07: Variance, Bernoulli, Binomial

Lisa Yan and Jerry Cain

September 28, 2020

Quick slide reference

Variance

Average annual weather

Stanford, CA

$E[\text{high}] = 68°\text{F}$

$E[\text{low}] = 52°\text{F}$

Washington, DC

$E[\text{high}] = 67°\text{F}$

$E[\text{low}] = 51°\text{F}$

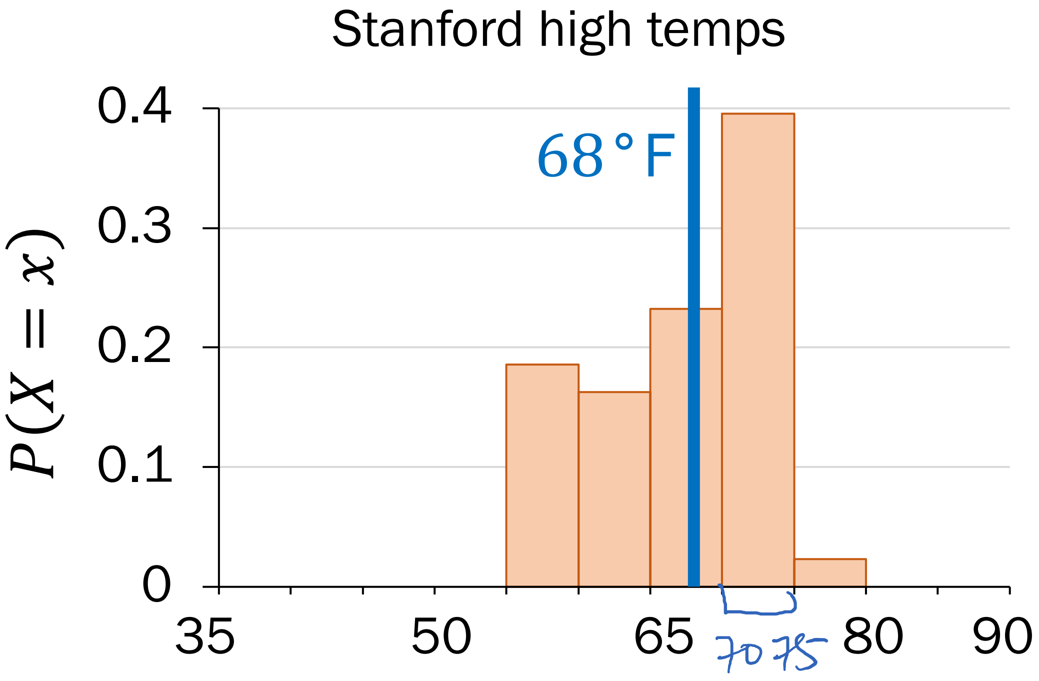
Is $E[X]$ enough?

Average annual weather

Stanford, CA

$E[\text{high}] = 68°\text{F}$

$E[\text{low}] = 52°\text{F}$



Washington, DC

$E[\text{high}] = 67°\text{F}$

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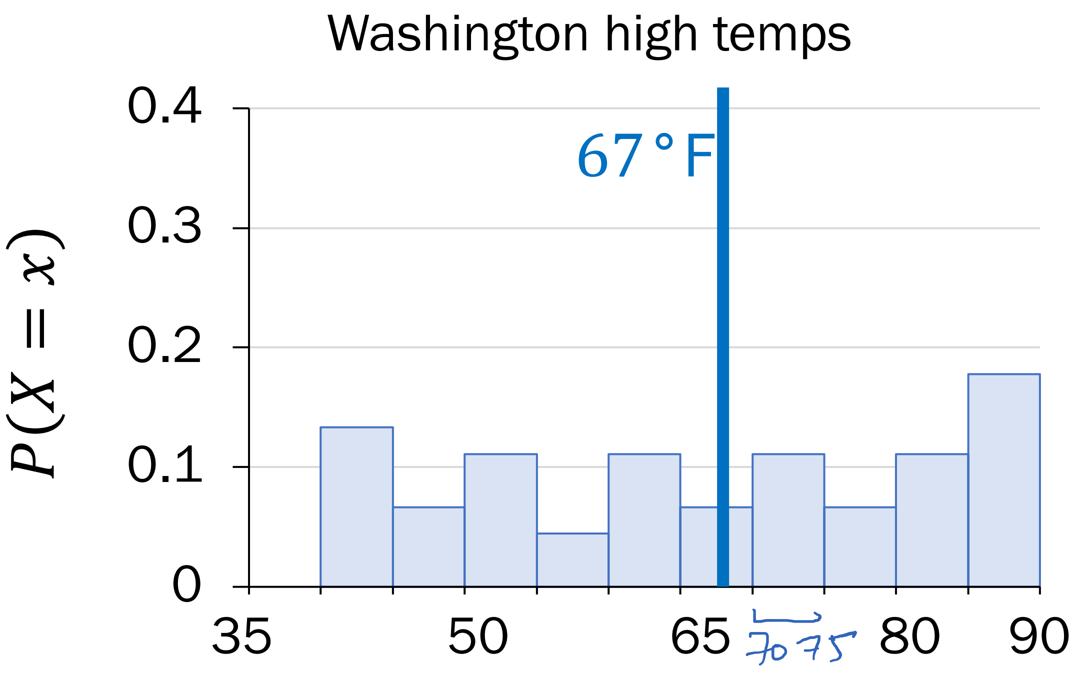
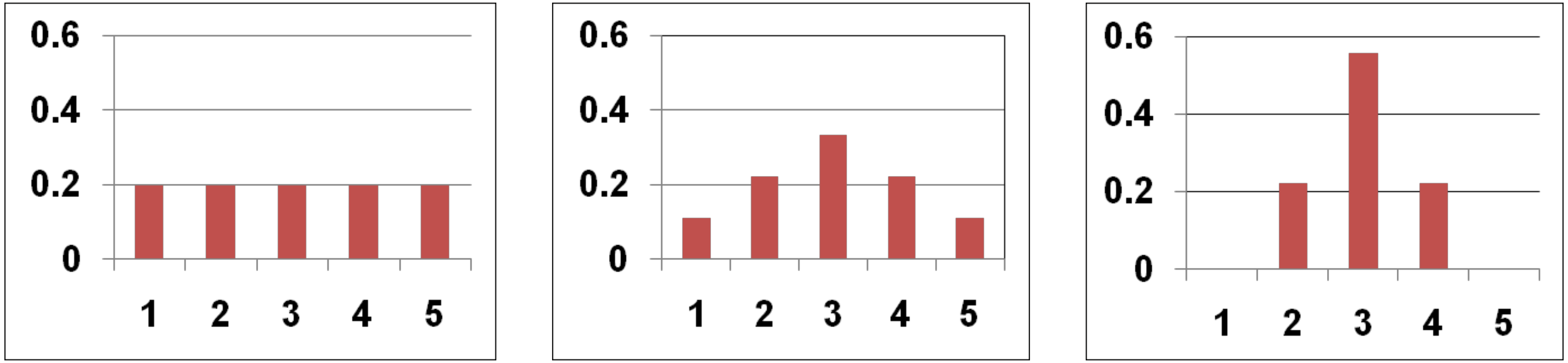


Normalized histograms are approximations of PMFs.

Variance = "spread"

Consider the following three distributions (PMFs):



- Expectation: $E[X] = 3$ for all distributions
- But the "spread" in the distributions is different!
- Variance, $\text{Var}(X)$: a formal quantification of "spread"

Variance RV

Variance

#, or value → (scalar)

The variance of a random variable X with mean $E[X] = \mu$ is

$$\text{Var}(X) = E[(X-\mu)^2]$$

average square of the difference b/t X & μ

- Also written as: $E[(X - E[X])^2]$ (value)
- Note: Var(X) ≥ 0
- Other names: **2nd central moment**, or square of the standard deviation

? $\mathbb{E}[(X-\mu)] = \mathbb{E}[X] - \mathbb{E}[\mu]$
$\mu \quad - \mu = 0$

	$\text{Var}(X)$	Units of X^2
def standard deviation	$\text{SD}(X) = \sqrt{\text{Var}(X)}$	Units of X ⇐

