{n(E)}{n} \approx \frac{\text{\# people who have watched movie}}{\text{\# people on Netflix}}$

$= 10{,}234{,}231 \;/\; 50{,}923{,}123 \approx 0.20$

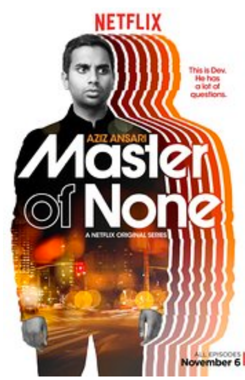
Netflix and Learn

$P(E|F) = \frac{P(EF)}{P(F)}$ Definition of Cond. Probability

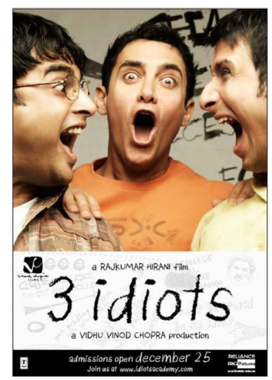
Let E be the event that a user watches the given movie.

$P(E) = 0.19$

$P(E) = 0.32$



$P(E) = 0.20$



$P(E) = 0.09$

$P(E) = 0.20$

Netflix and Learn

$P(E|F) = \frac{P(EF)}{P(F)}$ Definition of Cond. Probability

Let E = a user watches Life is Beautiful.

Let F = a user watches Amelie.

What is the probability that a user watches Life is Beautiful, given they watched Amelie?

$$P(E|F)$$

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{\text{\# people who have watched both}}{\text{\# people on Netflix}}}{\frac{\text{\# people who have watched Amelie}}{\text{\# people on Netflix}}}$$

$$= \frac{\text{\# people who have watched both}}{\text{\# people who have watched Amelie}}$$

$$\approx 0.42$$

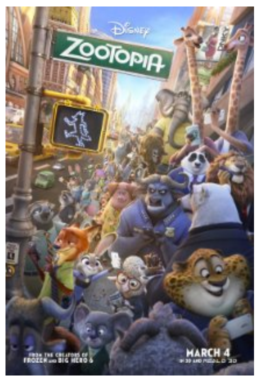
Netflix and Learn

$$P(E|F) = \frac{P(EF)}{P(F)}$$ Definition of Cond. Probability

Let E be the event that a user watches the given movie.
Let F be the event that the same user watches Amelie.

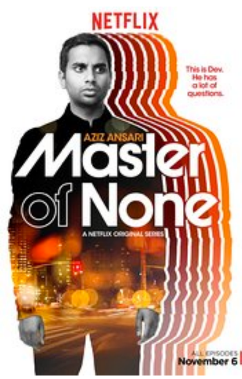
$P(E) = 0.19$

$P(E|F) = 0.14$



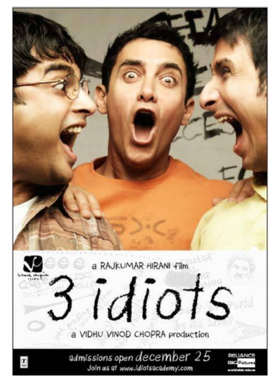
$P(E) = 0.32$

$P(E|F) = 0.35$



$P(E) = 0.20$

$P(E|F) = 0.20$



$P(E) = 0.09$

$P(E|F) = 0.72$

$P(E) = 0.20$

$P(E|F) = 0.42$

04b_total_prob

Law of Total Probability

Today's tasks

$$P(EF)$$

Chain rule (Product rule) ⇧ ⇩ Definition of conditional probability

$$P(E|F)$$

Law of Total Probability ⇙

$$P(E)$$

Law of Total Probability

Thm Let F be an event where $P(F) > 0$. For any event E,

$$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$$

Proof

1. F and F^C are disjoint s.t. $F \cup F^C = S$ — Def. of complement
2. $E = (EF) \cup (EF^C)$ — (see diagram)
3. $P(E) = P(EF) + P(EF^C)$ — Additivity axiom
4. $P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$ — Chain rule (product rule)

Note: disjoint sets by definition are mutually exclusive events

General Law of Total Probability

Thm For mutually exclusive events $F_1, F_2, \ldots, F_n$ s.t. $F_1 \cup F_2 \cup \cdots \cup F_n = S$,

$$P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i)$$

Finding $P(E)$ from $P(E|F)$

$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$ Law of Total Probability

- Flip a fair coin.
- If heads: roll a fair 6-sided die.
- Else: roll a fair 3-sided die.

You win if you roll a 6. What is P(winning)?

Finding $P(E)$ from $P(E|F)$

$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$ Law of Total Probability

- Flip a fair coin.
- If heads: roll a fair 6-sided die.
- Else: roll a fair 3-sided die.

You win if you roll a 6. What is P(winning)?

1. Define events & state goal
2. Identify <u>known</u> probabilities
3. Solve

Let: E: win, F: flip heads
Want: $P(\text{win})$
$= P(E)$

$P(\text{win}|\text{H}) = P(E|F) = 1/6$
$P(\text{H}) = P(F) = 1/2$
$P(\text{win}|\text{T}) = P(E|F^C) = 0$
$P(\text{T}) = P(F^C) = 1 - 1/2$

$$P(E) = (1/6)(1/2) + (0)(1/2) = \frac{1}{12} \approx 0.083$$

Finding $P(E)$ from $P(E|F)$, an understanding

- Flip a fair coin.
- If heads: roll a fair 6-sided die.
- Else: roll a fair 3-sided die.

