

o6: Random Variables

Lisa Yan and Jerry Cain
September 25, 2020

Quick slide reference

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Conditional Independence

Conditional Paradigm

For any events A, B, and E, you can condition consistently on E,
and all formulas still hold:

Axiom 1

$$0 \leq P(A|E) \leq 1$$

Corollary 1 (complement)

$$P(A|E) = 1 - P(A^c|E)$$

Transitivity

$$P(AB|E) = P(BA|E)$$

Chain Rule

$$P(AB|E) = P(B|E)P(A|BE)$$

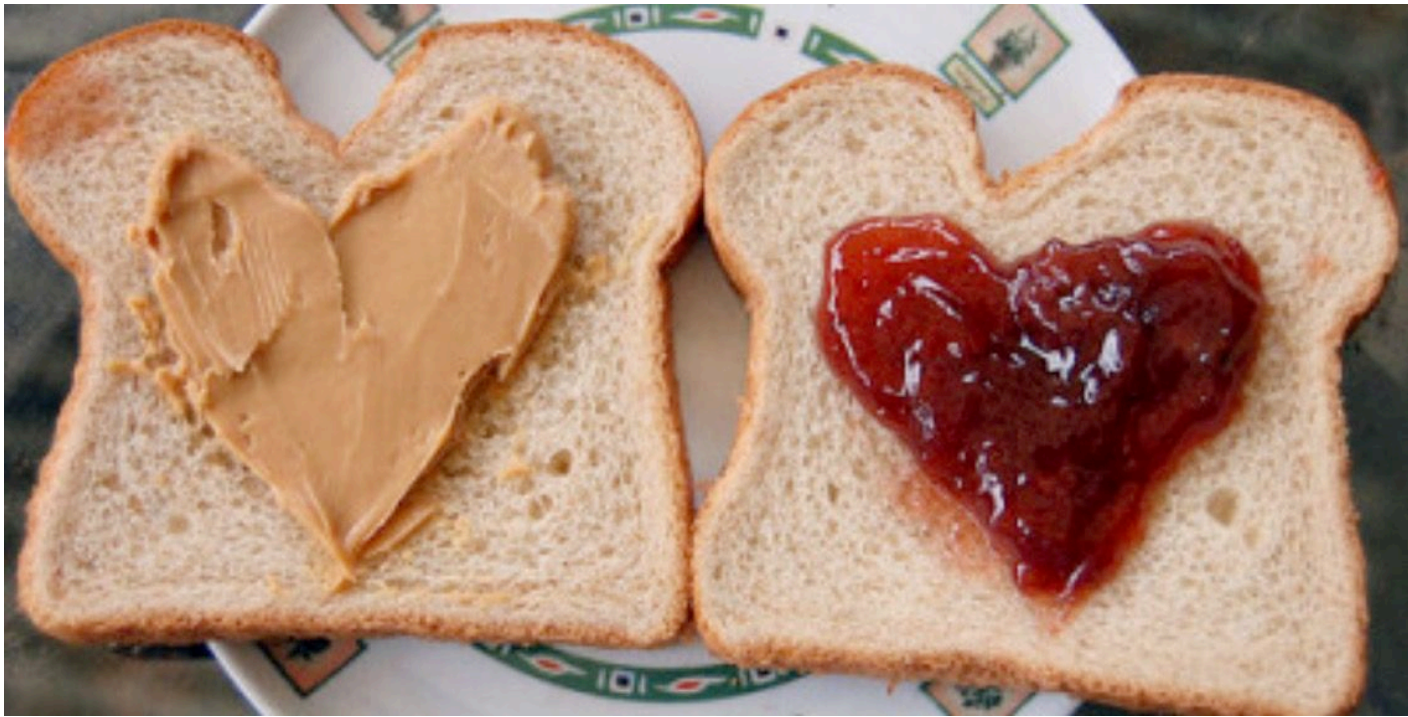
Bayes' Theorem

$$P(A|BE) = \frac{P(B|AE)P(A|E)}{P(B|E)}$$



BAE's theorem?

Conditional Independence



Conditional Probability

Independence

Conditional Independence

Independent events E and F \iff $P(EF) = P(E)P(F)$
 $P(E|F) = P(E)$

Two events A and B are defined as conditionally independent given E if:

$$P(AB|E) = P(A|E)P(B|E)$$

An equivalent definition:

- A. $P(A|B) = P(A)$
- B. $P(A|BE) = P(A)$
- C. $P(A|BE) = P(A|E)$



Conditional Independence

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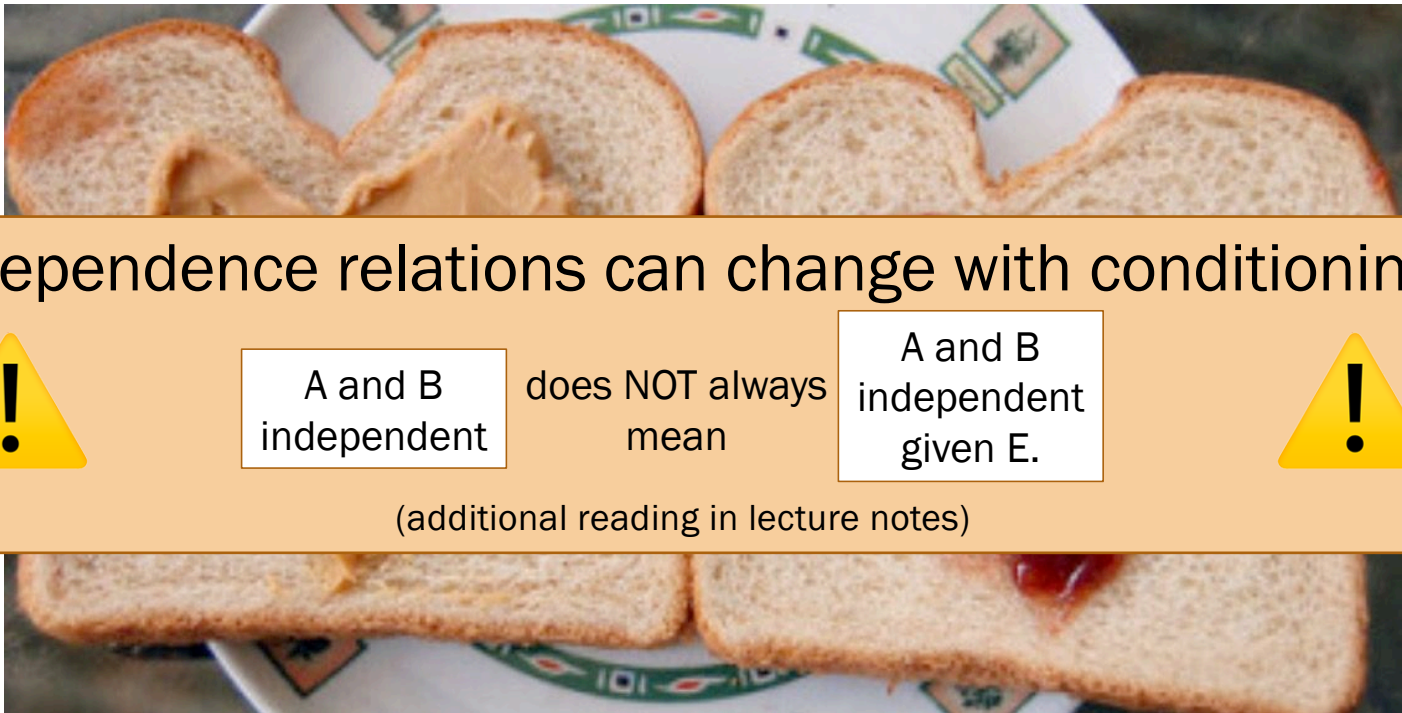
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$$P(AB|E) = P(A|E)P(B|E)$$

An equivalent definition:

- A. $P(A|B) = P(A)$
- B. $P(A|BE) = P(A)$
- C. $P(A|BE) = P(A|E)$

Conditional Independence



Independence relations can change with conditioning.