Variance of Stanford weather

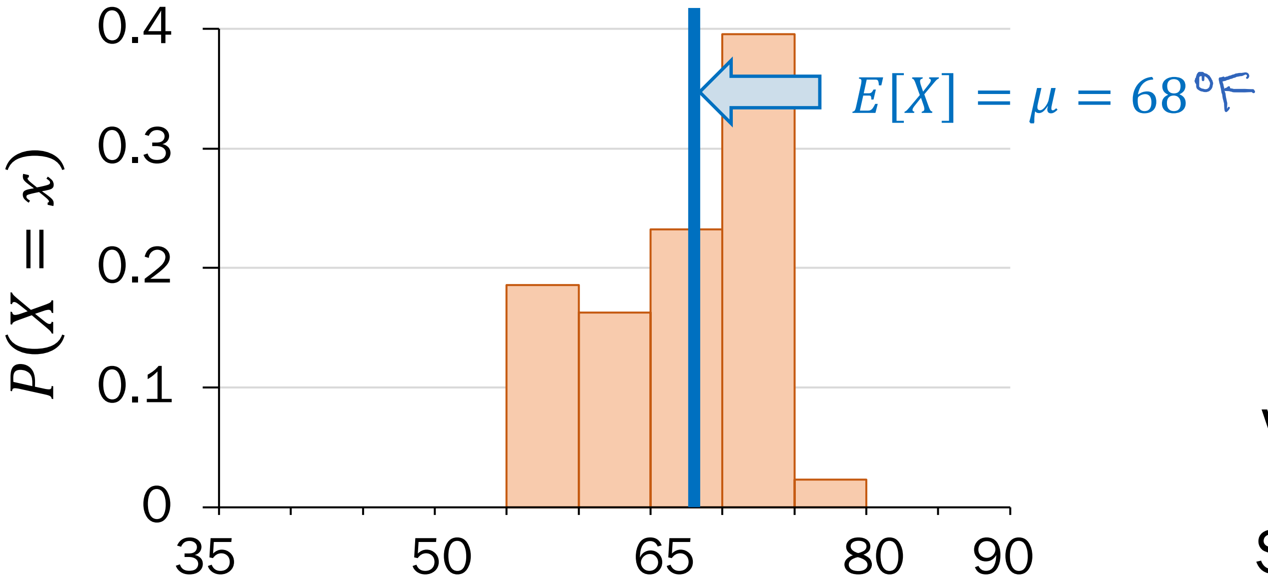
$\text{Var}(X) = E[(X - E[X])^2]$ Variance of X

Stanford, CA

$E[\text{high}] = 68°\text{F}$

$E[\text{low}] = 52°\text{F}$

Stanford high temps



X	$(X - \mu)^2$
57°F	124 (°F)^2
71°F	9 (°F)^2
75°F	49 (°F)^2
69°F	1 (°F)^2
…	…

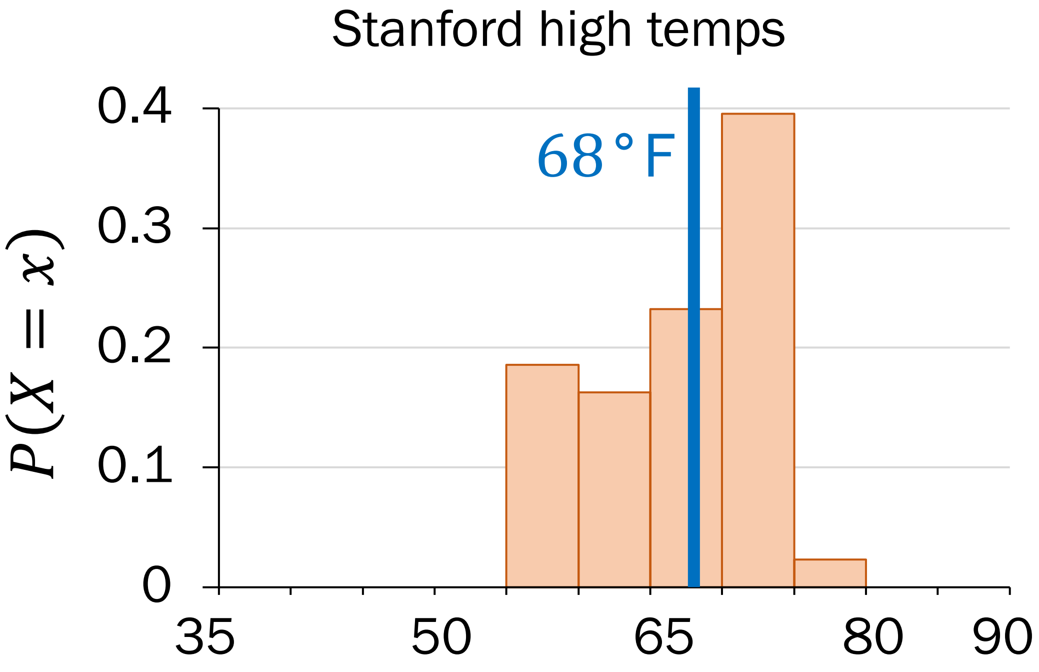
Variance $E[(X - \mu)^2] = 39\ (°\text{F})^2$

Standard deviation $= 6.2°\text{F}$

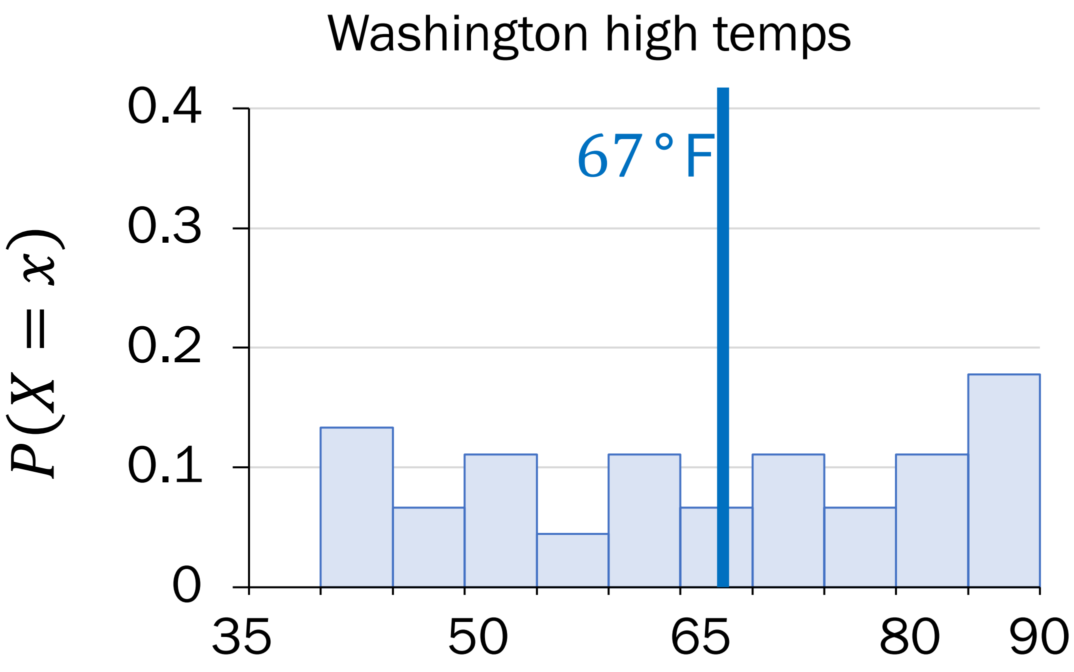
Comparing variance

$\text{Var}(X) = E[(X - E[X])^2]$ Variance of X

Stanford, CA	Washington, DC
$E[\text{high}] = 68°\text{F}$	$E[\text{high}] = 67°\text{F}$



$\text{Var}(X) = 39\ (°\text{F})^2$



$\text{Var}(X) = 248\ (°\text{F})^2$

Properties of Variance

Properties of variance

Definition $\text{Var}(X) = E[(X - E[X])^2]$ Units of X^2

def standard deviation $\text{SD}(X) = \sqrt{\text{Var}(X)}$ Units of X

Property 1 $\text{Var}(X) = E[X^2] - (E[X])^2$

Property 2 $\text{Var}(aX + b) = a^2\text{Var}(X)$

- Property 1 is often easier to compute than the definition
- Unlike expectation, variance is not linear

Properties of variance

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Computing variance, a proof

$$\text{Var}(X) = E[(X - E[X])^2]$$ Variance
$$= E[X^2] - (E[X])^2$$ of X

value

$$\text{Var}(X) = E[(X - E[X])^2] = E[(X-\mu)^2]$$

Let $E[X] = \mu$

$$= \sum_x (x-\mu)^2 p(x)$$

$P(X=x)$

$$= \sum_x (x^2 - 2\mu x + \mu^2) p(x)$$

$$= \sum_x x^2 p(x) - 2\mu \sum_x x p(x) + \mu^2 \sum_x p(x)$$

$$= E[X^2] - 2\mu E[X] + \mu^2 \cdot 1$$

$$= E[X^2] - 2\mu^2 + \mu^2$$

$$= E[X^2] - \mu^2$$

$$= E[X^2] - (E[X])^2$$

Everyone, please welcome the second moment!

Variance of a 6-sided die

$\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$ Variance of X

Let Y = outcome of a single die roll. Recall $E[Y] = 7/2$.

Calculate the variance of Y.