You win if you roll a 6. What is P(winning)?

1. Define events
   & state goal

Let: E: win, F: flip heads
Want: $P(\text{win})$
$= P(E)$

"Probability trees" can help connect your understanding of the experiment with the problem statement.

Bayes' Theorem I

Today's tasks

$$P(EF)$$

Chain rule (Product rule) ↑ ↓ Definition of conditional probability

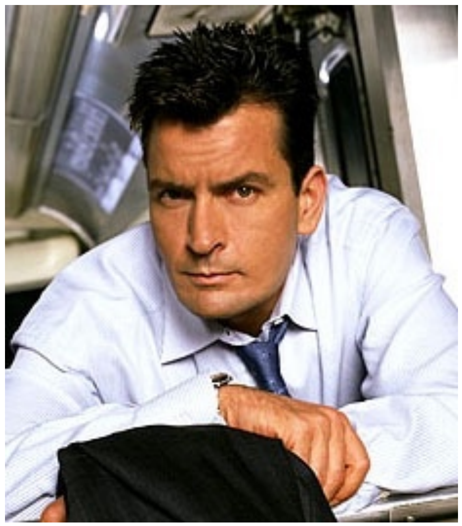
$$P(E|F)$$

Law of Total Probability ↙ ↘ Bayes' Theorem

$P(E)$ $P(F|E)$

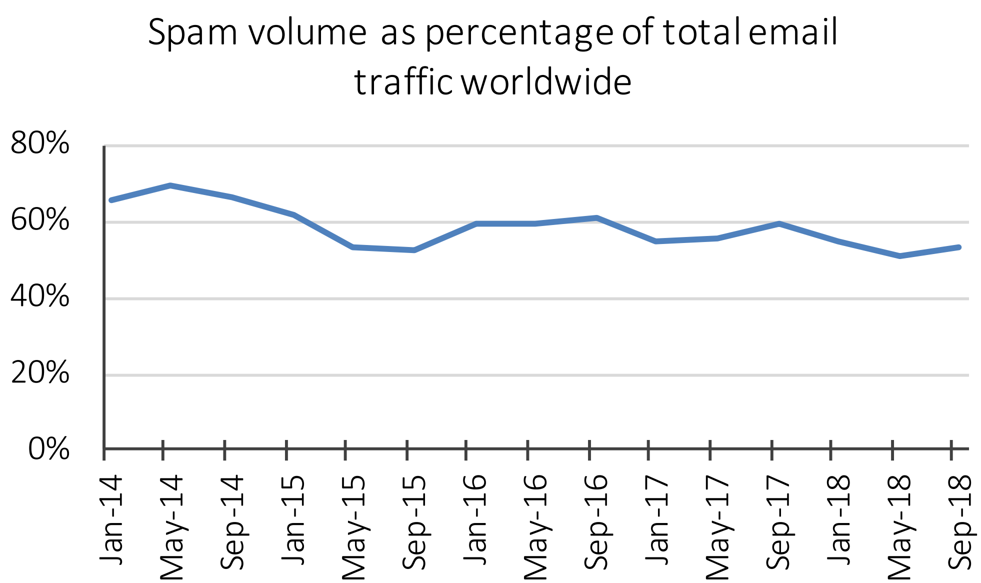
Thomas Bayes

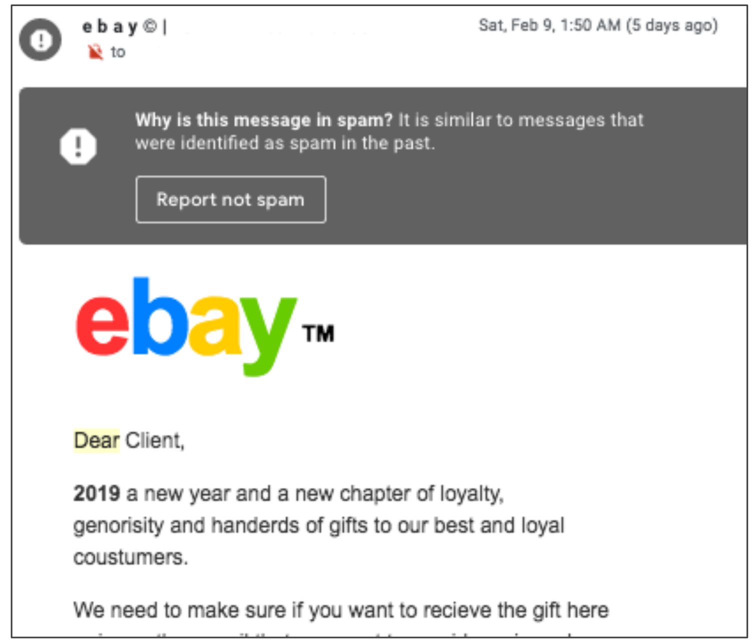
Rev. Thomas Bayes (~1701-1761):
British mathematician and Presbyterian minister



He looked remarkably similar to Charlie Sheen
(but that's not important right now)

Detecting spam email





We can easily calculate how many spam emails contain "Dear":

$$P(E|F) = P\left(\text{“Dear”} \,\middle|\, \begin{matrix}\text{Spam}\\ \text{email}\end{matrix}\right)$$