A and B
independent

does NOT always
mean

A and B
independent
given E.



(additional reading in lecture notes)

Conditional Probability

Independence

Netflix and Condition

Review

Let E = a user watches Life is Beautiful.

Let F = a user watches Amelie.

What is $P(E)$?

$$P(E) \approx \frac{\# \text{ people who have watched movie}}{\# \text{ people on Netflix}} = \frac{10,234,231}{50,923,123} \approx 0.20$$



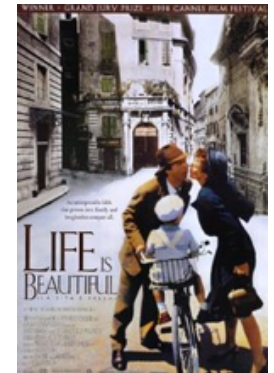
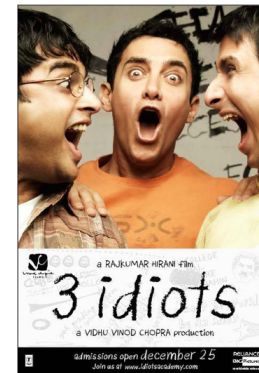
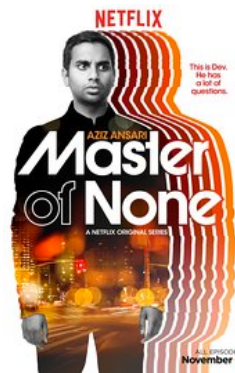
What is the probability that a user watches Life is Beautiful, given they watched Amelie?

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\# \text{ people who have watched both}}{\# \text{ people who have watched Amelie}} \approx 0.42$$

Netflix and Condition

Review

Let E be the event that a user watches the given movie.
Let F be the event that the same user watches Amelie.



$$P(E) = 0.19$$

$$P(E) = 0.32$$

$$P(E) = 0.20$$

$$P(E) = 0.09$$

$$P(E) = 0.20$$

$$P(E|F) = 0.14$$

$$P(E|F) = 0.35$$

$$P(E|F) = 0.20$$

$$P(E|F) = 0.72$$

$$P(E|F) = 0.42$$

Lisa Ye Independent!

Netflix and Condition (on many movies)

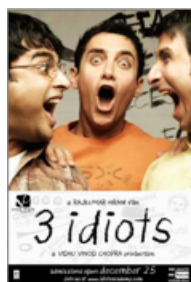
Watched:



E_1

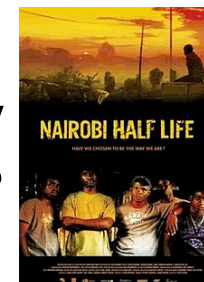


E_2



E_3

Will they
watch?



E_4

What if $E_1E_2E_3E_4$ are not independent? (e.g., all international emotional comedies)

$$P(E_4|E_1E_2E_3) = \frac{P(E_1E_2E_3E_4)}{P(E_1E_2E_3)} = \frac{\# \text{ people who have watched all 4}}{\# \text{ people who have watched those 3}}$$

We need to keep track of an exponential number of movie-watching statistics

Netflix and Condition (on many movies)

K : likes international emotional comedies

Watched:



E_1

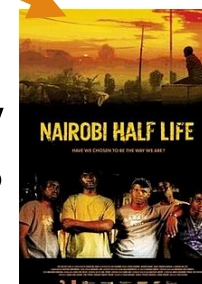


E_2



E_3

Will they watch?



E_4

Assume: $E_1 E_2 E_3 E_4$ are conditionally independent given K

$$P(E_4 | E_1 E_2 E_3) = \frac{P(E_1 E_2 E_3 E_4)}{P(E_1 E_2 E_3)} \quad P(E_4 | E_1 E_2 E_3 K) = \underbrace{P(E_4 | K)}$$

An easier probability to store and compute!

Conditional independence is a Big Deal

Conditional independence is a practical, real-world way of decomposing hard probability questions.

“Exploiting conditional independence to generate fast probabilistic computations is one of the main contributions CS has made to probability theory.”

–Judea Pearl wins 2011 Turing Award,
“For fundamental contributions to artificial intelligence through the development of a calculus for probabilistic and causal reasoning”

Netflix and Condition

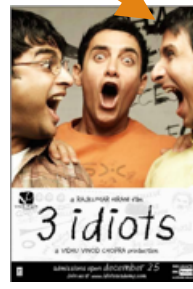
K : likes international emotional comedies



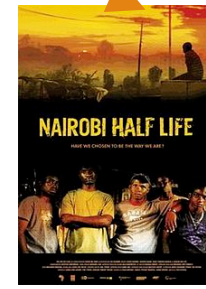
E_1



E_2



E_3



E_4

Challenge: How do we determine K ? Stay tuned in 6 weeks' time!

$E_1 E_2 E_3 E_4$ are
dependent

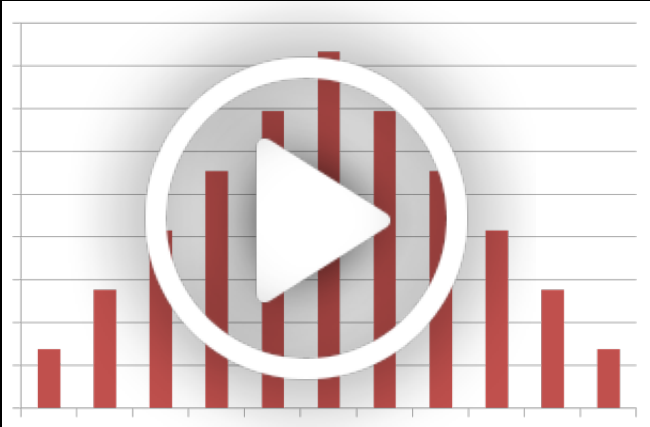
$E_1 E_2 E_3 E_4$ are
conditionally independent
given K

Dependent events can become conditionally independent.
And vice versa: Independent events can become conditionally dependent.