1. Approach #1: Definition

P(Y=1) $(y - 7/2)^2$

$$\text{Var}(Y) = \frac{1}{6}\left(1-\frac{7}{2}\right)^2 + \frac{1}{6}\left(2-\frac{7}{2}\right)^2 + \frac{1}{6}\left(3-\frac{7}{2}\right)^2 + \frac{1}{6}\left(4-\frac{7}{2}\right)^2 + \frac{1}{6}\left(5-\frac{7}{2}\right)^2 + \frac{1}{6}\left(6-\frac{7}{2}\right)^2$$

$$= 35/12$$

2. Approach #2: A property

2nd moment

$$E[Y^2] = \frac{1}{6}[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2] = 91/6$$

$$\text{Var}(Y) = 91/6 - (7/2)^2 = 35/12$$

Properties of variance

Definition $\text{Var}(X) = E[(X - E[X])^2]$ Units of X^2

def standard deviation $\text{SD}(X) = \sqrt{\text{Var}(X)}$ Units of X

Property 1 $\text{Var}(X) = E[X^2] - (E[X])^2$

Property 2 $\text{Var}(aX + b) = a^2\text{Var}(X)$

- Property 1 is often easier to compute than the definition
- Unlike expectation, variance is not linear

Property 2: A proof

Property 2 $\text{Var}(aX + b) = a^2\text{Var}(X)$

Proof: $\text{Var}(aX + b)$

$= E[(aX + b)^2] - (E[aX + b])^2$ Property 1

$= E[a^2X^2 + 2abX + b^2] - (aE[X] + b)^2$

$= a^2E[X^2] + 2abE[X] + b^2 - (a^2(E[X])^2 + 2abE[X] + b^2)$

Factoring/ Linearity of Expectation

$= a^2E[X^2] - a^2(E[X])^2$

$= a^2(E[X^2] - (E[X])^2)$

$= a^2\text{Var}(X)$ Property 1

$\text{Var}(aX+b) \neq a\,\text{Var}(X)+b$

Bernoulli RV

x	f(x)
0	3
1	-1

tables

⇒ linear ⇒ mx + b

quadratic

step

Bernoulli Random Variable

Consider an experiment with two outcomes: "success" and "failure."

E E^c

def A **Bernoulli** random variable X maps "success" to 1 and "failure" to 0.
Other names: **indicator** random variable, boolean random variable

$$X \sim \text{Ber}(p)$$

varies as

Support: $\{0,1\}$

PMF	$P(X = 1) = p(1) = p = P(E)$ $P(X = 0) = p(0) = 1 - p = P(E^c)$
Expectation	$E[X] = p$
Variance	$\text{Var}(X) = p(1 - p)$

$E[X] = 0 \cdot (1-p) + 1 \cdot p = p$
$E[X^2] = 0^2 \cdot (1-p) + 1^2 \cdot p = p$
$\text{Var}(X) = p - p^2$
$= p(1-p)$

Examples:
- Coin flip
- Random binary digit
- Whether a disk drive crashed

Remember this nice property of expectation. It will come back!

Jacob Bernoulli

Jacob Bernoulli (1654-1705), also known as “James”, was a Swiss mathematician

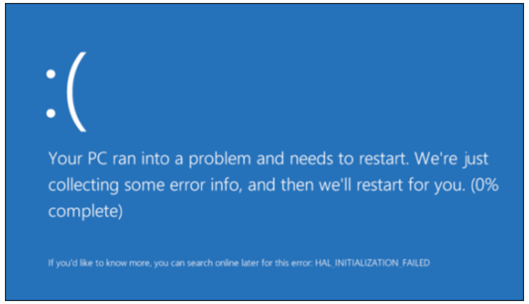
One of many mathematicians in Bernoulli family

The Bernoulli Random Variable is named for him

My academic great^{14} grandfather

Defining Bernoulli RVs

$X \sim \text{Ber}(p)$ $p_X(1) = p$
$E[X] = p$ $p_X(0) = 1 - p$



Run a program

- Crashes w.p. p
- Works w.p. $1 - p$

Let X: 1 if crash

$$X \sim \text{Ber}(p)$$

$$P(X = 1) = p$$

$$P(X = 0) = 1 - p$$

Serve an ad.

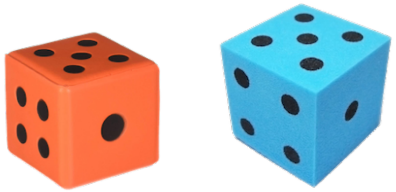
- User clicks w.p. 0.2
- Ignores otherwise

Let X: 1 if clicked

$$X \sim \text{Ber}(\_\_)$$

$$P(X = 1) = \_\_$$

$$P(X = 0) = \_\_$$



Roll two dice.

- Success: roll two 6's
- Failure: anything else

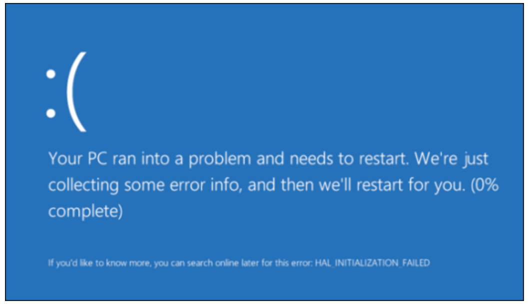
Let X : 1 if success

$$X \sim \text{Ber}(\_\_)$$

$$E[X] = \_\_$$

Defining Bernoulli RVs

$X \sim \text{Ber}(p)$ $\quad p_X(1) = p$
$E[X] = p$ $\quad p_X(0) = 1 - p$



Run a program

- Crashes w.p. p
- Works w.p. $1 - p$

Let X: 1 if crash

$$X \sim \text{Ber}(p)$$

$$P(X = 1) = p$$

$$P(X = 0) = 1 - p$$

Serve an ad.

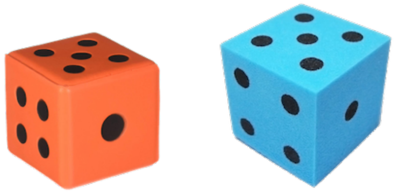
- User clicks w.p. 0.2
- Ignores otherwise

Let X: 1 if clicked

$$X \sim \text{Ber}(\underline{0.2})$$

$$P(X = 1) = \underline{0.2}$$

$$P(X = 0) = \underline{0.8}$$



Roll two dice.

- Success: roll two 6's 1/36
- Failure: anything else

Let X : 1 if success

$$X \sim \text{Ber}(\underline{1/36})$$

$$E[X] = \underline{1/36}$$

Binomial RV

Binomial Random Variable

Consider an experiment: n independent trials of Ber(p) random variables.

def A Binomial random variable X is the number of successes in n trials.

trials p(success)

$$X \sim \text{Bin}(n, p)$$

Support: $\{0,1,\dots,n\}$

PMF $k = 0, 1, \dots, n$:

$$P(X = k) = p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Expectation $E[X] = np$

Variance $\text{Var}(X) = np(1-p)$

Examples:

- # heads in n coin flips
- # of 1's in randomly generated length n bit string
- # of disk drives crashed in 1000 computer cluster (assuming disks crash independently)


SLOW
DOWN

Reiterating notation

1. The random variable

$$X \sim \text{Bin}(n, p)$$

2. is distributed as a

3. Binomial

4. with parameters

The parameters of a Binomial random variable:

- n: number of independent trials
- p: probability of success on each trial

Reiterating notation

$$X \sim \text{Bin}(n, p)$$

If X is a binomial with parameters n and p, the PMF of X is

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Probability that X takes on the value k

Probability Mass Function for a Binomial

Three coin flips

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

Three fair ("heads" with $p = 0.5$) coins are flipped.

- X is number of heads
- $X \sim \text{Bin}(3, 0.5)$ $n = 3$, $p = 0.5$

Compute the following event probabilities:

$$P(X = 0)$$

$$P(X = 1)$$

$$P(X = 2)$$

$$P(X = 3)$$

$$P(X = 7)$$

P(event)

Three coin flips

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

$k = \{0, \dots, n\}$

Three fair ("heads" with $p = 0.5$) coins are flipped.

- X is number of heads
- $X \sim \text{Bin}(3, 0.5)$ $n=3, p=0.5$

Compute the following event probabilities:

$$P(X=0) = p(0) = \binom{3}{0} p^0 (1-p)^3 = \frac{1}{8}$$

$$P(X=1) = p(1) = \binom{3}{1} p^1 (1-p)^2 = \frac{3}{8}$$

$$P(X=2) = p(2) = \binom{3}{2} p^2 (1-p)^1 = \frac{3}{8}$$

$$P(X=3) = p(3) = \binom{3}{3} p^3 (1-p)^0 = \frac{1}{8}$$

(sum) $= 1$

$$P(X=7) = p(7) = 0$$

P(event) PMF

k=0
TTT

HTT ← $\frac{1}{8}$
THT
TTH

Extra math note:
By Binomial Theorem, we can prove $\sum_{k=0}^{n} P(X=k) = 1$

$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k}$

$1^n = (p+(1-p))^n = \sum_{k=0}^{n} \binom{n}{k} p^n (1-p)^{n-k}$

Binomial Random Variable

Consider an experiment: n independent trials of Ber(p) random variables.

def A **Binomial** random variable X is the number of successes in n trials.