But what is the probability that an email containing "Dear" is spam?

$$P(F|E) = P\left(\begin{matrix}\text{Spam}\\ \text{email}\end{matrix} \,\middle|\, \text{“Dear”}\right)$$

(silent drumroll)

Bayes' Theorem

$P(E|F) \Rightarrow P(F|E)$

Thm For any events E and F where $P(E) > 0$ and $P(F) > 0$,

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

Proof

2 steps! See board

Expanded form:

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$

Proof

1 more step! See board

Detecting spam email

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$ Bayes' Theorem

- 60% of all email in 2016 is spam.
- 20% of spam has the word "Dear"
- 1% of non-spam (aka ham) has the word "Dear"

You get an email with the word "Dear" in it.

What is the probability that the email is spam?

1. Define events & state goal
2. Identify known probabilities
3. Solve

Let: E: "Dear", F: spam

Want: $P(\text{spam} \mid \text{"Dear"})$

$= P(F|E)$

Detecting spam email, an understanding

- 60% of all email in 2016 is spam.
- 20% of spam has the word "Dear"
- 1% of non-spam (aka ham) has the word "Dear"

You get an email with the word "Dear" in it.

What is the probability that the email is spam?

1. Define events & state goal

Let: E: "Dear", F: spam

Want: P(spam | "Dear")

$= P(F|E)$

Note: You should still know how to use Bayes/ Law of Total Probab., but drawing a probability tree can help you identify which probabilities you have. The branches are determined using the problem setup.

Bayes' Theorem terminology

- 60% of all email in 2016 is spam. $P(F)$
- 20% of spam has the word "Dear" $P(E|F)$
- 1% of non-spam (aka ham) has the word "Dear" $P(E|F^C)$

You get an email with the word "Dear" in it.

What is the probability that the email is spam? Want: $P(F|E)$

likelihood prior

posterior

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

normalization constant

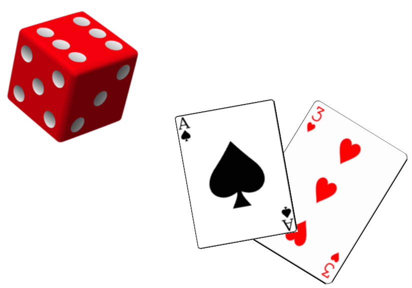
(live)
04: Conditional Probability and Bayes

Lisa Yan and Jerry Cain
September 21, 2020

This class going forward

Last week

Equally likely events



$P(E \cap F)$ $P(E \cup F)$

(counting, combinatorics)

Today and for most of this course

Not equally likely events

$P(E = \text{Evidence} \mid F = \text{Fact})$

(collected from data)

Bayes'

$P(F = \text{Fact} \mid E = \text{Evidence})$

(categorize a new datapoint)

Conditional probability in general

Review

General definition of conditional probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

The Chain Rule (aka Product rule):

$$P(EF) = P(F)P(E|F)$$

These properties hold even when outcomes are not equally likely.

Think, then Breakout Rooms

Then check out the question on the next slide (Slide 35). Post any clarifications here!

https://us.edstem.org/courses/2678/discussion/128395

Think by yourself: 1 min

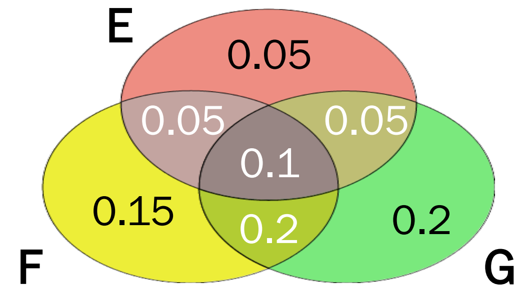
Breakout rooms: 4 min. Introduce yourself!

🤔

Think, then groups

You have a flowering plant.

Let E = Flowers bloom
F = Plant was watered
G = Plant got sun

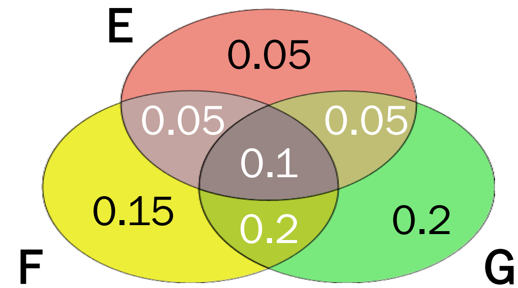


1. How would you write
   i. the probability that the plant got sun, given that it was watered and flowers bloomed?
   ii. the probability that the plant got sun and flowers bloomed given that it was watered?
2. Using the Venn diagram, compute the above probabilities.
3. Chain Rule: Fill in the blanks.
   i. $P(GE) = \_\_\_\_\_ \cdot P(E)$
   ii. $P(GE|F) = P(G|EF) \cdot \_\_\_\_\_$

Think, then groups

You have a flowering plant.

Let E = Flowers bloom
F = Plant was watered
G = Plant got sun



1. How would you write
   i. the probability that the plant got sun, given that it was watered and flowers bloomed?
   ii. the probability that the plant got sun and flowers bloomed given that it was watered?
2. Using the Venn diagram, compute the above probabilities.
3. Chain Rule: Fill in the blanks.
   i. $P(GE) = \_\_\_\_\_ \cdot P(E)$
   ii. $P(GE|F) = P(G|EF) \cdot \_\_\_\_\_$

Bayes' Theorem II

Why is Bayes' so important?

