

10: The Normal (Gaussian) Distribution

Lisa Yan and Jerry Cain
October 5, 2020

Quick slide reference

| | | |
|----|----------------------------------|------------------|
| 3 | Normal RV | 10a_normal |
| 15 | Normal RV: Properties | 10b_normal_props |
| 21 | Normal RV: Computing probability | 10c_normal_prob |
| 30 | Exercises | LIVE |

Normal RV

Today's the Big Day



the big day noun phrase

Definition of *the big day*

{ : the day that something important happens

// Today *is the big day.*

also : the day someone is to be married

// So, when's *the big day?*

Normal Random Variable

def An **Normal** random variable X is defined as follows:

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Support: $(-\infty, \infty)$

PDF

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

Expectation

$$E[X] = \mu$$

Variance

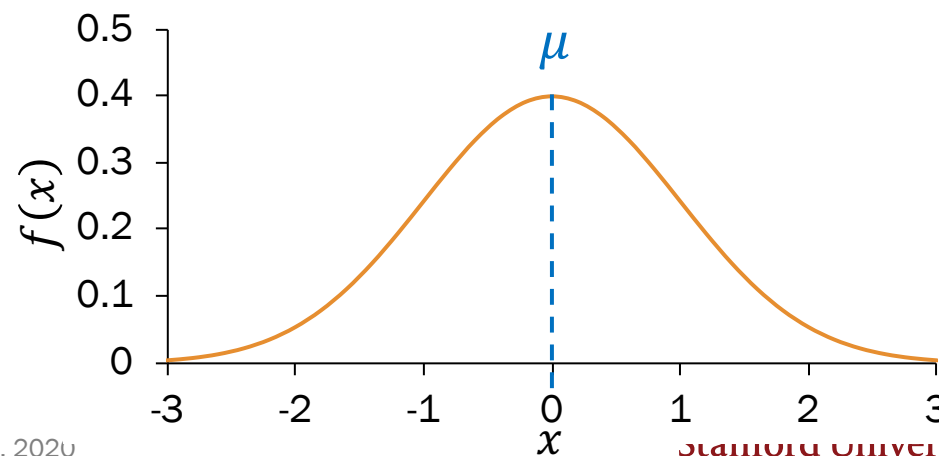
$$\text{Var}(X) = \sigma^2$$

σ : std deviation

Other names: **Gaussian** random variable

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

mean variance



Carl Friedrich Gauss

Carl Friedrich Gauss (1777-1855) was a remarkably influential German mathematician.



Johann Carl Friedrich Gauss ([/ɡaʊs/](#); **German:** *Gauß* [ɡaʊs] [\(listen\)](#); **Latin:** *Carolus Fridericus Gauss*; 30 April 1777 – 23 February 1855) was a German mathematician and physicist who made significant contributions to many fields, including [algebra](#), [analysis](#), [astronomy](#), [differential geometry](#), [electrostatics](#), [geodesy](#), [geophysics](#), [magnetic fields](#), [matrix theory](#), [mechanics](#), [number theory](#), [optics](#) and [statistics](#).

Sometimes referred to as the *Princeps mathematicorum*^[1] (**Latin** for "the foremost of mathematicians") and "[the greatest mathematician since antiquity](#)". Gauss had an exceptional influence in many fields of mathematics and science, and is ranked among history's most influential mathematicians.^[2]

Did not invent Normal distribution but rather popularized it

Why the Normal?

- Common for natural phenomena: height, weight, etc.
- Most noise in the world is Normal
- Often results from the sum of many random variables
- Sample means are distributed normally

That's what they
want you to believe...



Why the Normal?

- Common for natural phenomena: height, weight, etc. Actually log-normal
- Most noise in the world is Normal Just an assumption
- Often results from the sum of many random variables Only if equally weighted
- **Sample means are distributed normally** (okay this one is true, we'll see this in 3 weeks)

Okay, so why the Normal?