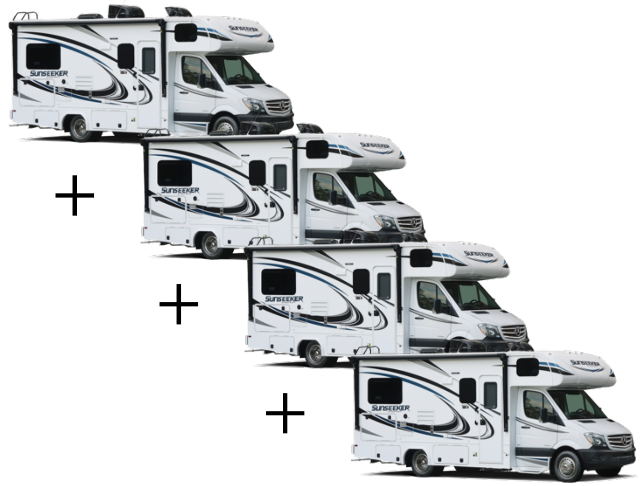
$X \sim \text{Bin}(n, p)$	PMF	$k = 0, 1, \ldots, n$: $P(X = k) = p(k) = \binom{n}{k} p^k (1-p)^{n-k}$
	Expectation	$E[X] = np$
Range: $\{0, 1, \ldots, n\}$	Variance	$\text{Var}(X) = np(1-p)$

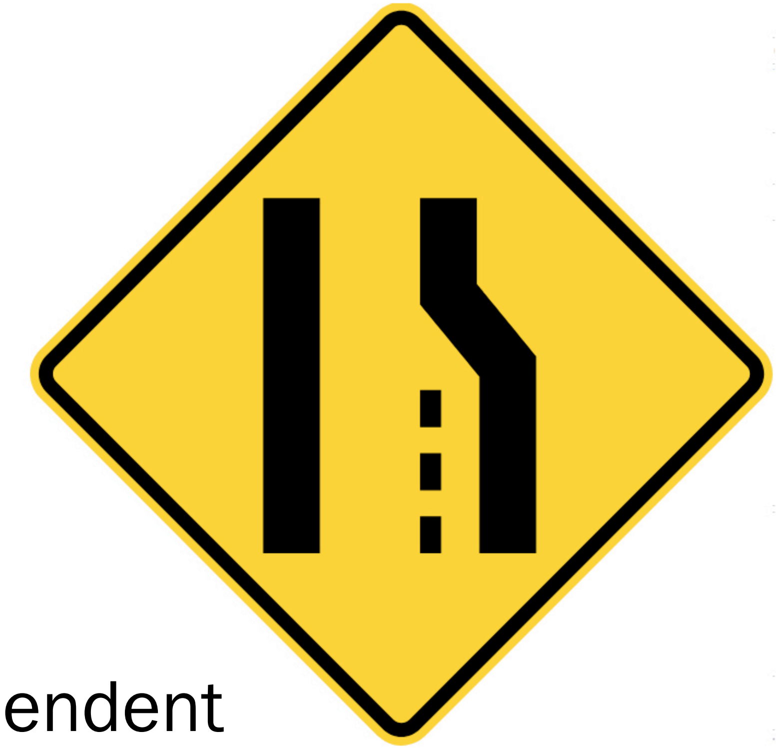
Examples:

- # heads in n coin flips
- # of 1's in randomly generated length n bit string
- # of disk drives crashed in 1000 computer cluster (assuming disks crash independently)

Binomial RV is sum of Bernoulli RVs

Bernoulli
- $X \sim \text{Ber}(p)$





Binomial
- $Y \sim \text{Bin}(n, p)$
- The sum of n independent Bernoulli RVs

$$Y = \sum_{i=1}^{n} X_i, \qquad X_i \sim \text{Ber}(p)$$

↑ Binomial RV

H T T T H H H
1 0 0 0 1 1 1

n=7

$\text{Ber}(p) = \text{Bin}(1, p)$

Binomial Random Variable

Consider an experiment: n independent trials of Ber(p) random variables.

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$$X \sim \text{Bin}(n, p)$$

Range: $\{0,1,\ldots,n\}$

PMF $k = 0, 1, \ldots, n$:

$$P(X = k) = p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Expectation $E[X] = np$

Variance $\text{Var}(X) = np(1-p)$

Examples:

- # heads in n coin flips
- # of 1's in randomly generated length n bit string
- # of disk drives crashed in 1000 computer cluster (assuming disks crash independently)

Proof: $X = \sum_{i=1}^{n} X_i$, $X_i \sim \text{Ber}(p)$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

$$= \sum_{i=1}^{n} p, \quad E[X_i] = p$$

$$= np$$

Binomial Random Variable

Consider an experiment: n independent trials of Ber(p) random variables.

def A **Binomial** random variable X is the number of successes in n trials.

$X \sim \text{Bin}(n, p)$	PMF	$k = 0, 1, \ldots, n$: $P(X = k) = p(k) = \binom{n}{k} p^k (1-p)^{n-k}$
	Expectation	$E[X] = np$
Range: $\{0, 1, \ldots, n\}$	Variance	$\text{Var}(X) = np(1-p)$

We'll prove this later in the course

Examples:

- # heads in n coin flips
- # of 1's in randomly generated length n bit string
- # of disk drives crashed in 1000 computer cluster (assuming disks crash independently)

No, give me the variance proof right now

To simplify the algebra a bit, let $q = 1 - p$, so $p + q = 1$.

So:

$$\begin{aligned}
\mathrm{E}(X^2) &= \sum_{k \geq 0}^{n} k^2 \binom{n}{k} p^k q^{n-k} && \text{Definition of Binomial Distribution: } p+q=1 \\
&= \sum_{k=0}^{n} kn \binom{n-1}{k-1} p^k q^{n-k} && \text{Factors of Binomial Coefficient: } k\binom{n}{k} = n\binom{n-1}{k-1} \\
&= np \sum_{k=1}^{n} k \binom{n-1}{k-1} p^{k-1} q^{(n-1)-(k-1)} && \text{Change of limit: term is zero when } k-1=0 \\
&= np \sum_{j=0}^{m} (j+1) \binom{m}{j} p^j q^{m-j} && \text{putting } j = k-1, m = n-1 \\
&= np \left(\sum_{j=0}^{m} j \binom{m}{j} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right) && \text{splitting sum up into two} \\
&= np \left(\sum_{j=0}^{m} m \binom{m-1}{j-1} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right) && \text{Factors of Binomial Coefficient: } j\binom{m}{j} = m\binom{m-1}{j-1} \\
&= np \left((n-1)p \sum_{j=1}^{m} \binom{m-1}{j-1} p^{j-1} q^{(m-1)-(j-1)} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right) && \text{Change of limit: term is zero when } j-1=0 \\
&= np\left((n-1)p(p+q)^{m-1} + (p+q)^m\right) && \text{Binomial Theorem} \\
&= np((n-1)p+1) && \text{as } p+q=1 \\
&= n^2p^2 + np(1-p) && \text{by algebra}
\end{aligned}$$

Then:

$$\begin{aligned}
\mathrm{var}(X) &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 \\
&= np(1-p) + n^2p^2 - (np)^2 && \text{Expectation of Binomial Distribution: } \mathrm{E}(X) = np \\
&= np(1-p)
\end{aligned}$$

as required.

proofwiki.org

(live)
07: Variance, Bernoulli, and Binomial

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September 28, 2020

Our first common RVs

1. The random variable

$$X \sim \text{Ber}(p)$$

2. is distributed as/varies as a

3. Bernoulli

4. with parameter

Example: Heads in one coin flip,
P(heads) = 0.8 = p

$$Y \sim \text{Bin}(n, p)$$

Example: # heads in 40 coin flips,
P(heads) = 0.8 = p

otherwise

Identify PMF, or
identify as a function of an
existing random variable

Breakout Rooms

Check out the questions on the next slide (Slide 37). Post any clarifications here!

https://us.edstem.org/courses/2678/discussion/134630

Breakout rooms: 5 min. Introduce yourself!

🤔

Statistics: Expectation and variance

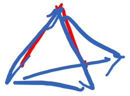
1. a. Let X = the outcome of a fair 4-sided die roll. What is $E[X]$?

   b. Let Y = the sum of three rolls of a fair 4-sided die. What is $E[Y]$?