Random Variables

Next Episode Playing in 5 seconds

$P(X = k)$



$E[X]$

[Back to Browse](#)

[More Episodes](#)

Random variables are like typed variables

type name value
int a = 5;

double b = 4.2;

bit c = 1;

CS variables

A is the number of Pokemon we bring to our *future* battle.

$$A \in \{1, 2, \dots, 6\}$$



B is the amount of money we get *after* we win a battle.

$$B \in \mathbb{R}^+$$



C is 1 if we successfully beat the Elite Four. 0 otherwise.

$$C \in \{0, 1\}$$

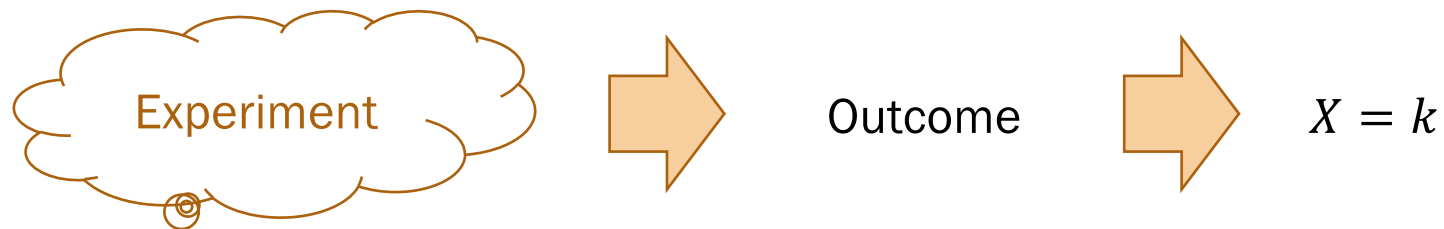


Random variables are like typed variables (with uncertainty)

Random variables

Random Variable

A **random variable** is a real-valued function defined on a sample space.



Example:

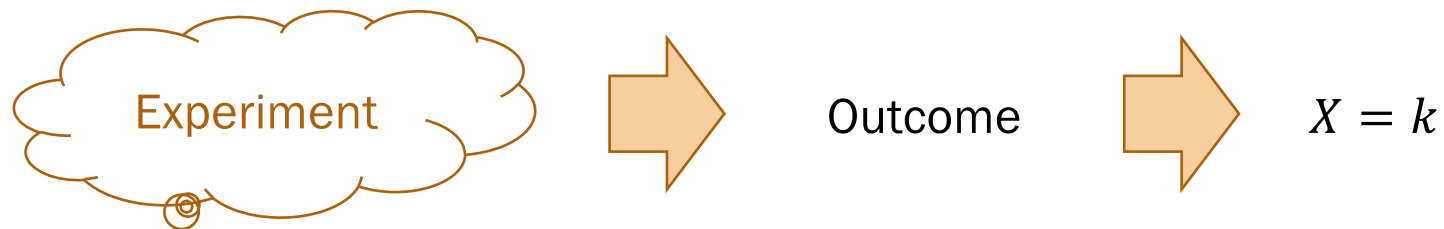
3 coins are flipped.
Let $X = \#$ of heads.
 X is a **random variable**.

1. What is the value of X for the outcomes:
 - (T,T,T)?
 - (H,H,T)?
2. What is the event (set of outcomes) where $X = 2$?
3. What is $P(X = 2)$?



Random Variable

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Random variables are **NOT** events!

It is confusing that random variables and events use the same notation.

- Random variables \neq events.
- We can define an event to be a particular assignment of a random variable.

Example:

3 coins are flipped.
Let $X = \#$ of heads.
 X is a **random variable**.

$X = 2$
event

$P(X = 2)$
probability
(**number** b/t 0 and 1)

Random variables are **NOT** events!

It is confusing that random variables and events use the same notation.

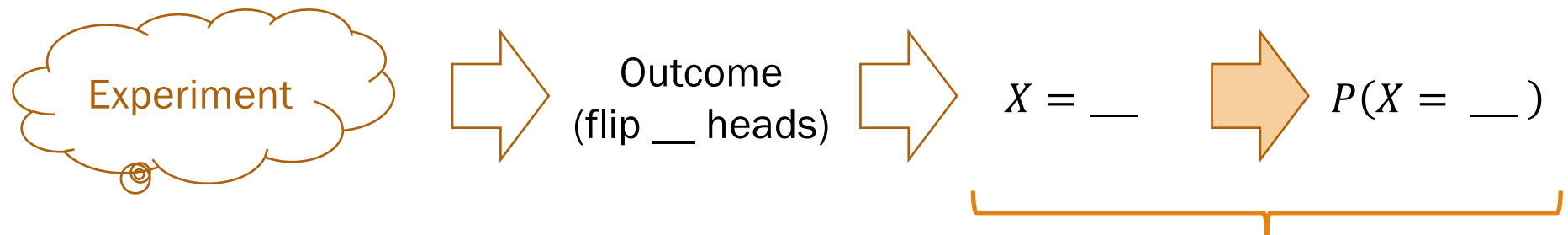
- Random variables \neq events.
- We can define an event to be a particular assignment of a random variable.

	$X = x$	Set of outcomes	$P(X = k)$
Example: 3 coins are flipped. Let $X = \#$ of heads. X is a random variable.	$X = 0$	$\{(T, T, T)\}$	$1/8$
	$X = 1$	$\{(H, T, T), (T, H, T), (T, T, H)\}$	$3/8$
	$X = 2$	$\{(H, H, T), (H, T, H), (T, H, H)\}$	$3/8$
	$X = 3$	$\{(H, H, H)\}$	$1/8$
	$X \geq 4$	$\{\}$	0

PMF/CDF

So far

3 coins are flipped. Let $X = \#$ of heads. X is a random variable.



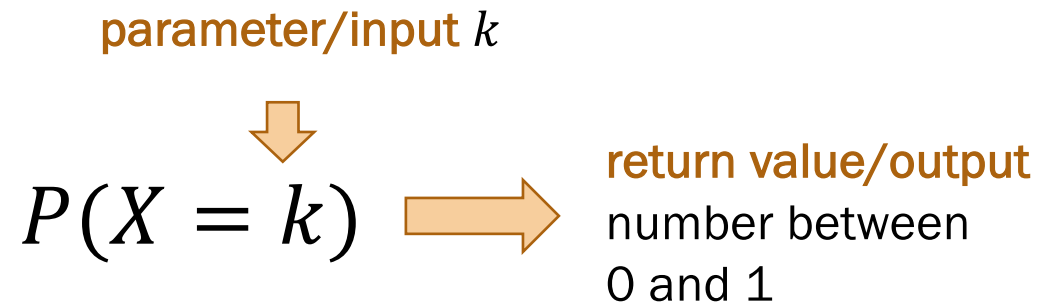
$X = x$	$P(X = k)$	Set of outcomes
$X = 0$	$1/8$	$\{(T, T, T)\}$
$X = 1$	$3/8$	$\{(H, T, T), (T, H, T), (T, T, H)\}$
$X = 2$	$3/8$	$\{(H, H, T), (H, T, H), (T, H, H)\}$
$X = 3$	$1/8$	$\{(H, H, H)\}$
$X \geq 4$	0	$\{\}$

Can we get a “shorthand” for this last step?
Seems like it might be useful!

Probability Mass Function

3 coins are flipped. Let $X = \#$ of heads. X is a random variable.

A **function** on k
with range $[0,1]$



What would be a *useful* function to define?