👉 It links belief to evidence in probability!

Bayes' Theorem

Review

posterior likelihood prior

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

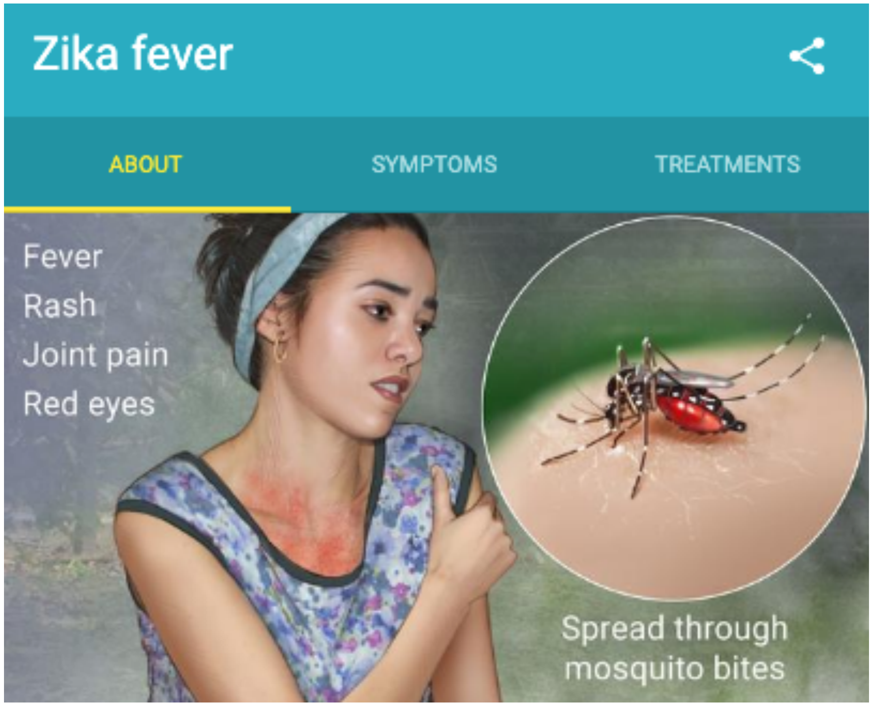
Mathematically:

$$P(E|F) \rightarrow P(F|E)$$

Real-life application:

Given new evidence E, update belief of fact F
Prior belief → Posterior belief
$P(F) \rightarrow P(F|E)$

Zika, an autoimmune disease



A disease spread through mosquito bites.
Usually no symptoms; worst case paralysis. During pregnancy: may cause birth defects

Ziika Forest, Uganda

Rhesus monkeys

If a test returns positive, what is the likelihood you have the disease?

Taking tests: Confusion matrix

Fact, F — Has disease
or F^C — No disease

Take test

Evidence, E — Test positive
or E^C — Test negative

		Fact	
		F, disease +	F^C, disease –
Evidence	E, Test +	True positive $P(E\|F)$	False positive $P(E\|F^C)$
	E^C, Test –	False negative $P(E^C\|F)$	True negative $P(E^C\|F^C)$

If a test returns positive, what is the likelihood you have the disease?

Taking tests: Confusion matrix

Take test

Fact, F Has disease
or F^C No disease

Evidence, E Test positive
or E^C Test negative

		Fact	
		F, disease +	F^C, disease –
Evidence	E, Test +	True positive $P(E\|F)$	False positive $P(E\|F^C)$
	E^C, Test –	False negative $P(E^C\|F)$	True negative $P(E^C\|F^C)$

If a test returns positive, what is the likelihood you have the disease?

Breakout Rooms

Check out the question on the next slide (Slide 43). Post any clarifications here!

https://us.edstem.org/courses/2678/discussion/128395

Breakout rooms: 5 minutes

🤔

Zika Testing

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$ Bayes' Theorem

- A test is 98% effective at detecting Zika ("true positive").
- However, the test has a "false positive" rate of 1%.
- 0.5% of the US population has Zika.

What is the likelihood you have Zika if you test positive?

Why would you expect this number?

1. Define events & state goal

Let: E = you test positive
F = you actually have the disease

Want:
P(disease | test+)
$= P(F|E)$

Zika Testing

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$ Bayes' Theorem

- A test is 98% effective at detecting Zika ("true positive").
- However, the test has a "false positive" rate of 1%.
- 0.5% of the US population has Zika.

What is the likelihood you have Zika if you test positive?

Why would you expect this number?