2. Let Z = # of ***tails*** on 10 flips of a biased coin (w.p. 0.4 of heads). What is $E[Z]$?

3. Compare the variances of $B_1 \sim \text{Ber}(0.1)$ and $B_2 \sim \text{Ber}(0.5)$.

Statistics: Expectation and variance



1. a. Let X = the outcome of a fair 4-sided die roll. What is $E[X]$?

   $\mathbb{E}[X] = \sum_{x=1}^{4} x\,p(x) = \sum_{x=1}^{4} \frac{1}{4} x = 2.5$

   b. Let Y = the sum of three rolls of a fair 4-sided die. What is $E[Y]$?

   $\mathbb{E}[Y] \xrightarrow{\text{def}} \sum_y y\,p(y)$ or ...

   linearity $\rightarrow \mathbb{E}[X_1 + X_2 + X_3]$

   $= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3]$

   $= 7.5$

2. Let Z = # of ***tails*** on 10 flips of a biased coin (w.p. 0.4 of heads). What is $E[Z]$?

   Z: # of tails

   $Z \sim \text{Bin}(10, 0.6)$, where $n = 10$, $p = 0.6$ $\quad \mathbb{E}[Z] = np = 6$

3. Compare the variances of $B_1 \sim \text{Ber}(0.1)$ and $B_2 \sim \text{Ber}(0.5)$.

   ($0.1 = p$)

   $\text{Var}(B_1) = p(1-p) = 0.1(0.9) = 0.09$ lower var

   $\text{Var}(B_2) = (0.5)^2 = 0.25$ high var

If you can identify common RVs, just look up statistics instead of re-deriving from definitions.

Think

Slide 40 has a matching question to go over by yourself. We'll go over it together afterwards.

Post any clarifications here!

https://us.edstem.org/courses/2678/discussion/134630

Think by yourself: 1 min

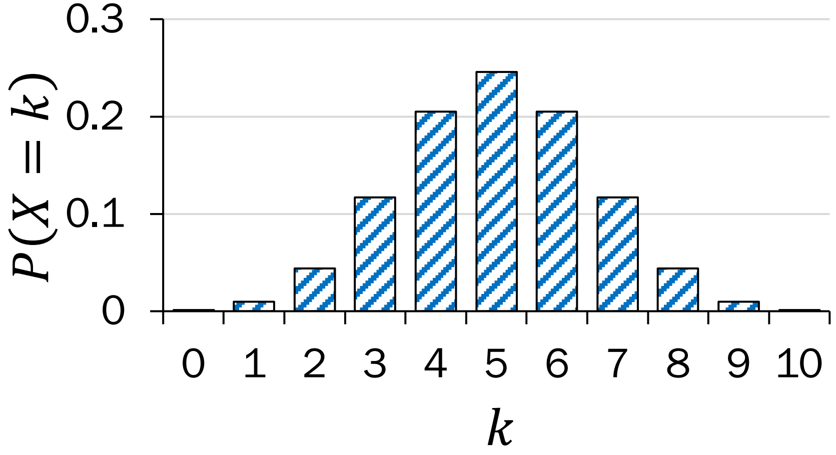
(by yourself)

Visualizing Binomial PMFs

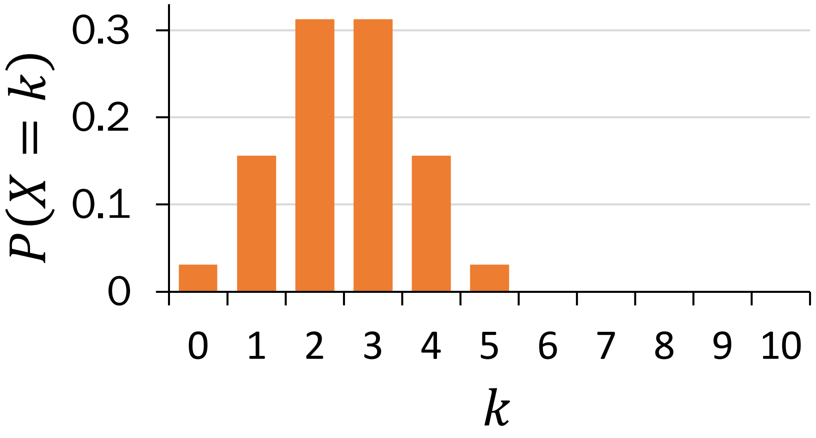
$X \sim \text{Bin}(n, p)$ $\quad E[X] = np \quad p(i) = \binom{n}{k} p^k (1-p)^{n-k}$

Type & hold your answer @ end of 1 min
1-A, 2-B, 3-C, 4-D

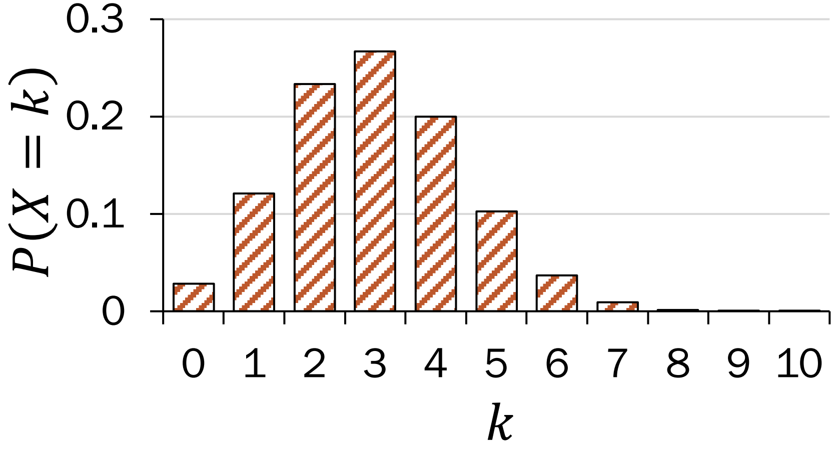
A.



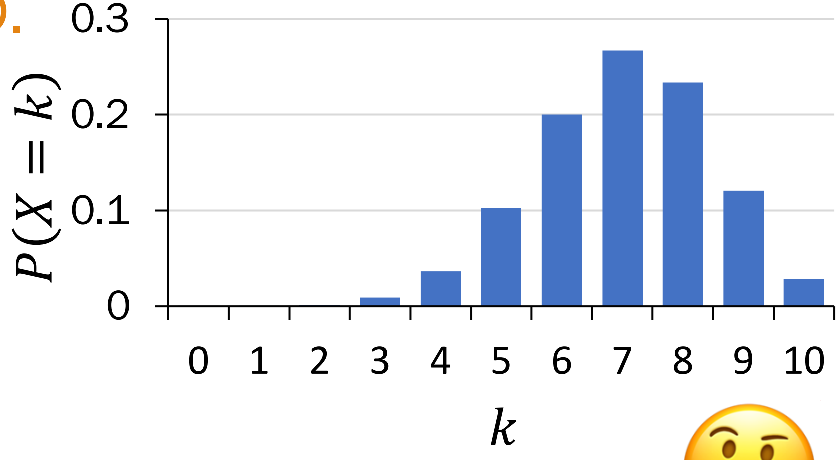
B.



C.



D.



Match the distribution of X to the graph:

1. $\text{Bin}(10, 0.5)$
2. $\text{Bin}(10, 0.3)$
3. $\text{Bin}(10, 0.7)$
4. $\text{Bin}(5, 0.5)$