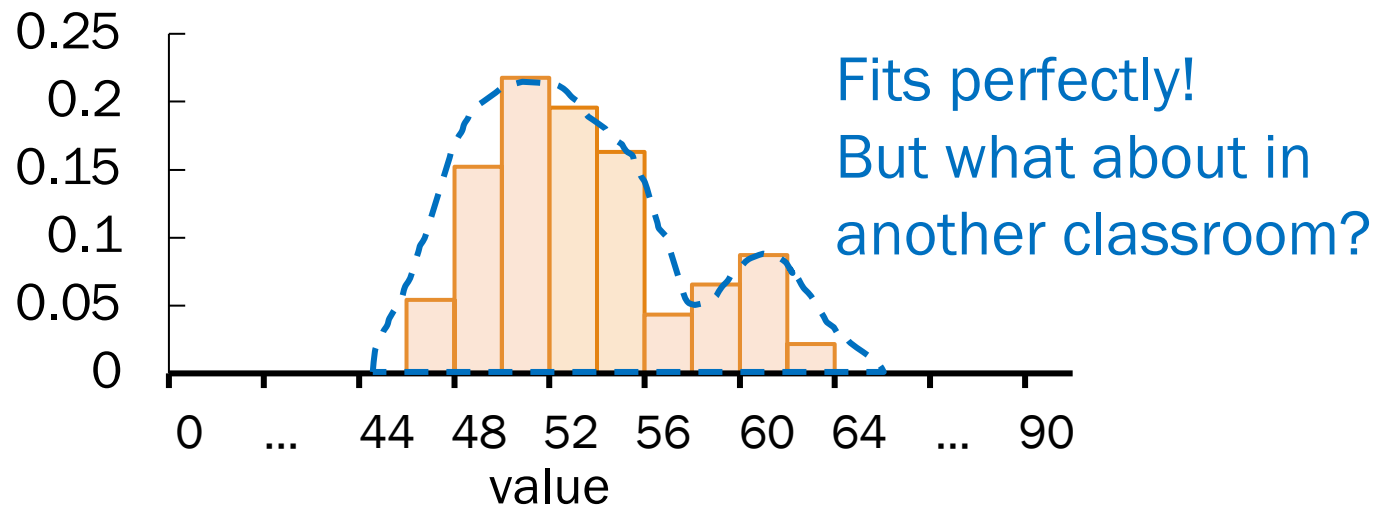
Part of CS109 learning goals:

- Translate a problem statement into a random variable

In other words: **model real life situations** with probability distributions

How do you model student heights?

- Suppose you have data from one classroom.



Okay, so why the Normal?

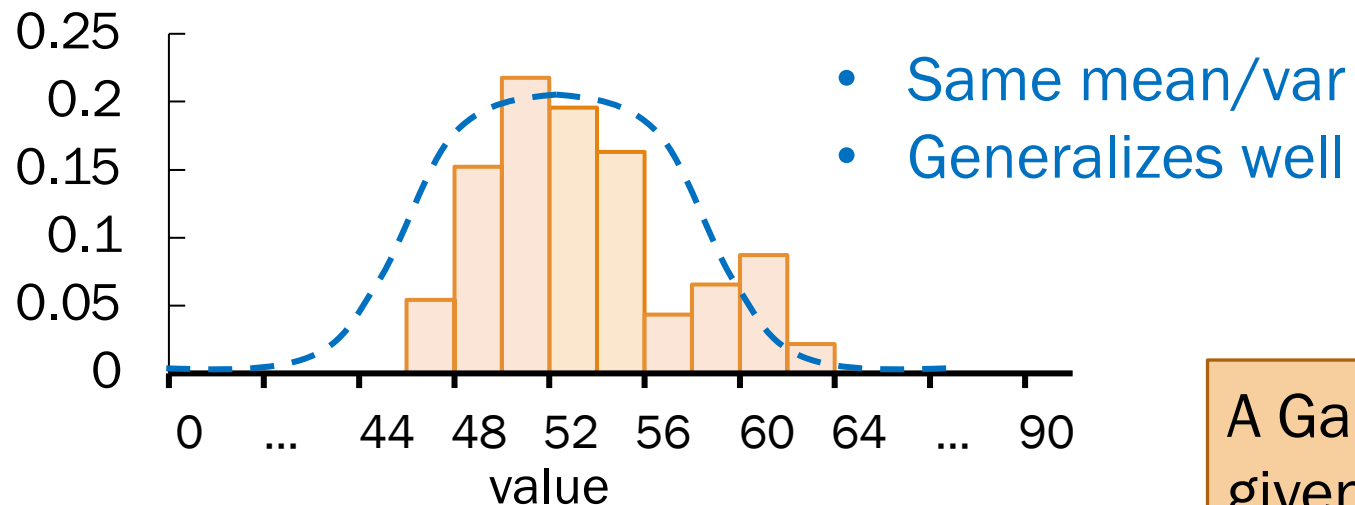
Part of CS109 learning goals:

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In other words: **model real life situations** with probability distributions

How do you model student heights?

- Suppose you have data from one classroom.



Occam's Razor:

"Non sunt multiplicanda entia sine necessitate."

Entities should not be multiplied without necessity.

A Gaussian maximizes entropy for a given mean and variance.