1. Define events & state goal
2. Identify known probabilities
3. Solve

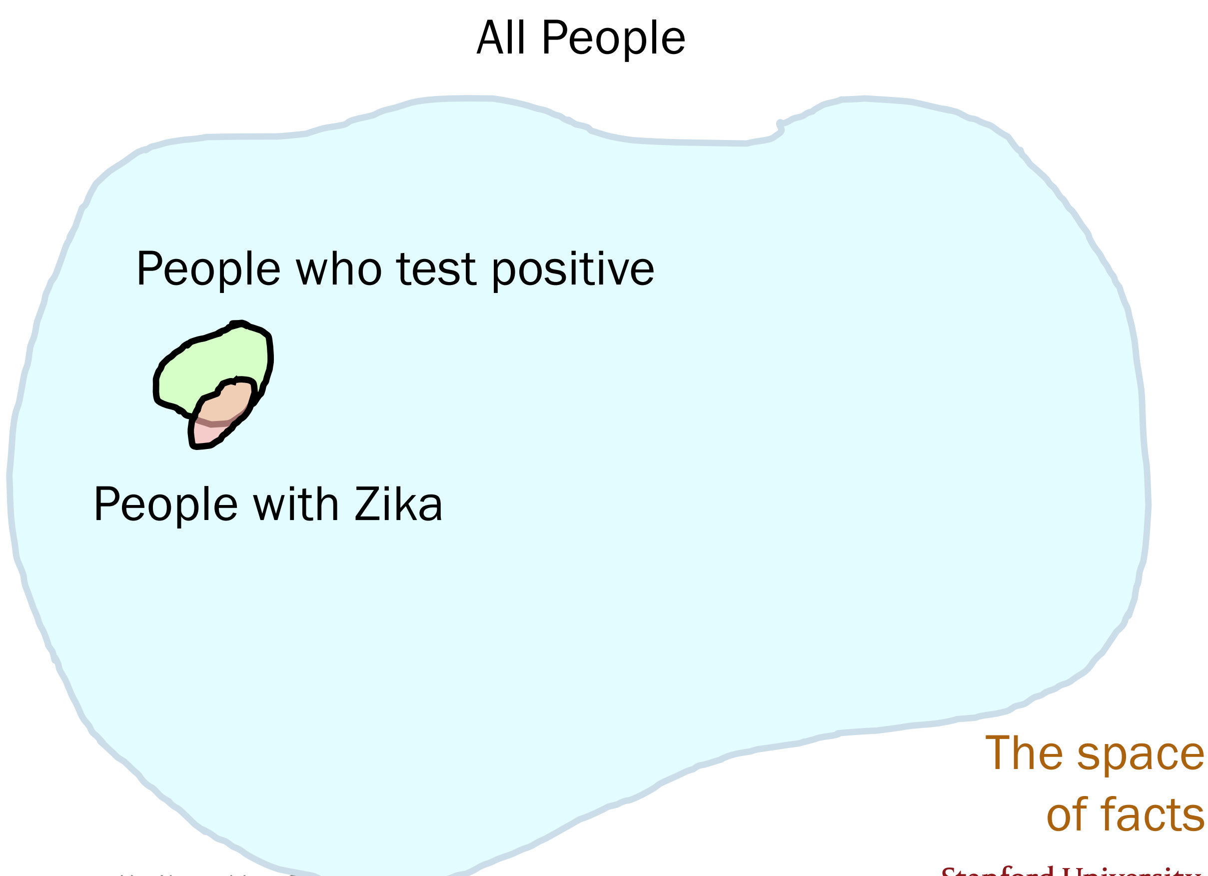
Let: E = you test positive
F = you actually have the disease

Want:
P(disease | test+)
$= P(F|E)$

Bayes' Theorem intuition

<u>Original question</u>:

What is the likelihood you have Zika if you test positive for the disease?