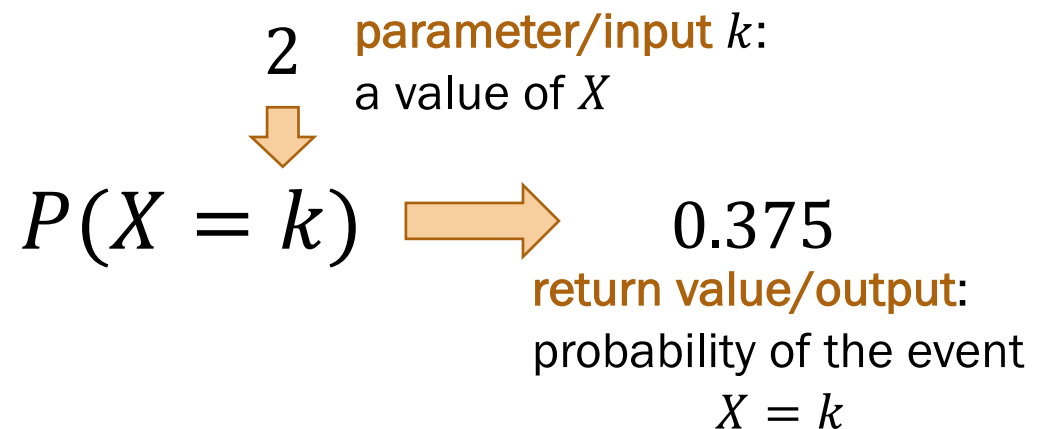
The probability of the event that a random variable X takes on the value k !

For **discrete random variables**, this is a **probability mass function**.

Probability Mass Function

3 coins are flipped. Let $X = \#$ of heads. X is a random variable.

A function on k
with range $[0,1]$



```
N = 3
P = 0.5

def prob_event_y_equals(k):
    n_ways = probability mass function
    p_way = probability mass function - P, N-k)
    return n_ways * p_way
```

Discrete RVs and Probability Mass Functions

A random variable X is **discrete** if it can take on countably many values.

- $X = x$, where $x \in \{x_1, x_2, x_3, \dots\}$

The **probability mass function** (PMF) of a discrete random variable is

$$P(X = x) = \underbrace{p(x)}_{\text{shorthand notation}} = \underbrace{p_X(x)}$$

- Probabilities must sum to 1:

$$\sum_{i=1}^{\infty} p(x_i) = 1$$

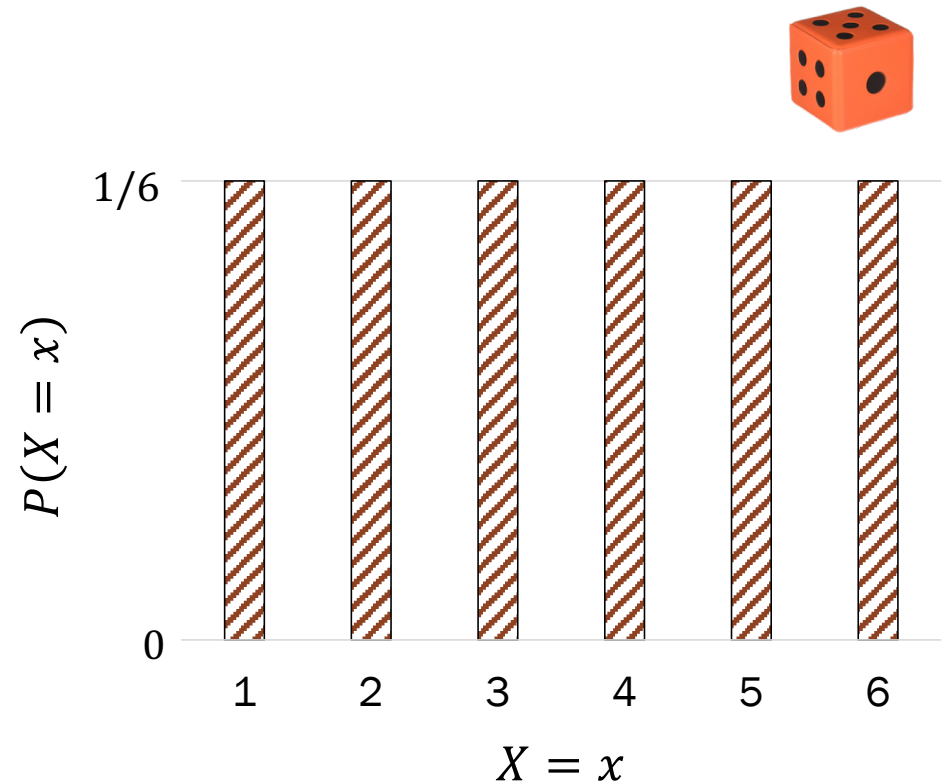
This last point is a good way to verify any PMF you create.

PMF for a single 6-sided die

Let X be a random variable that represents the result of a single dice roll.

- **Support** of X : $\{1, 2, 3, 4, 5, 6\}$
- Therefore X is a **discrete** random variable.
- PMF of X :

$$p(x) = \begin{cases} 1/6 & x \in \{1, \dots, 6\} \\ 0 & \text{otherwise} \end{cases}$$



Cumulative Distribution Functions

For a random variable X , the **cumulative distribution function** (CDF) is defined as