Why the Normal?

- Common for natural phenomena: height, weight, etc.
- Most noise in the world is Normal
- Often results from the sum of many random variables
- Sample means are distributed normally

Actually log-normal

because it's easy to use

description

Only if equally weighted

(okay this one is true, we'll see this in 3 weeks)

I encourage you to stay critical of how to model real-world phenomena.

Anatomy of a beautiful equation

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

The PDF of X is defined as:

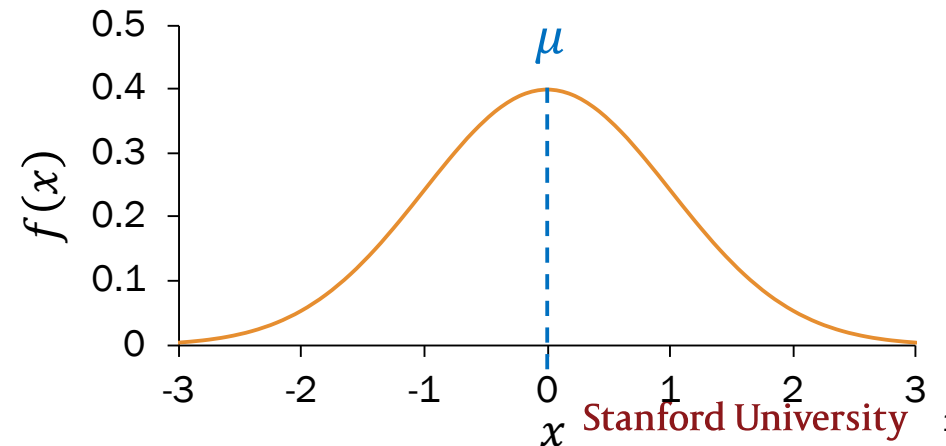
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

normalizing constant

exponential tail

symmetric around μ

variance σ^2 manages spread



Campus bikes

You spend some minutes, X , traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes²

Suppose X is normally distributed. What is the probability you spend ≥ 6 minutes traveling?

$$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2)$$

$$P(X \geq 6) = \int_6^{\infty} f(x) dx = \int_6^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

(call me if you analytically solve this)



Loving, not scary
...except this time

Computing probabilities with Normal RVs

For a Normal RV $X \sim \mathcal{N}(\mu, \sigma^2)$, its CDF has no closed form.

$$P(X \leq x) = F(x) = \int_{-\infty}^x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

! Cannot be solved analytically

However, we can solve for probabilities numerically using a function Φ :

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

To get here, we'll first need to know some properties of Normal RVs.

CDF of $X \sim \mathcal{N}(\mu, \sigma^2)$

A function that has been solved for numerically

Normal RV: Properties

Properties of Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$.

1. Linear transformations of Normal RVs are also Normal RVs.

$$\text{If } Y = aX + b, \text{ then } Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2).$$

2. The PDF of a Normal RV is symmetric about the mean μ .

$$F(\mu - x) = 1 - F(\mu + x)$$

1. Linear transformations of Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$.

Linear transformations of X are also Normal.

$$\text{If } Y = aX + b, \text{ then } Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

Proof:

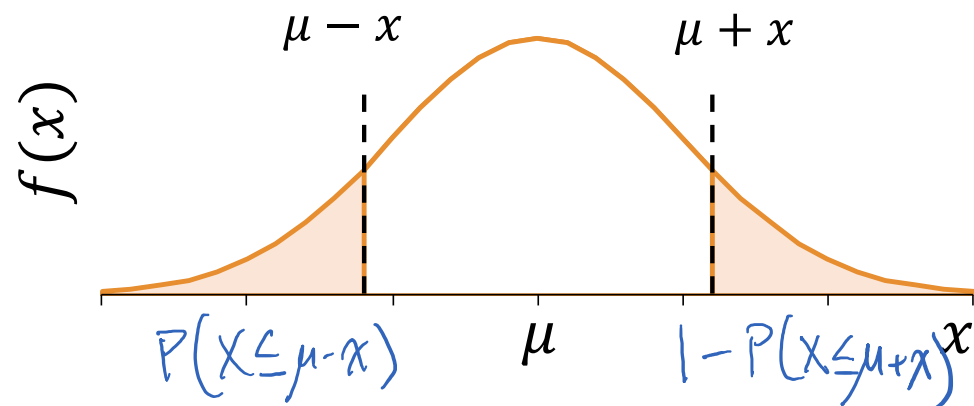
- $E[Y] = E[aX + b] = aE[X] + b = a\mu + b$ Linearity of Expectation
- $\text{Var}(Y) = \text{Var}(aX + b) = a^2\text{Var}(X) = a^2\sigma^2$ $\text{Var}(aX + b) = a^2\text{Var}(X)$
- Y is also Normal Proof in Ross,
10th ed (Section 5.4)