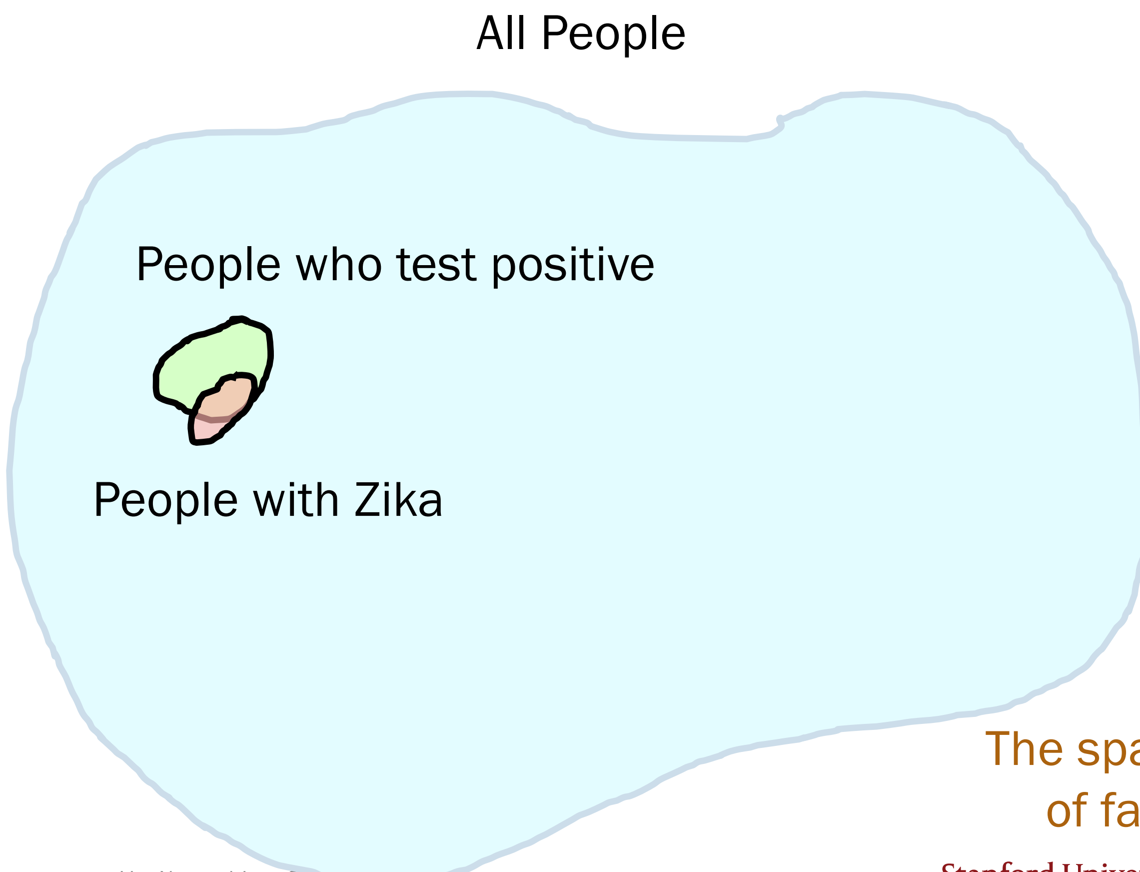
Bayes' Theorem intuition

Original question:

What is the likelihood you have Zika if you test positive for the disease?

Interpret

Interpretation:
Of the people who test positive, how many actually have Zika?



The space of facts

Bayes' Theorem intuition

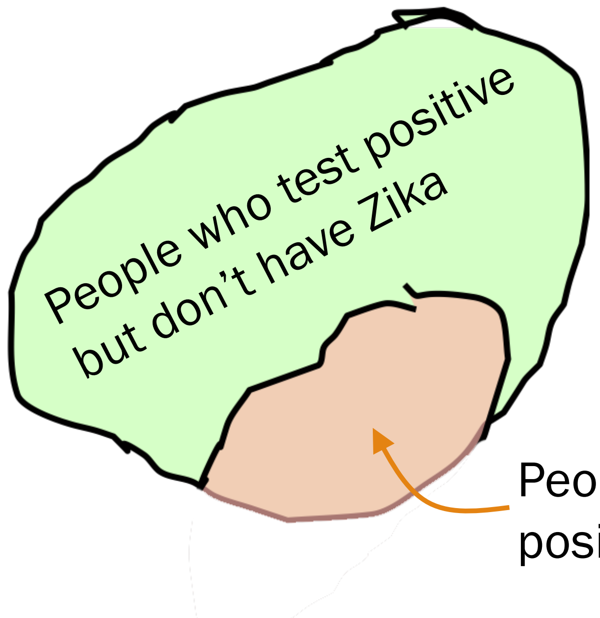
Original question:

What is the likelihood you have Zika if you test positive for the disease?

Interpret

Interpretation:
Of the people who test positive, how many actually have Zika?

People who test positive



People who test positive and have Zika

The space of facts, **conditioned** on a positive test result

Zika Testing

- A test is 98% effective at detecting Zika ("true positive").
- However, the test has a "false positive" rate of 1%.
- 0.5% of the US population has Zika.

What is the likelihood you have Zika if you test positive?

Say we have 1000 people:

5 have Zika
and test positive

985 **do not** have Zika
and test negative.

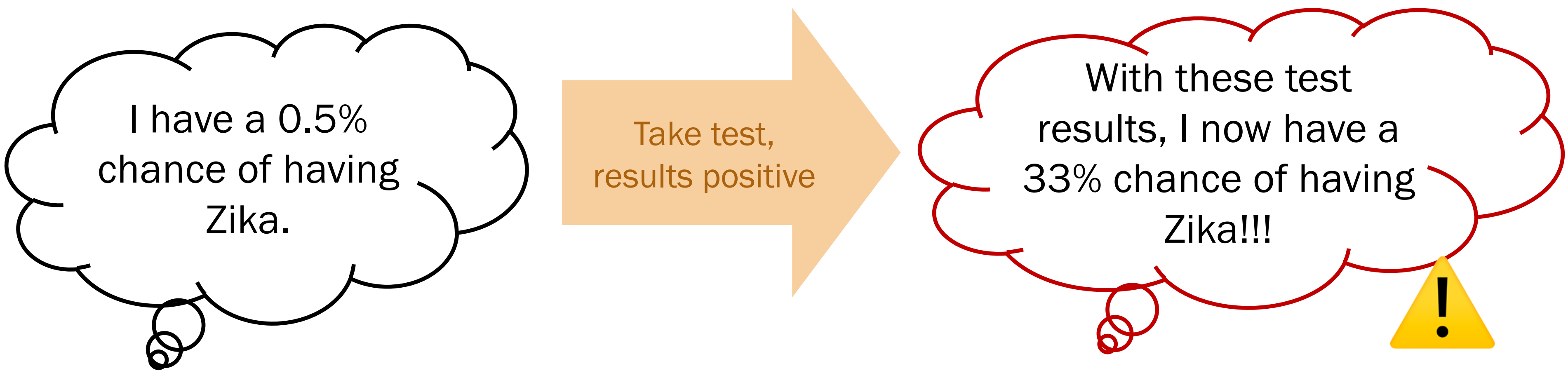
10 **do not** have Zika
and test positive.

$$\approx 0.333$$

Demo (class website)