$$F(a) = F_X(a) = P(X \leq a), \text{ where } -\infty < a < \infty$$

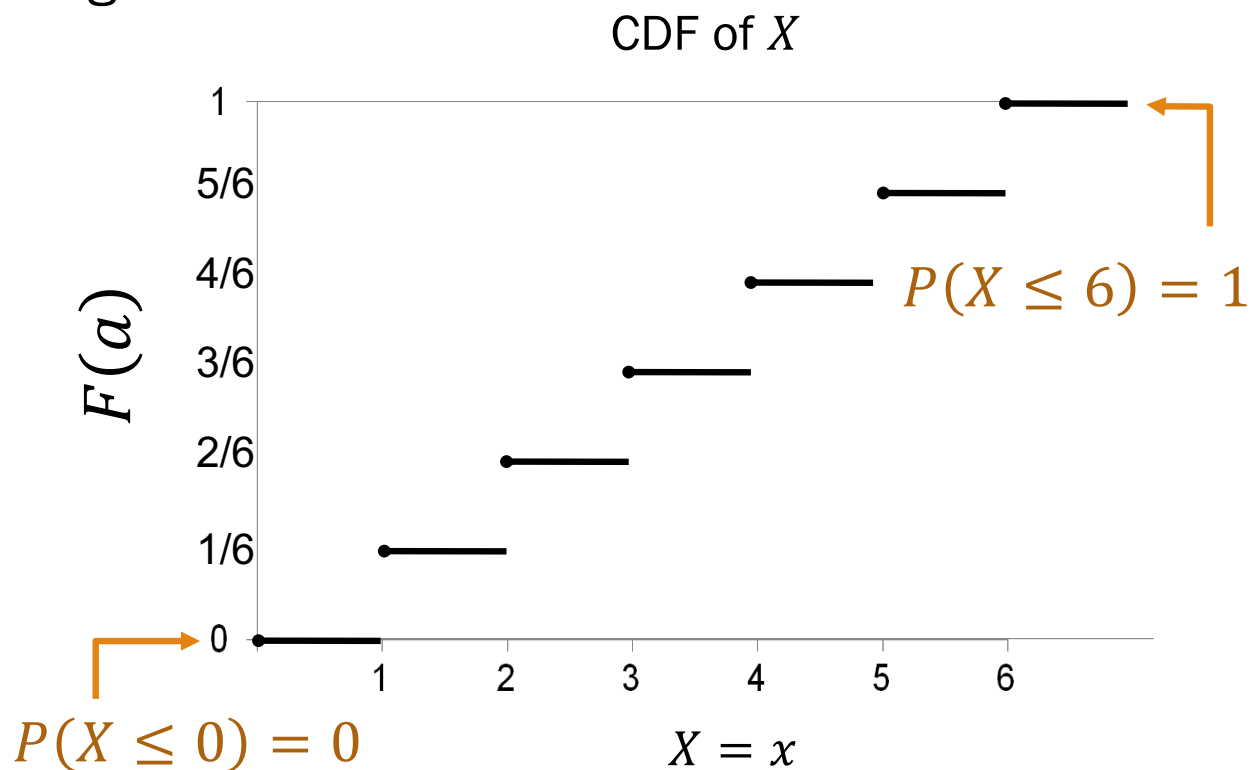
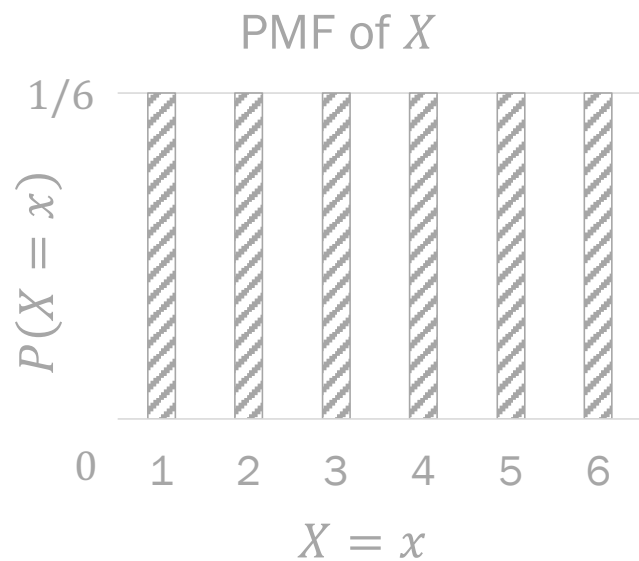
For a discrete RV X , the CDF is:

$$F(a) = P(X \leq a) = \sum_{\text{all } x \leq a} p(x)$$

CDFs as graphs

CDF (cumulative distribution function) $F(a) = P(X \leq a)$

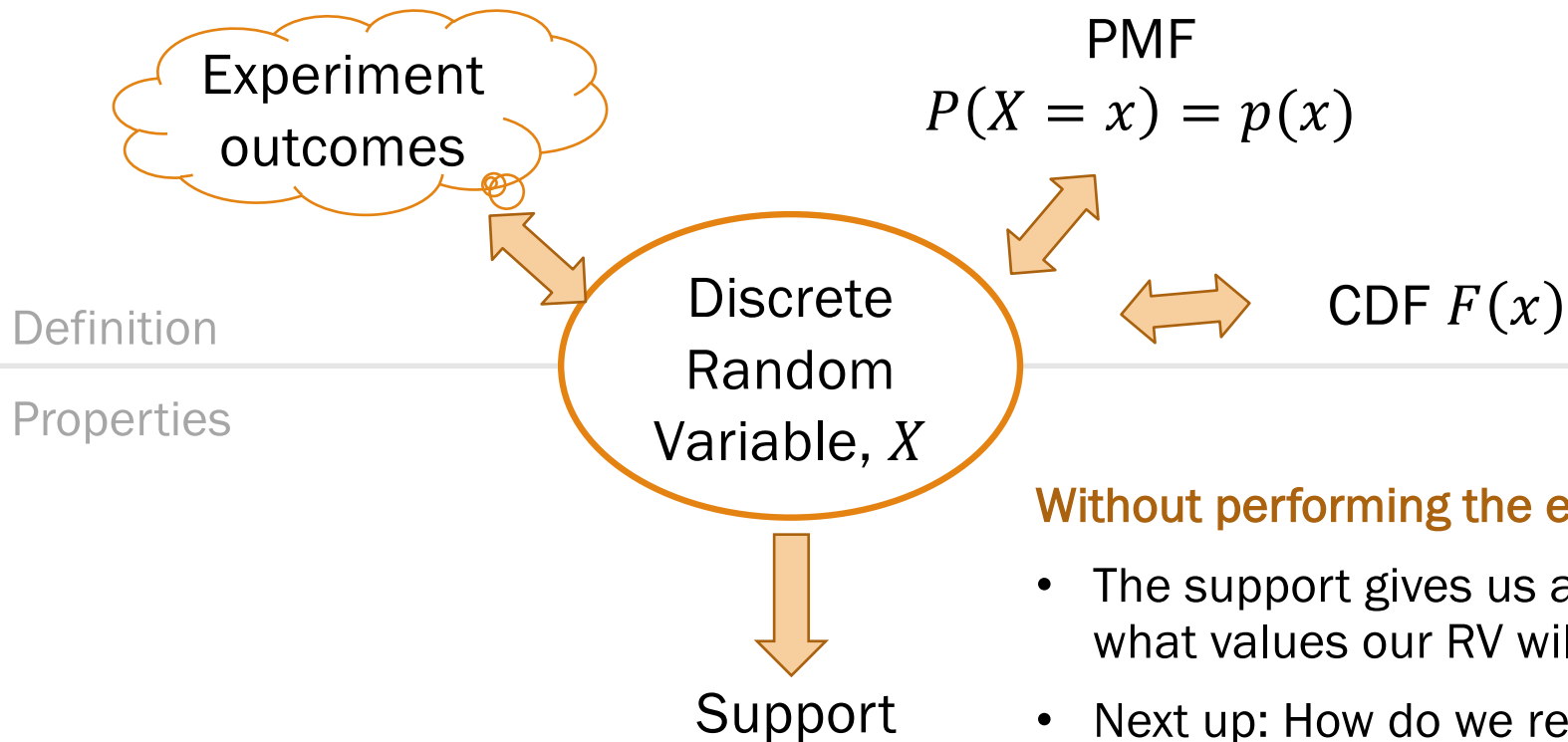
Let X be a random variable that represents the result of a single dice roll.



06d_expectation

Expectation

Discrete random variables



Without performing the experiment:

- The support gives us a ballpark of what values our RV will take on
- Next up: How do we report the “average” value?

Expectation

The **expectation** of a discrete random variable X is defined as:

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

- Note: sum over all values of $X = x$ that have non-zero probability.
- Other names: **mean**, expected value, **weighted average**, center of mass, first moment

Expectation of a die roll

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x \quad \text{Expectation of } X$$



What is the expected value of a 6-sided die roll?