2. Symmetry of Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$.

The PDF of a Normal RV is symmetric about the mean μ .

$$F(\mu - x) = 1 - F(\mu + x)$$



Using symmetry of the Normal RV

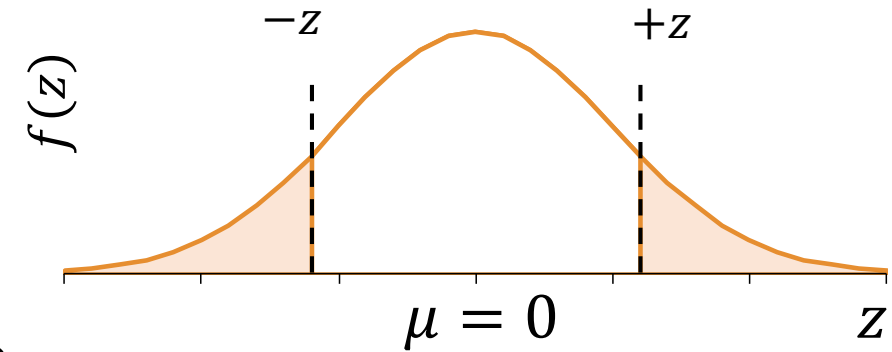
$$F(\mu - x) = 1 - F(\mu + x)$$

Let $Z \sim \mathcal{N}(0,1)$ with CDF $P(Z \leq z) = F(z)$.

Suppose we only knew numeric values for $F(z)$ and $F(y)$, for some $z, y \geq 0$.

How do we compute the following probabilities?

1. $P(Z \leq z) = F(z)$
2. $P(Z < z)$
3. $P(Z \geq z)$
4. $P(Z \leq -z)$
5. $P(Z \geq -z)$
6. $P(y < Z < z)$



- A. $F(z)$
- B. $1 - F(z)$
- C. $F(z) - F(y)$



Using symmetry of the Normal RV

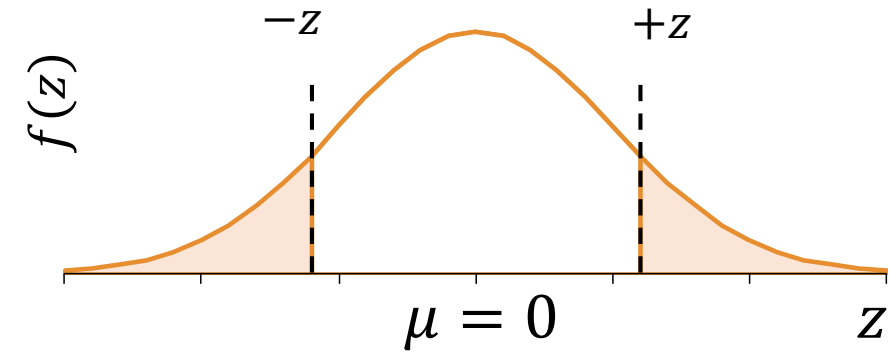
$$F(\mu - x) = 1 - F(\mu + x)$$

Let $Z \sim \mathcal{N}(0, 1)$ with CDF $P(Z \leq z) = F(z)$.

Suppose we only knew numeric values for $F(z)$ and $F(y)$, for some $z, y \geq 0$.

How do we compute the following probabilities?

1. $P(Z \leq z) = F(z)$
2. $P(Z < z) = F(z)$
3. $P(Z \geq z) = 1 - F(z)$
4. $P(Z \leq -z) = 1 - F(z)$
5. $P(Z \geq -z) = F(z)$
6. $P(y < Z < z) = F(z) - F(y)$



- A. $F(z)$
- B. $1 - F(z)$
- C. $F(z) - F(y)$

Symmetry is particularly useful when computing probabilities of zero-mean Normal RVs.