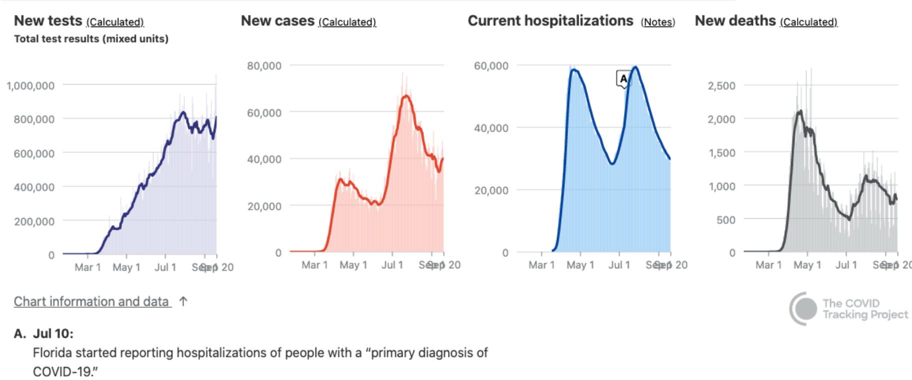
Update your beliefs with Bayes' Theorem

E = you test positive for Zika
F = you actually have the disease



$P(F)$ $P(F|E)$

Topical probability news: Bayes for COVID-19 testing



How representative are today's testing rates?

How do we know if a positive test is a true positive or a false positive?

Reasonable Question:Why test if there are errors?

https://covidtracking.com/data
http://med.stanford.edu/news.html

Think

Slide 53 is a question to think over by yourself.
We'll go over it together afterwards.

Post any clarifications here!

https://us.edstem.org/courses/2678/discussion/128395

Think by yourself: 2 minutes

(by yourself)

Why it's still good to get tested

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$ Bayes' Theorem

- A test is 98% effective at detecting Zika ("true positive").
- However, the test has a "false positive" rate of 1%.
- 0.5% of the US population has Zika.

Let: E = you test positive
F = you actually have the disease

Let: E^C = you test negative for Zika with this test.

	F, disease +	F^C, disease –
E, Test +	True positive $P(E|F) = 0.98$	False positive $P(E|F^C) = 0.01$

What is $P(F|E^C)$?

(by yourself)

Why it's still good to get tested

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$ Bayes' Theorem

- A test is 98% effective at detecting Zika ("true positive").
- However, the test has a "false positive" rate of 1%.
- 0.5% of the US population has Zika.

Let: E = you test positive
F = you actually have the disease

Let: E^C = you test negative for Zika with this test.

What is $P(F|E^C)$?

	F, disease +	F^C, disease –
E, Test +	True positive $P(E|F) = 0.98$	False positive $P(E|F^C) = 0.01$

Why it's still good to get tested

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$ Bayes' Theorem

- A test is 98% effective at detecting Zika ("true positive").
- However, the test has a "false positive" rate of 1%.
- 0.5% of the US population has Zika.

Let: E = you test positive
F = you actually have the disease

Let: E^C = you test negative for Zika with this test.

What is $P(F|E^C)$?

	F, disease +	F^C, disease –
E, Test +	True positive $P(E\|F) = 0.98$	False positive $P(E\|F^C) = 0.01$
E^C, Test –	False negative $P(E^C\|F) = 0.02$	True negative $P(E^C\|F^C) = 0.99$

$$P(F|E^C) = \frac{P(E^C|F)P(F)}{P(E^C|F)P(F) + P(E^C|F^C)P(F^C)}$$

Why it's still good to get tested

E = you test positive for Zika
F = you actually have the disease
E^C = you test negative for Zika

I have a **0.5%** chance of having Zika disease.

$P(F)$

Take test, results positive

With these test results, I now have a **33%** chance of having Zika!!!

$P(F|E)$

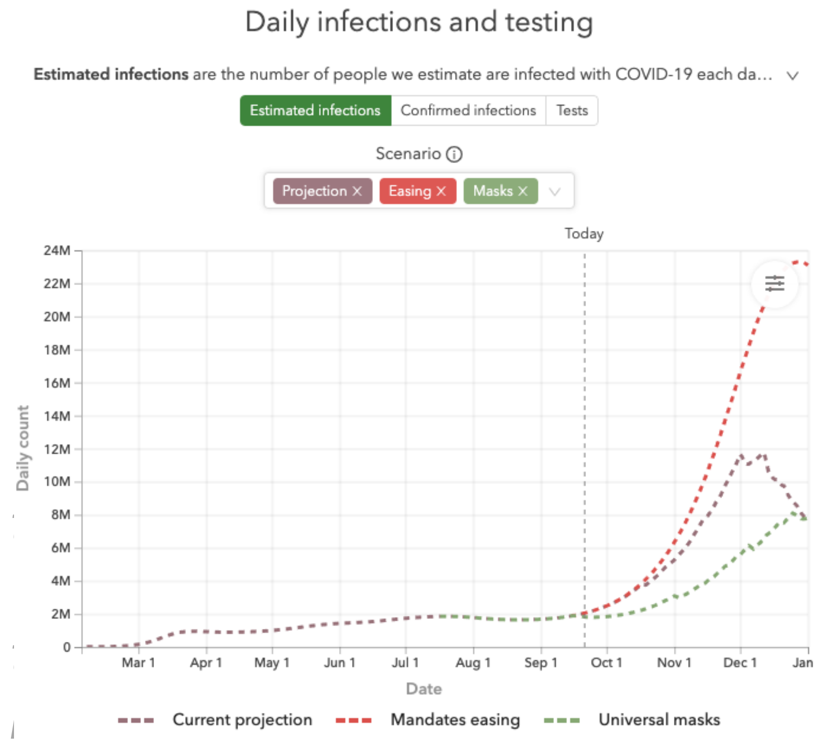
Take test, results negative

With these test results, I now have a **0.01%** chance of having Zika disease!!!