1. Define random variables

$X =$ RV for value of roll

$$P(X = x) = \begin{cases} 1/6 & x \in \{1, \dots, 6\} \\ 0 & \text{otherwise} \end{cases}$$

2. Solve

$$E[X] = 1 \left(\frac{1}{6}\right) + 2 \left(\frac{1}{6}\right) + 3 \left(\frac{1}{6}\right) + 4 \left(\frac{1}{6}\right) + 5 \left(\frac{1}{6}\right) + 6 \left(\frac{1}{6}\right) = \frac{7}{2}$$

Important properties of expectation

1. Linearity:

$$E[aX + b] = aE[X] + b$$

- Let $X = 6$ -sided dice roll,
 $Y = 2X - 1$.
- $E[X] = 3.5$
- $E[Y] = 6$

2. Expectation of a sum = sum of expectation:

$$E[X + Y] = E[X] + E[Y]$$

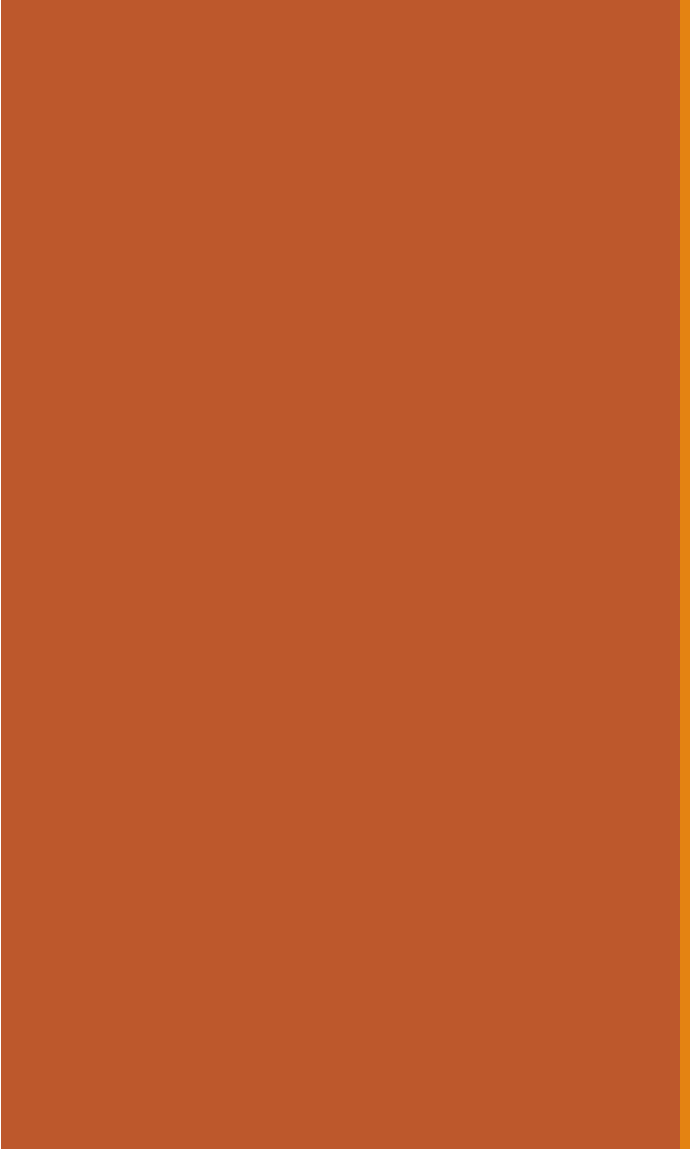
Sum of two dice rolls:

- Let $X =$ roll of die 1
 $Y =$ roll of die 2
- $E[X + Y] = 3.5 + 3.5 = 7$

3. Unconscious statistician:

$$E[g(X)] = \sum_x g(x)p(x)$$

These properties let you avoid defining difficult PMFs.



Proofs (OK to
stop here)

Important properties of expectation

Review

1. Linearity:

$$E[aX + b] = aE[X] + b$$

2. Expectation of a sum = sum of expectation:

$$E[X + Y] = E[X] + E[Y]$$

3. Unconscious statistician:

$$E[g(X)] = \sum_x g(x)p(x)$$

Linearity of Expectation proof

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

$$E[aX + b] = aE[X] + b$$

Proof:

$$\begin{aligned} E[aX + b] &= \sum_x (ax + b)p(x) = \sum_x axp(x) + bp(x) \\ &= a \sum_x xp(x) + b \sum_x p(x) \\ &= aE[X] + b \cdot 1 \end{aligned}$$

Expectation of Sum intuition

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

$$E[X + Y] = E[X] + E[Y] \quad (\text{we'll prove this in two weeks})$$

Intuition
for now:

X	Y	$X + Y$
3	6	9
2	4	6
6	12	18
10	20	30
-1	-2	-3
0	0	0
8	16	24

Average:

$$\frac{1}{n} \sum_{i=1}^n x_i + \frac{1}{n} \sum_{i=1}^n y_i = \frac{1}{n} \sum_{i=1}^n (x_i + y_i)$$
$$\frac{1}{7} (28) + \frac{1}{7} (56) = \frac{1}{7} (84)$$

LOTUS proof

$$E[g(X)] = \sum_x g(x)p(x) \quad \text{Expectation of } g(X)$$

Let $Y = g(X)$, where g is a real-valued function.

$$\begin{aligned} E[g(X)] &= E[Y] = \sum_j y_j p(y_j) \\ &= \sum_j y_j \sum_{i:g(x_i)=y_j} p(x_i) \\ &= \sum_j \sum_{i:g(x_i)=y_j} y_j p(x_i) \\ &= \sum_j \sum_{i:g(x_i)=y_j} g(x_i) p(x_i) \\ &= \sum_i g(x_i) p(x_i) \end{aligned}$$