Normal RV: Computing probability

Computing probabilities with Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

To compute the CDF, $P(X \leq x) = F(x)$:

- We cannot analytically solve the integral (it has no closed form)
- ...but we **can** solve numerically using a function Φ :

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

CDF of the
Standard Normal, Z

Standard Normal RV, Z

The **Standard Normal** random variable Z is defined as follows:

$$Z \sim \mathcal{N}(0, 1)$$

Expectation

$$E[Z] = \mu = 0 \quad \text{"zero mean"}$$

Variance

$$\text{Var}(Z) = \sigma^2 = 1 \quad \text{"unit variance"}$$

Note: not a new distribution; just a special case of the Normal

Other names: **Unit Normal**

CDF of Z defined as: $P(Z \leq z) = \Phi(z)$

Φ has been numerically computed

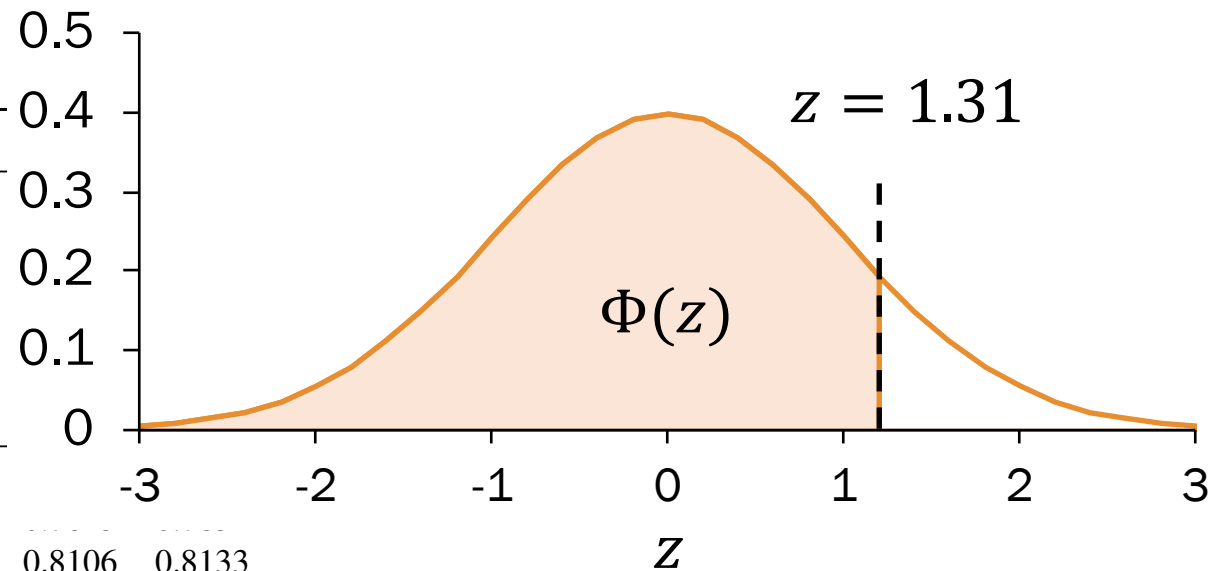
Standard Normal Table

An entry in the table is the area under the curve to the left of z , $P(Z \leq z) = \Phi(z)$.



| Z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | | |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0 | $f(z)$ | |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0 | | |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0 | | |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0 | | |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | | |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | | |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | | |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7703 | 0.7734 | 0.7764 | 0.7793 | | |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8906 | 0.8925 | 0.8943 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |

$$P(Z \leq 1.31) = \Phi(1.31)$$



Standard Normal Table only has probabilities $\Phi(z)$ for $z \geq 0$.

History fact: Standard Normal Table

T A B L E S
S E R V A N T
A U C A L C U L D E S R É F R A C T I O N S
A P P R O C H A N T E S D E L ' H O R I Z O N .

T A B L E P R E M I È R E .

Intégrales de $e^{-t^2} dt$, depuis une valeur quelconque de t jusqu'à t infinie.

$\int_a^\infty e^{-t^2} dt$

| t | Intégrale. | Diff. prem. | Diff. II. | Diff. III. |
|------|------------|-------------|-----------|------------|
| 0,00 | 0,88622692 | 999968 | 201 | 199 |
| 0,01 | 0,87622724 | 999767 | 400 | 199 |
| 0,02 | 0,86622057 | 999367 | 599 | 200 |
| 0,03 | 0,85623590 | 998768 | 799 | 199 |
| 0,04 | 0,84624822 | 997969 | 998 | 197 |
| 0,05 | 0,83626853 | 996971 | 1195 | 199 |
| 0,06 | 0,82629882 | 995776 | 1394 | 196 |

The first Standard Normal Table was computed by Christian Kramp, French astronomer (1760–1826), in *Analyse des Réfractions Astronomiques et Terrestres*, 1799

Used a Taylor series expansion to the third power

integral from x = 0.03 to infinity of e^{-x^2}

\int_a^∞ Extended Keyboard ↑ Upload

Definite integral:

$$\int_{0.03}^{\infty} e^{-x^2} dx = 0.856236$$

Probabilities for a general Normal RV

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. To compute the CDF $P(X \leq x) = F(x)$, we use Φ , the CDF for the Standard Normal $Z \sim \mathcal{N}(0, 1)$:

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

Proof:

$$F(x) = P(X \leq x)$$

Definition of CDF

$$= P(X - \mu \leq x - \mu) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right)$$

Algebra + $\sigma > 0$

$$= P\left(Z \leq \frac{x - \mu}{\sigma}\right)$$

$$= \Phi\left(\frac{x - \mu}{\sigma}\right)$$

- $\frac{X - \mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$ is a linear transform of X .
- This is distributed as $\mathcal{N}\left(\frac{1}{\sigma}\mu - \frac{\mu}{\sigma}, \frac{1}{\sigma^2}\sigma^2\right) = \mathcal{N}(0, 1)$
- In other words, $\frac{X - \mu}{\sigma} = Z \sim \mathcal{N}(0, 1)$ with CDF Φ .

Probabilities for a general Normal RV

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. To compute the CDF $P(X \leq x) = F(x)$, we use Φ , the CDF for the Standard Normal $Z \sim \mathcal{N}(0, 1)$:

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

Proof:

$$F(x) = P(X \leq x)$$

Definition of CDF

$$= P(X - \mu \leq x - \mu) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right)$$

Algebra + $\sigma > 0$

$$= P\left(Z \leq \frac{x - \mu}{\sigma}\right)$$

$$= \Phi\left(\frac{x - \mu}{\sigma}\right)$$

• $\frac{X - \mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$ is a linear transform of X .

• This distribution is $\mathcal{N}\left(\frac{1}{\sigma}(1 \cdot \mu - 1 \cdot \mu), \frac{1}{\sigma^2} \cdot \sigma^2\right) = \mathcal{N}(0, 1)$.

1. Compute $z = (x - \mu)/\sigma$.

• 2. Look up $\Phi(z)$ in Standard Normal table.

Campus bikes

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

You spend some minutes, X , traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes²

Suppose X is normally distributed. What is the probability you spend ≥ 6 minutes traveling?



$$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2) \quad \times \quad P(X \geq 6) = \int_6^{\infty} f(x) dx \quad (\text{no analytic solution})$$

1. Compute $z = \frac{(x-\mu)}{\sigma}$

2. Look up $\Phi(z)$ in table

$$\begin{aligned} P(X \geq 6) &= 1 - F_x(6) \\ &= 1 - \Phi\left(\frac{6-4}{\sqrt{2}}\right) \\ &\approx 1 - \Phi(1.41) \end{aligned}$$

$$\begin{aligned} &1 - \Phi(1.41) \\ &\approx 1 - 0.9207 \\ &= \mathbf{0.0793} \end{aligned}$$

Is there an easier way? (yes)

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. What is $P(X \leq x) = F(x)$?

- Use Python

```
from scipy import stats
X = stats.norm(mu, std)
X.cdf(x)
```

Handwritten notes:
- $X.cdf(x)$ points to $1 - X.cdf(b)$
 μ points to μ
 σ points to σ

SciPy reference:

<https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.norm.html>

- Use website tool

Calculator

x:

mu:

std:

norm.cdf(x, mu, std)

= 0.5000

Navigation menu: CS109, Lectures, Problem Sets, Section, Handouts/Demos (selected), Administrivia, Calculation Ref, Python for Probability, Python Session Slides, Standard Normal Table, Normal CDF Calculator

Website tool:

<https://web.stanford.edu/class/cs109/handouts/normalCDF.html>

(live)

10: The Normal (Gaussian) Distribution

Lisa Yan and Jerry Cain
October 5, 2020

The Normal (Gaussian) Random Variable

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

$$g(x) = \frac{(x - \mu)^2}{2\sigma^2}$$

The PDF of X is defined as:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x - \mu)^2}{2\sigma^2}}$$