$P(F|E^C)$

Topical probability news: Bayes for COVID-19 testing

- Antibody tests (blood samples) have higher false negative, false positive rates than RT-PCR tests (nasal swab). However, they help explain/identify our body's reaction to the virus.
- The real world has many more "givens" (current symptoms, existing medical conditions) that improve our belief prior to testing.
- Most importantly, testing gives us a noisy signal of the spread of a disease.



Why test if there are errors?

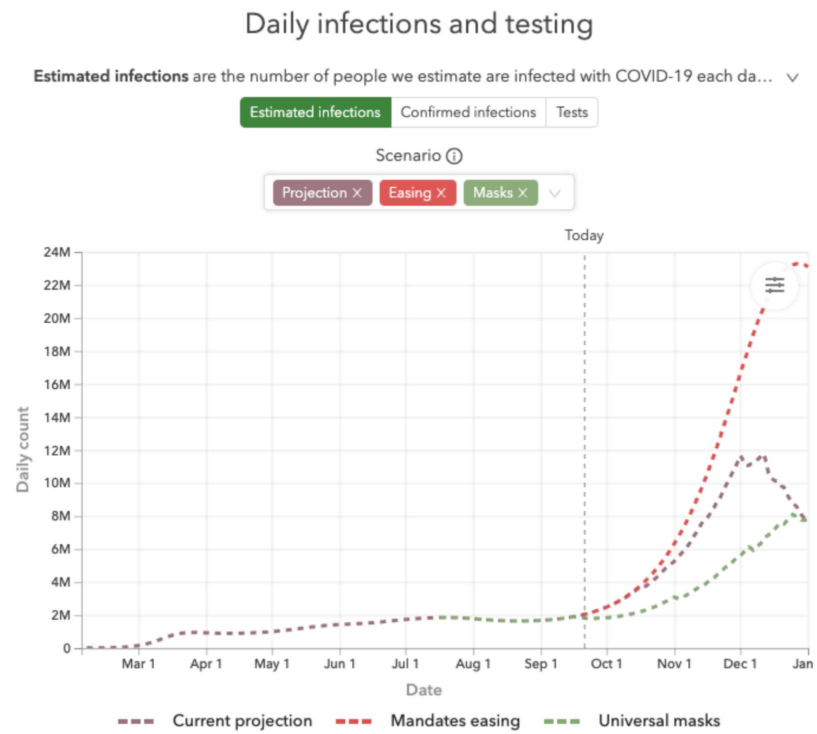
Topical probability news: Sources

COVID-19 Projections
http://covid19.healthdata.org/

Stanford Medicine (Sept 9 2020)
http://med.stanford.edu/news/all-news/2020/09/researchers-test-antibodies-as-covid-19-treatment.html

Overview of different testing types
https://www.globalbiotechinsights.com/articles/20247/the-worldwide-test-for-covid-19

Compilation of scientific publications on COVID-19 https://rega.kuleuven.be/if/corona_covid-19



LIVE

Monty Hall Problem

Monty Hall Problem

and Wayne Brady

Monty Hall Problem aka Let's Make a Deal

Behind one door is a prize (equally likely to be any door).

Behind the other two doors is nothing

1. We choose a door
2. Host opens 1 of other 2 doors, revealing nothing
3. We are given an option to change to the other door.

Doors A,B,C

Should we switch?

Note: If we don't switch, P(win) = 1/3 (random)

We are comparing P(win) and P(win|switch).

(by yourself)

If we switch

Without loss of generality, say we pick A (out of Doors A,B,C).

1/3 1/3 1/3

A = prize	B = prize	C = prize
• Host opens B or C • We switch • We always lose	• Host must open C • We switch to B • We always win	• Host must open B • We switch to C • We always win
P(win | A prize, picked A, switched) = 0	P(win | B prize, picked A, switched) = 1	P(win | C prize, picked A, switched) = 1

P(win | picked A, switched) = 1/3 * 0 + 1/3 * 1 + 1/3 * 1 = 2/3

You should switch.

Monty Hall, 1000 envelope version

Start with 1000 envelopes (of which 1 is the prize).

1. You choose 1 envelope.

   $\frac{1}{1000}$ = P(envelope is prize)

   $\frac{999}{1000}$ = P(other 999 envelopes have prize)

2. I open 998 of remaining 999 (showing they are empty).

   $\frac{999}{1000}$ = P(998 empty envelopes had prize) + P(last other envelope has prize)

   = P(last other envelope has prize)

3. Should you switch?

   No: P(win without switching) = $\frac{1}{\text{original \# envelopes}}$

   Yes: P(win with new knowledge) = $\frac{\text{original \# envelopes - 1}}{\text{original \# envelopes}}$