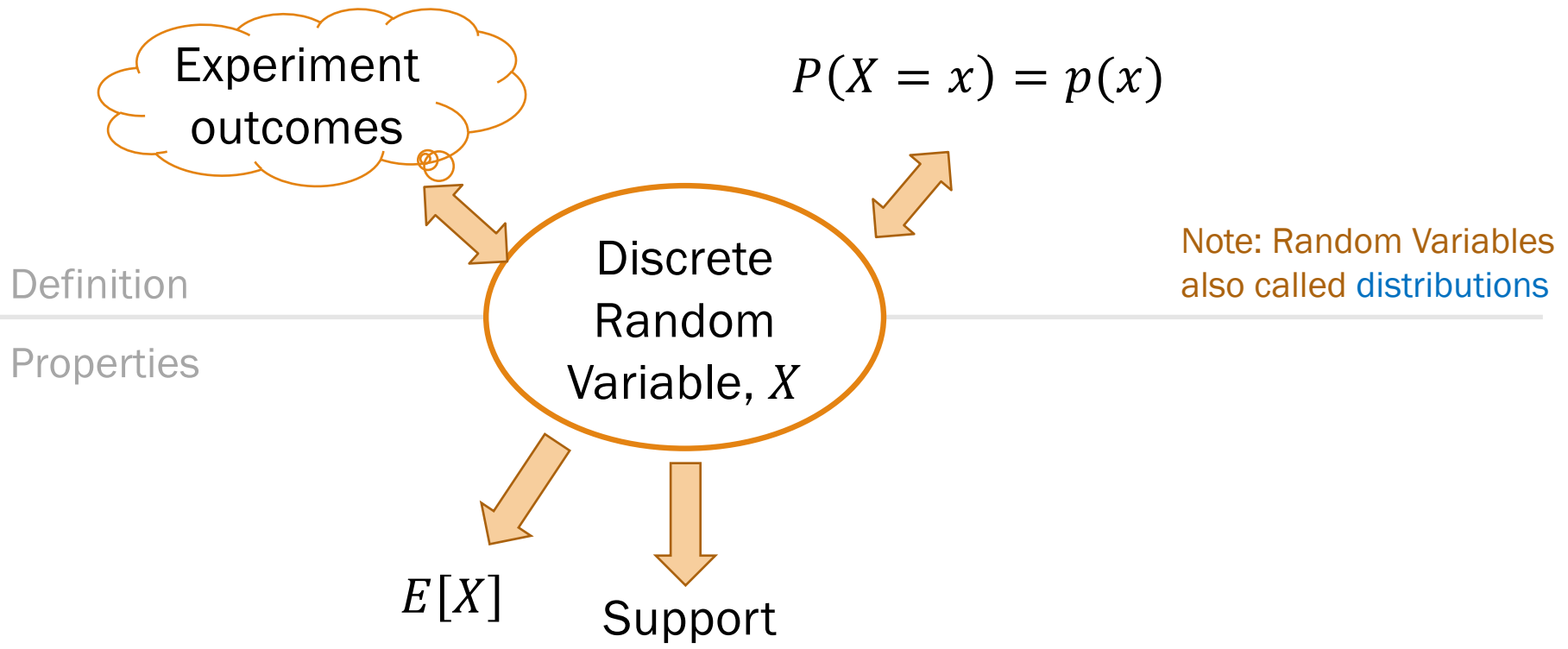
Lisa Yan and Jerry Cain, CS109, 2020

For you to review
so that you can
sleep at night

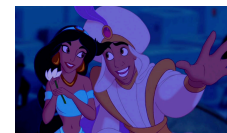
o6: Random Variables (live)

Lisa Yan and Jerry Cain
September 25, 2020

Discrete random variables



A Whole New World with Random Variables



Event-driven probability

- Relate only binary events
 - Either happens (E)
 - or doesn't happen (E^C)
- Can only report probability
- Lots of combinatorics



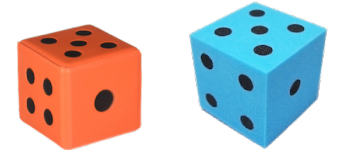
Random Variables

- Link multiple similar events together ($X = 1, X = 2, \dots, X = 6$)
- Can compute statistics: report the “average” outcome
- Once we have the PMF (discrete RVs), we can do regular math



PMF for the sum of two dice

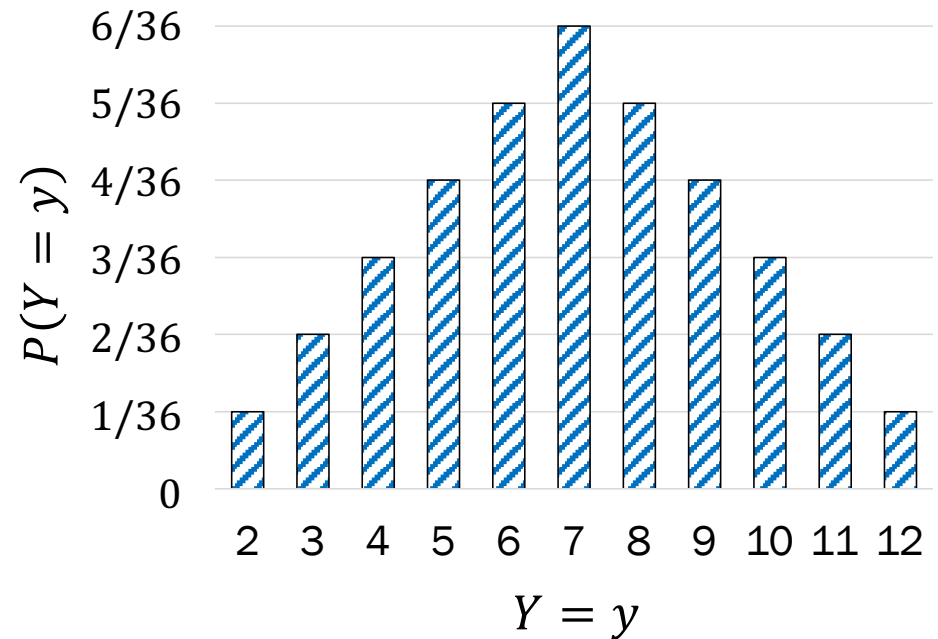
Let Y be a random variable that represents the sum of two independent dice rolls.



Support of Y : $\{2, 3, \dots, 11, 12\}$

$$p(y) = \begin{cases} \frac{y-1}{36} & y \in \mathbb{Z}, 2 \leq y \leq 6 \\ \frac{13-y}{36} & y \in \mathbb{Z}, 7 \leq y \leq 12 \\ 0 & \text{otherwise} \end{cases}$$

Sanity check: $\sum_{y=2}^{12} p(y) = 1$



Think, then Breakout Rooms

Then check out the question on the next slide (Slide 45). Post any clarifications here!

<https://us.edstem.org/courses/2678/discussion/128397>

Think by yourself: 1 min

Breakout rooms: 3 min. Introduce yourself!
(though feel free to leave at any time)



Example random variable

Consider 5 flips of a coin which comes up heads with probability p . Each coin flip is an independent trial. **Let $Y = \#$ of heads on 5 flips.**

1. What is the **support** of Y ? In other words, what are the values that Y can take on with non-zero probability?
2. Define the event $Y = 2$. What is $P(Y = 2)$?
3. What is the PMF of Y ? In other words, what is $P(Y = k)$, for k in the support of Y ?



Example random variable

Consider 5 flips of a coin which comes up heads with probability p . Each coin flip is an independent trial. Let $Y = \#$ of heads on 5 flips.

1. What is the **support** of Y ? In other words, what are the values that Y can take on with non-zero probability?

$$\{0, 1, 2, 3, 4, 5\}$$

2. Define the event $Y = 2$. What is $P(Y = 2)$?

$$\begin{aligned} \text{HHTTT} &\rightarrow p^2(1-p)^3 \\ \text{HTTTH} &\rightarrow p^2(1-p)^3 \end{aligned}$$

$$P(Y = k) = \binom{5}{k} p^k (1-p)^{5-k}$$

3. What is the PMF of Y ? In other words, what is $P(Y = k)$, for k in the support of Y ?

$$P(Y = k) = \binom{5}{k} p^k (1-p)^{5-k}$$

Expectation

Review

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

Expectation: The **average value** of a random variable

Remember that the expectation of a die roll is 3.5.