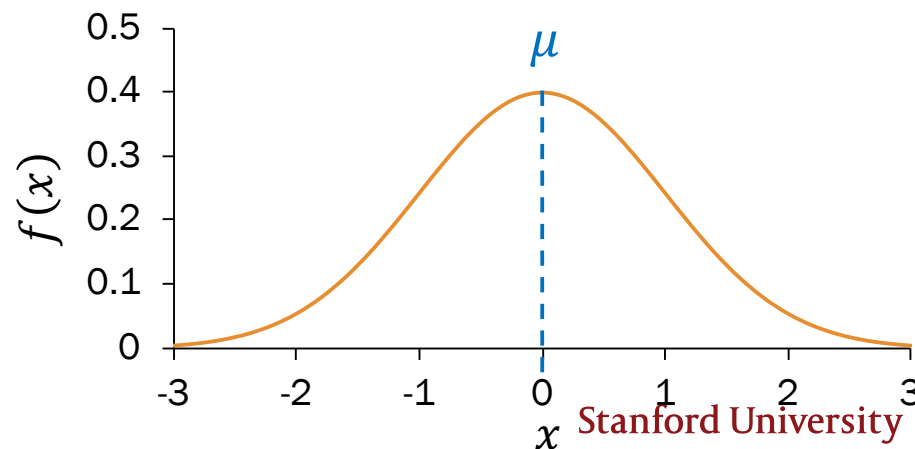
normalizing constant

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

symmetric
around μ

variance σ^2
manages spread

exponential
tail



Think

Slide 33 has a question to go over by yourself.

[Type and wait: 1: A, 2: B, ...]

Post any clarifications here or in chat!

<https://us.edstem.org/courses/2678/discussion/141411>

Think by yourself: 2 min

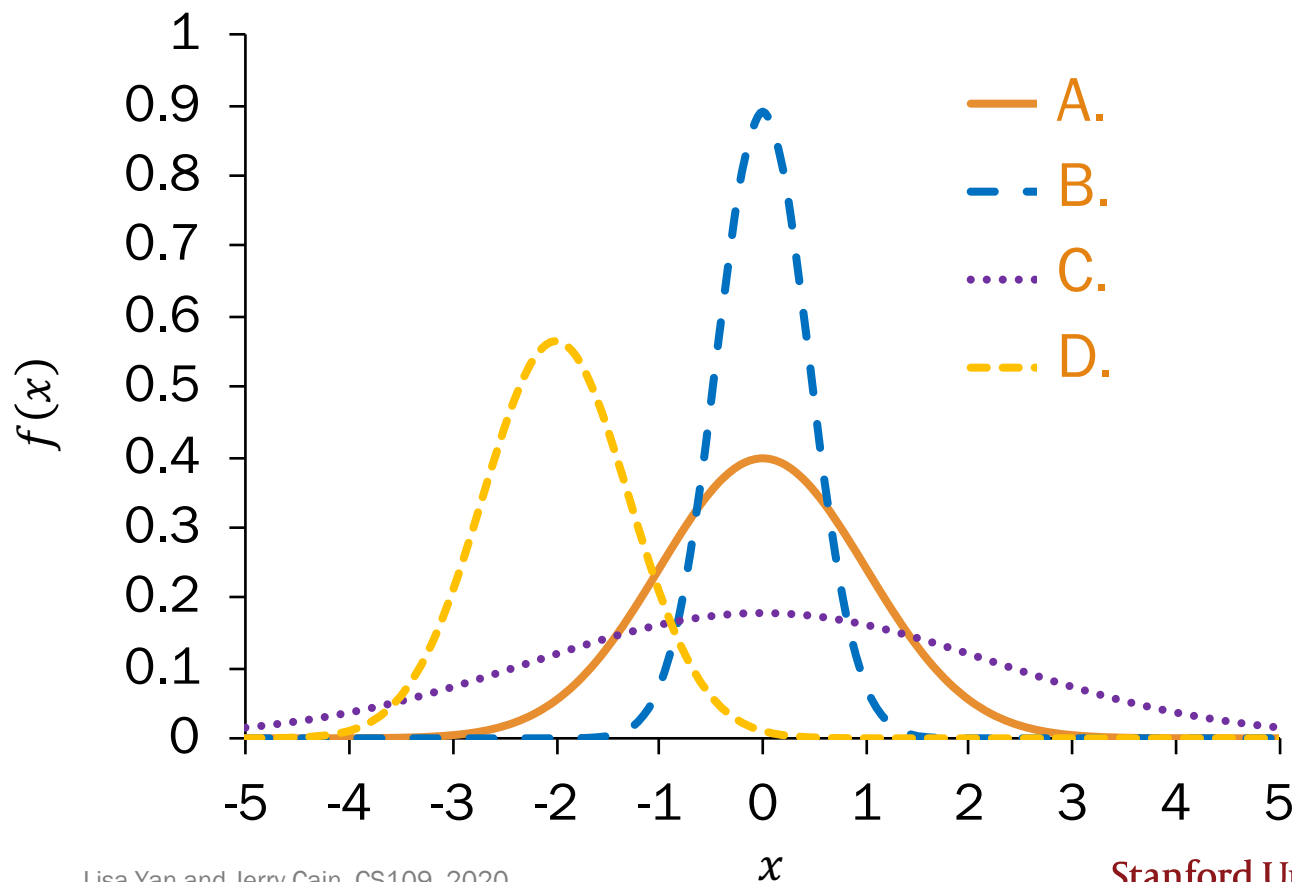


Normal Random Variable

$$X \sim \mathcal{N}(\overset{\text{mean}}{\mu}, \overset{\text{variance}}{\sigma^2})$$