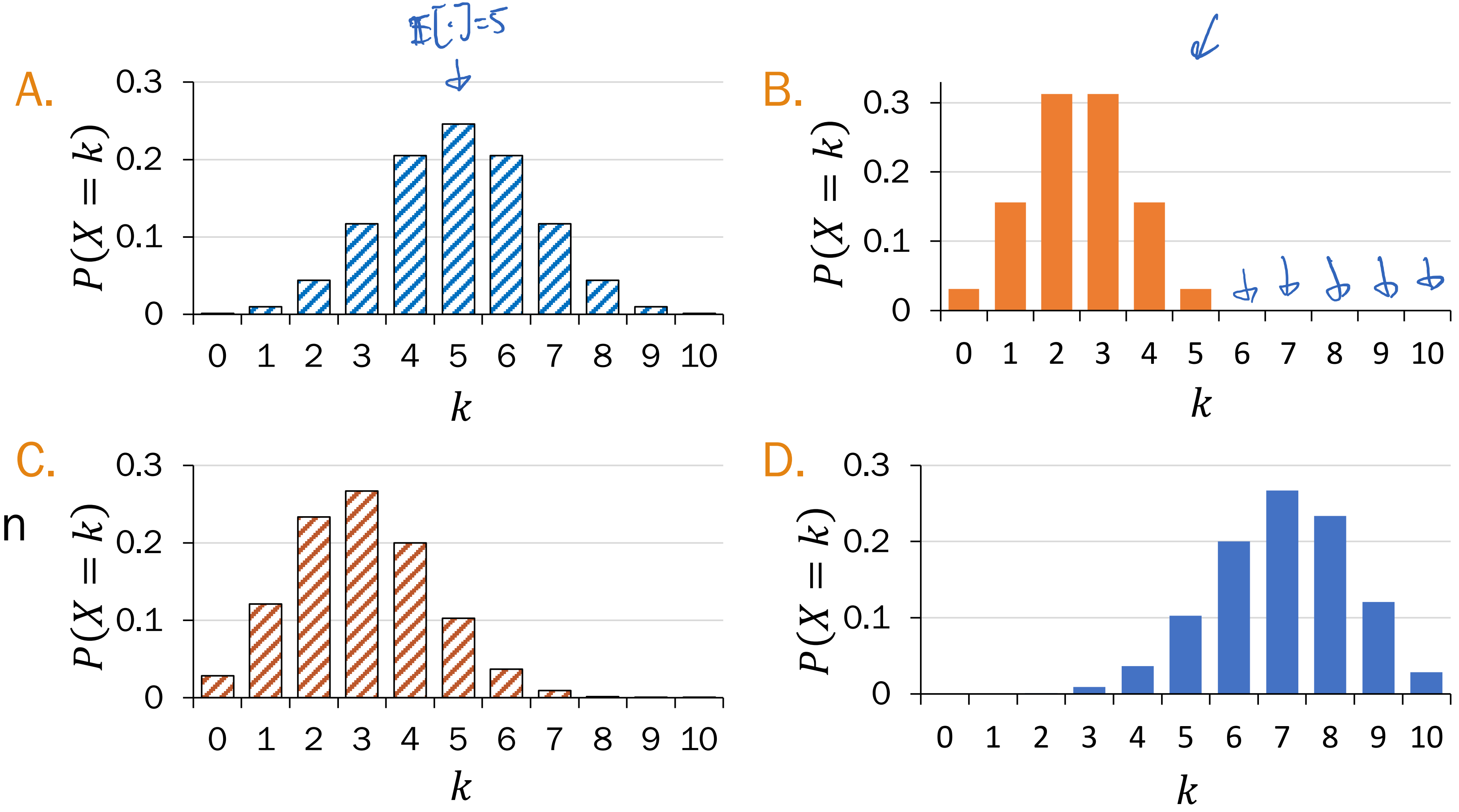
(by yourself)

Visualizing Binomial PMFs

$X \sim \text{Bin}(n, p)$ $\quad E[X] = np \quad p(i) = \binom{n}{k} p^k (1-p)^{n-k}$

Bin(n, p)

1. E[·] = 5
2. 3 ←
3. 7 ←
4. 2.5



Match the distribution of X to the graph:

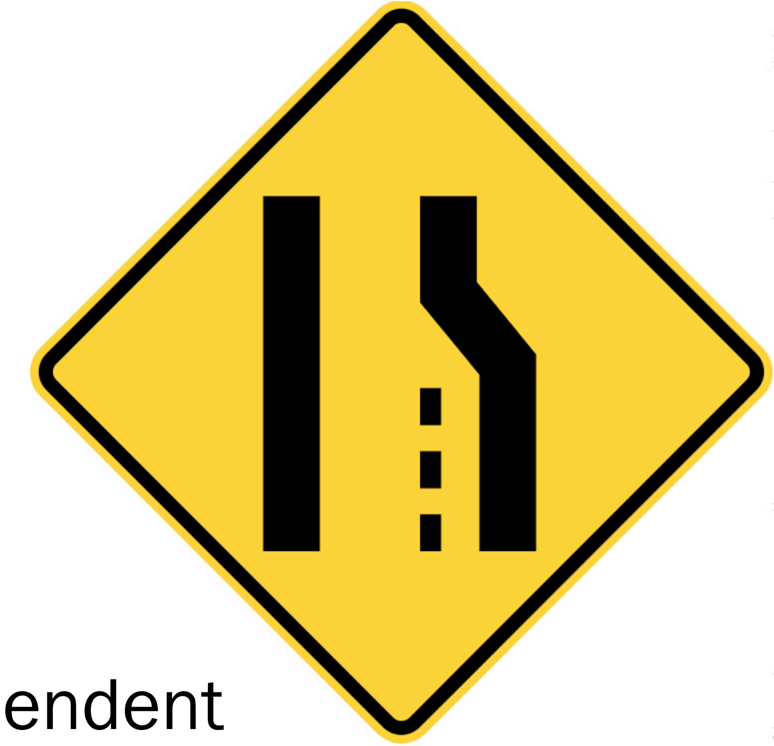
1. Bin(10,0.5) A
2. Bin(10,0.3) C
3. Bin(10,0.7) D
4. Bin(5,0.5) B

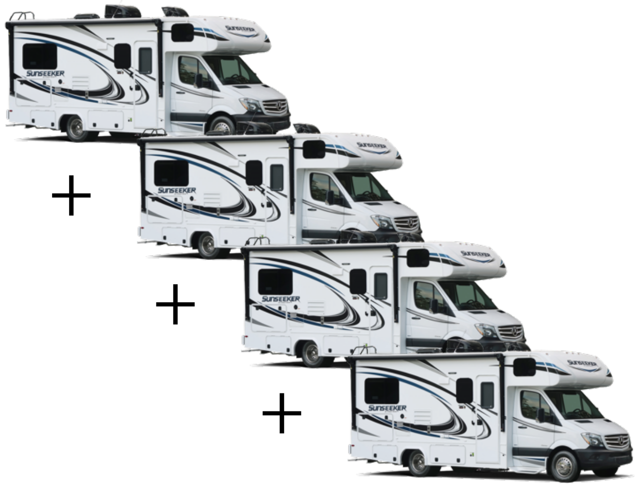
Binomial RV is sum of Bernoulli RVs

Review

Bernoulli

- $X \sim \text{Ber}(p)$



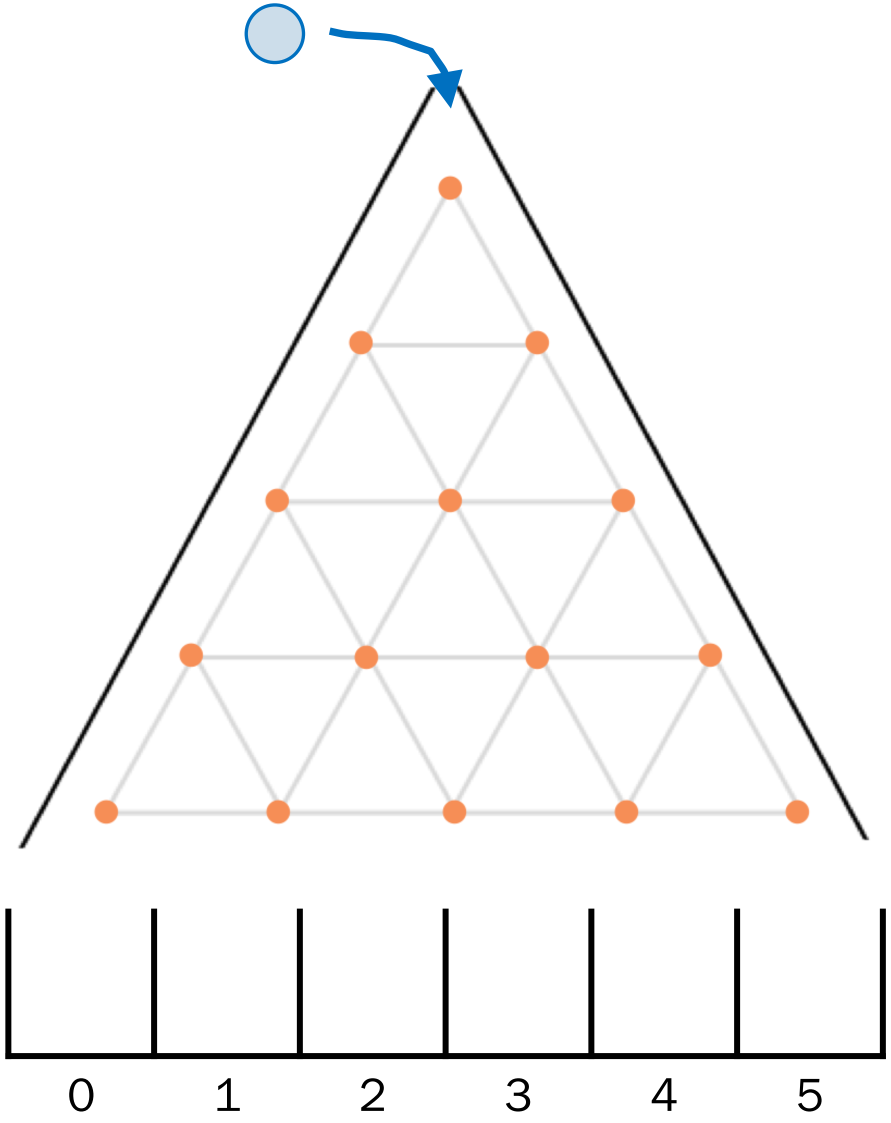


Binomial

- $Y \sim \text{Bin}(n, p)$
- The sum of n independent Bernoulli RVs

$$Y = \sum_{i=1}^{n} X_i, \qquad X_i \sim \text{Ber}(p)$$

Galton Board



http://web.stanford.edu/class/cs109/demos/galton.html

Think

Slide 45 has a question to go over by yourself.