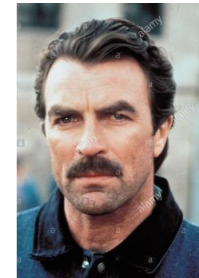
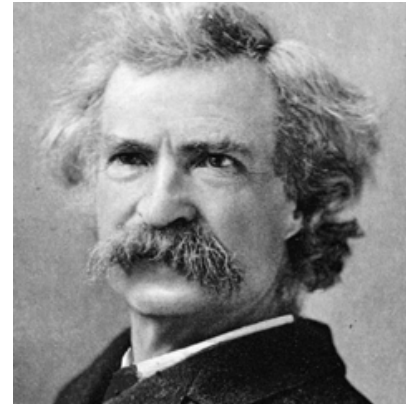


X = RV for value of roll

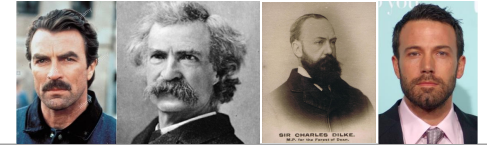
$$E[X] = 1 \left(\frac{1}{6}\right) + 2 \left(\frac{1}{6}\right) + 3 \left(\frac{1}{6}\right) + 4 \left(\frac{1}{6}\right) + 5 \left(\frac{1}{6}\right) + 6 \left(\frac{1}{6}\right) = \frac{7}{2}$$

Lying with statistics

“There are three kinds of lies:
lies, damned lies, and statistics”
–popularized by Mark Twain, 1906
–generally attributed to Sir
Charles Dilke, 1891



Lying with statistics



A school has 3 classes with 5, 10, and 150 students.
What is the average class size?

1. Interpretation #1

- Randomly choose a class with equal probability.
- X = size of chosen class

$$\begin{aligned} E[X] &= 5 \left(\frac{1}{3} \right) + 10 \left(\frac{1}{3} \right) + 150 \left(\frac{1}{3} \right) \\ &= \frac{165}{3} = 55 \end{aligned}$$

What universities usually report

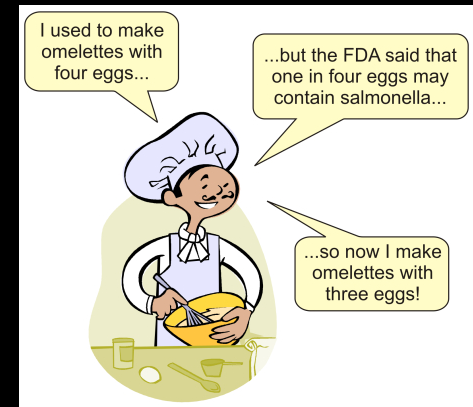
2. Interpretation #2

- Randomly choose a student with equal probability.
- Y = size of chosen class

$$\begin{aligned} E[Y] &= 5 \left(\frac{5}{165} \right) + 10 \left(\frac{10}{165} \right) + 150 \left(\frac{150}{165} \right) \\ &= \frac{22635}{165} \approx 137 \end{aligned}$$

Average student perception of class size

Interlude for announcements



Announcements

Problem Set #2

Out: today
Due: **Monday** 10/5, 1:00pm
Covers: through today

Python tutorial #2

When: Wed 9/30 3:30-4:30pm PT
Recorded? Yes
Covers: PS2 content
Notes: to be posted [online](#)

Important properties of expectation

Review

1. Linearity:

$$E[aX + b] = aE[X] + b$$

Roll a die, outcome is X . You win $\$2X - 1$.
What are your expected winnings?

Let $X = 6$ -sided dice roll.

$$E[2X - 1] = 2(3.5) - 1 = 6$$

2. Expectation of a sum = sum of expectation:

$$E[X + Y] = E[X] + E[Y]$$

3. Unconscious statistician:

$$E[g(X)] = \sum_x g(x)p(x)$$

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What is the expectation of the sum of two dice rolls?

Let $X =$ roll of die 1, $Y =$ roll of die 2.

$$E[X + Y] = 3.5 + 3.5 = 7$$

3. Unconscious statistician:

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(next up)

Think, then Breakout Rooms

Then check out the question on the next slide (Slide 56). Post any clarifications here!

<https://us.edstem.org/courses/2678/discussion/128397>

Think by yourself: 2 min



Being a statistician unconsciously

$$E[g(X)] = \sum_x g(x)p(x) \quad \text{Expectation of } g(X)$$

Let X be a discrete random variable.

- $P(X = x) = \frac{1}{3}$ for $x \in \{-1, 0, 1\}$

Let $Y = |X|$. What is $E[Y]$?

A. $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot -1 = 0$

B. $E[Y] = E[0] = 0$

C. $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 = \frac{2}{3}$

D. $\frac{1}{3} \cdot |-1| + \frac{1}{3} \cdot |0| + \frac{1}{3} \cdot |1| = \frac{2}{3}$

E. C and D



Being a statistician unconsciously

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A. $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot -1 = 0 \quad E[X]$

B. $E[Y] = E[0] = 0 \quad E[E[X]]$

C. $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 = \frac{2}{3}$

D. $\frac{1}{3} \cdot |-1| + \frac{1}{3} \cdot |0| + \frac{1}{3} \cdot |1| = \frac{2}{3}$

E. C and D

1. Find PMF of Y : $p_Y(0) = \frac{1}{3}, p_Y(1) = \frac{2}{3}$
2. Compute $E[Y]$

- Use LOTUS by using PMF of X :
1. $P(X = x) \cdot |x|$
2. Sum up

Think

Then check out the question on the next slide (Slide 59). Post any clarifications here!

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Think by yourself: 2 min



St. Petersburg Paradox

$$E[g(x)] = \sum_x g(x)p(x) \quad \text{Expectation of } g(X)$$

- A fair coin (comes up “heads” with $p = 0.5$)
- Define Y = number of coin flips (“heads”) before first “tails”
- Casino pays you $\$2^Y$

How much would you bet to play? (How much can you expect to win?)

$$\frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 8$$

$$+ \dots + \frac{1}{2^{100}} \cdot 2^{100}$$

- A. ~~\$0.50~~
- B. \$1
- C. \$2
- D. \$4
- E. $\$ \infty$
- F. I wouldn't play



St. Petersburg Paradox

$$E[g(x)] = \sum_x g(x)p(x) \quad \text{Expectation of } g(X)$$

- A fair coin (comes up “heads” with $p = 0.5$)
- Define Y = number of coin flips (“heads”) before first “tails”
- Casino pays you $\$2^Y$

How much would you bet to play? (How much can you expect to win?)

1. Define random variables
For $i \geq 0$: $P(Y = i) = \left(\frac{1}{2}\right)^{i+1}$
Let W = your winnings, 2^Y .
2. Solve
$$E[W] = E[2^Y] = \left(\frac{1}{2}\right)^1 2^0 + \left(\frac{1}{2}\right)^2 2^1 + \left(\frac{1}{2}\right)^3 2^2 + \dots$$
$$= \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{i+1} 2^i = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right) = \infty$$

St. Petersburg + Reality

$$E[g(x)] = \sum_x g(x)p(x) \quad \text{Expectation of } g(X)$$

What if the casino has only \$65,536?

- Same game
- Define $Y = \#$ heads before first tails
- You win $W = \$2^Y$
- If you win \$65,536, the casino stops the game and closes.

1. Define random variables

For $i \geq 0$: $P(Y = i) = \left(\frac{1}{2}\right)^{i+1}$

Let $W =$ your winnings, 2^Y .

2. Solve

$$E[W] = \left(\frac{1}{2}\right)^1 2^0 + \left(\frac{1}{2}\right)^2 2^1 + \left(\frac{1}{2}\right)^3 2^2 + \dots$$

$$k = \log_2(65,536) = 16$$

$$\begin{aligned} &\longrightarrow \sum_{i=0}^k \left(\frac{1}{2}\right)^{i+1} 2^i = \sum_{i=0}^{16} \left(\frac{1}{2}\right) = 8.5 \end{aligned}$$