Post any clarifications here!

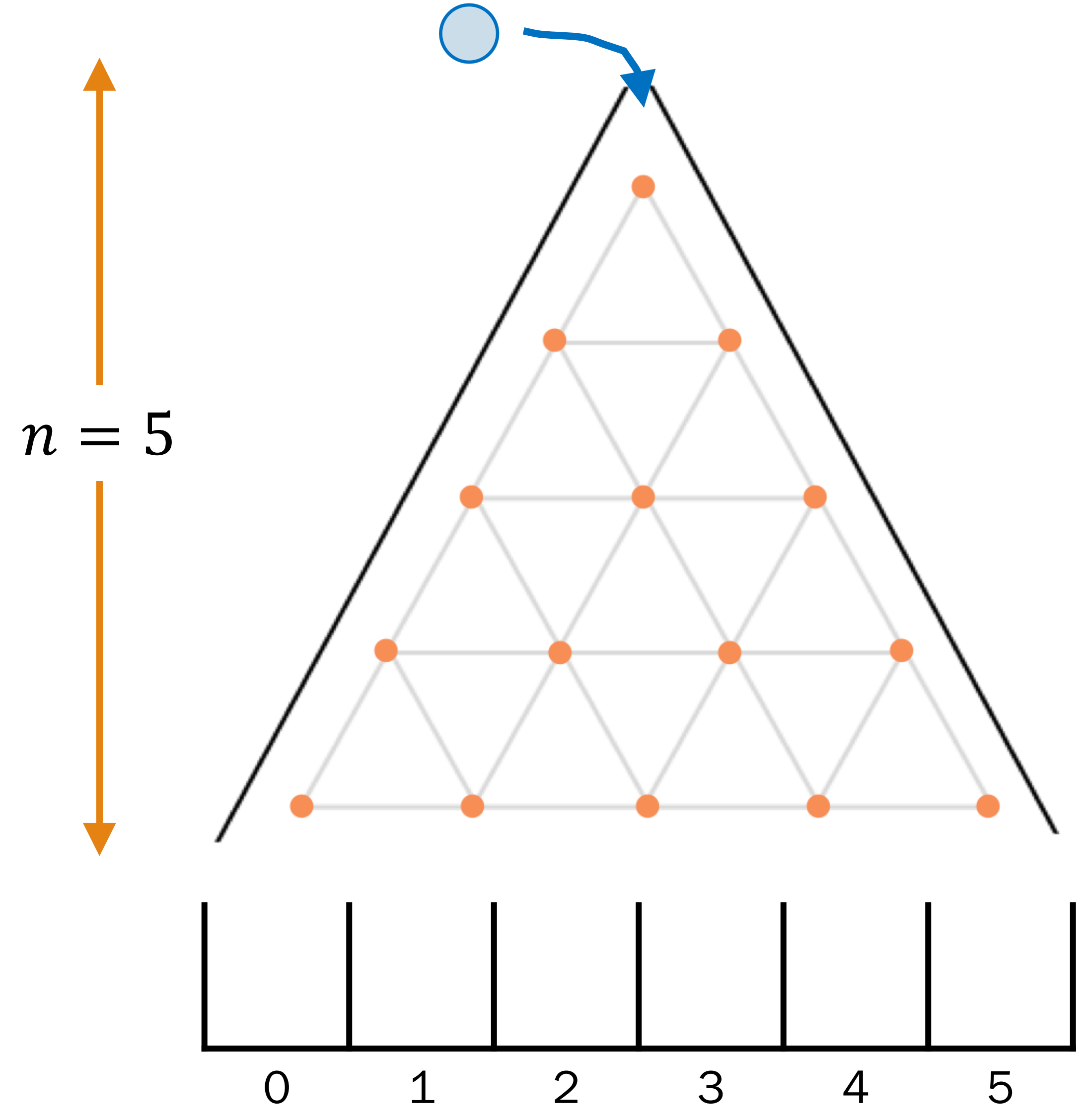
https://us.edstem.org/courses/2678/discussion/134630

Think by yourself: 1 min

(by yourself)

Galton Board

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$



When a marble hits a pin, it has an equal chance of going left or right.

Let B = the bucket index a ball drops into.

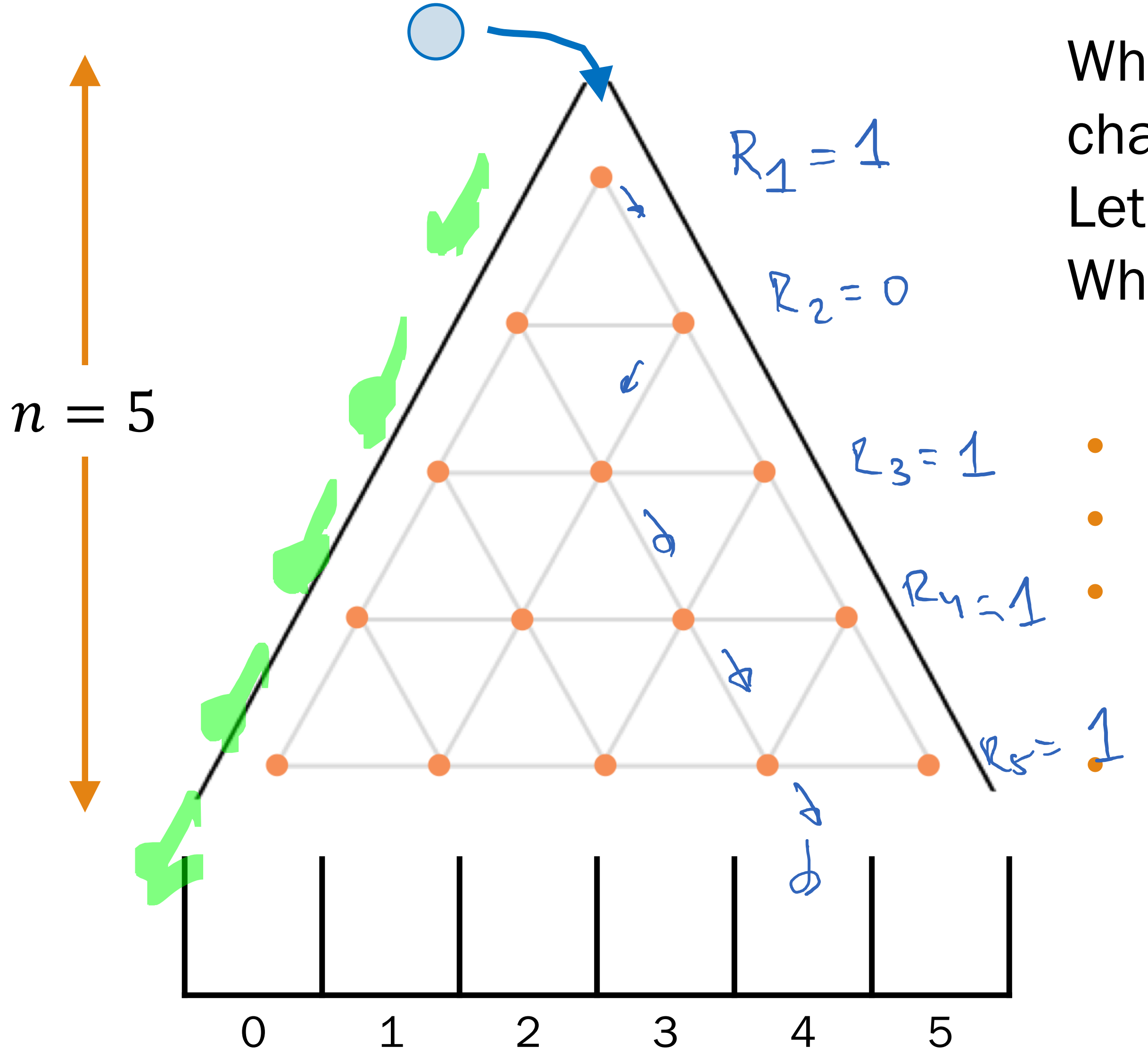
What is the distribution of B?

(Interpret: If B is a common random variable, report it, otherwise report PMF)

(by yourself)

Galton Board

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$



When a marble hits a pin, it has an equal chance of going left or right.
Let B = the bucket index a ball drops into.
What is the **distribution** of B?

decision of left or right

- Each pin is an independent trial
- One decision made for level $i = 1, 2, \ldots, 5$
- Consider a Bernoulli RV with success R_i if ball went right on level i

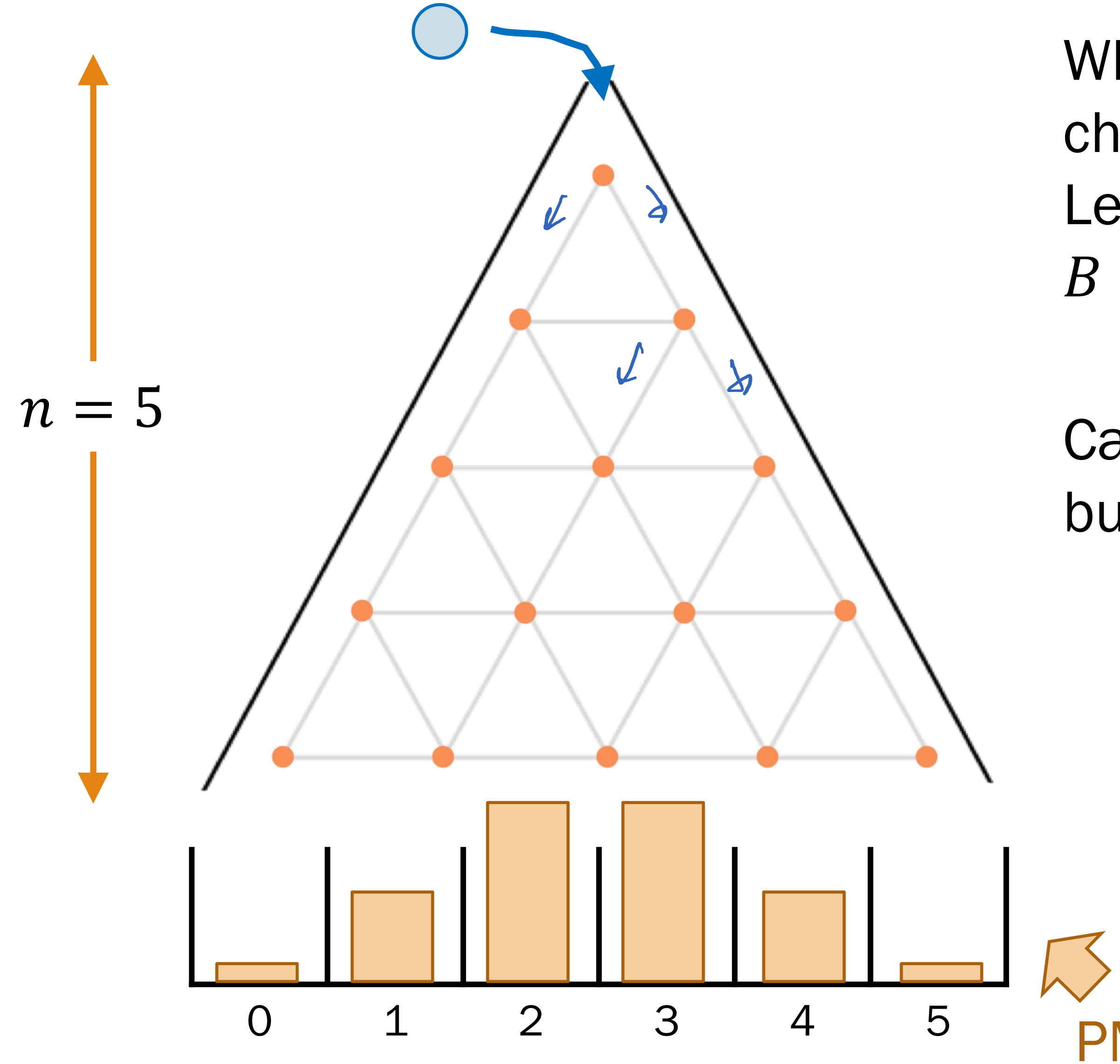
Bucket index B = # times ball went right

$B = \sum_{i=1}^{5} R_i \qquad R_i \sim \text{Ber}(0.5)$

$$B \sim \text{Bin}(n = 5, p = 0.5)$$

Galton Board

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$



When a marble hits a pin, it has an equal chance of going left or right.

Let B = the bucket index a ball drops into.

B is distributed as a Binomial RV,

$$B \sim \text{Bin}(n = 5, p = 0.5)$$

Calculate the probability of a ball landing in bucket k.

$$P(B = 0) = \binom{5}{0} 0.5^5 \approx 0.03$$

$$P(B = 1) = \binom{5}{1} 0.5^5 \approx 0.16$$

$$P(B = 2) = \binom{5}{2} 0.5^5 \approx 0.31$$

PMF of Binomial RV!

Interlude for jokes/announcements

Python tutorial #2

When: Wed 9/30 3:30-4:30pm PT
Recorded? Yes
Covers: csv PS2 coding
Notes: to be posted online

no idea

Think, then Breakout Rooms

Check out the questions on the next slide (Slide 50). Post any clarifications here!

https://us.edstem.org/courses/2678/discussion/134630

By yourself: 2 min

Breakout rooms: 5 min.

Genetics and NBA Finals

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

1. Each person has 2 genes per trait (e.g., eye color).

- Child receives 1 gene (equally likely) from each parent
- Brown is "dominant", blue is "recessive":
  - Child has brown eyes if either (or both) genes are brown
  - Blue eyes only if both genes are blue.
- Parents each have 1 brown and 1 blue gene.

A family has 4 children. What is P(exactly 3 children with brown eyes)?

2. The LA Lakers are going to play the Miami Heat in a 7-game series during the 2019 NBA finals.

- The Lakers have a probability of 58% of winning each game, independently.
- A team wins the series if they win at least 4 games (we play all 7 games).

What is P(Lakers winning)?

Genetic inheritance

Each person has 2 genes per trait (e.g., eye color).

- Child receives 1 gene (equally likely) from each parent
- Brown is “dominant”, blue is ”recessive”:
  - Child has brown eyes if either (or both) genes are brown
  - Blue eyes only if both genes are blue.
- Parents each have 1 brown and 1 blue gene.

A family has 4 children. What is P(exactly 3 children with brown eyes)?

<u>Big Q</u>: Fixed parameter or random variable?

Parameters What is **common** among all outcomes of our experiment?

Random variable What **differentiates** our event from the rest of the sample space?

$n = 4$

$P(\text{brown}) = p$

\# of brown-eyed children

$X \in \{0, 1, 2, 3, 4\}$

Genetic inheritance

$X \sim Bin(n, p)$ $P(k) = \binom{n}{k} p^k (1-p)^{n-k}$

Each person has 2 genes per trait (e.g., eye color).

- Child receives 1 gene (equally likely) from each parent
- Brown is "dominant", blue is "recessive":
  - Child has brown eyes if either (or both) genes are brown
  - Blue eyes only if both genes are blue.
- Parents each have 1 brown and 1 blue gene.

A family has 4 children. What is P(exactly 3 children with brown eyes)?

1. Define events/ RVs & state goal
2. Identify known probabilities
3. Solve

X: # brown-eyed children,

$X \sim \text{Bin}(4, p)$

p: $P(\text{brown}-\text{eyed child})$ $= 1 - P(\text{blue})$

$= 1 - 1/4 = 3/4 = p$

Want: $P(X = 3)$

$P(X=3) = \binom{4}{3} 0.75^3 (0.25)^1$

NBA Finals

The LA Lakers are going to play the Miami Heat in a 7-game series during the ~~2020~~ NBA finals.

- The Lakers have a probability of 58% of winning each game, independently.
- A team wins the series if they win at least 4 games (we play all 7 games).

What is P(Lakers winning)?

1. Define events/ RVs & state goal

X: # games Lakers win

$X \sim \text{Bin}(7, 0.58)$

Want:

Big Q: Fixed parameter or random variable?

Parameters

\# of total games $n=7$

prob. Lakers winning a game $p=0.58$

Random variable

\# of games Lakers win $X \in \{0,1,2,3,4,5,6,7\}$

Event based on RV

$P(X \geq 4) = P(X > 3) = 1 - P(X \leq 3)$

NBA Finals

The LA Lakers are going to play the Miami Heat in a 7-game series during the 2020 NBA finals.

- The Lakers have a probability of 58% of winning each game, independently.
- A team wins the series if they win at least 4 games (we play all 7 games).

What is P(Lakers winning)?

1. Define events/ RVs & state goal

X: # games Lakers win
$X \sim \text{Bin}(7, 0.58)$

Want: $P(X \geq 4)$

2. Solve

$= P(X=4) + P(X=5) + P(X=6) + P(X=7)$

$$P(X \geq 4) = \sum_{k=4}^{7} P(X = k) = \sum_{k=4}^{7} \binom{7}{k} 0.58^k (0.42)^{7-k}$$

Cool Algebra/Probability Fact: this is identical to the probability of winning if we define winning = first to win 4 games

See you next time