Match PDF to distribution:

1. $\mathcal{N}(0, 1)$
2. $\mathcal{N}(-2, 0.5)$
3. $\mathcal{N}(0, 5)$
4. $\mathcal{N}(0, 0.2)$

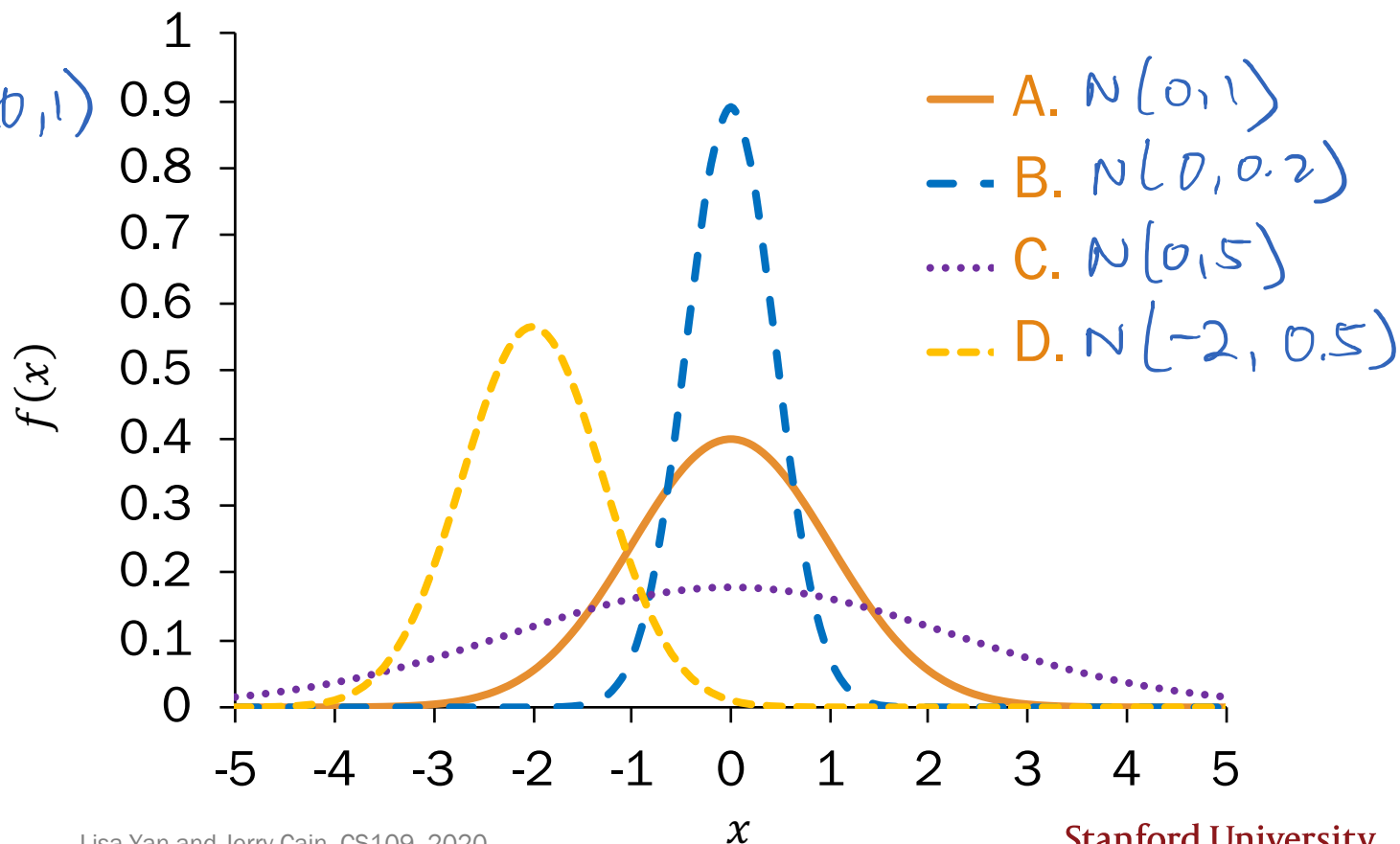


Normal Random Variable

$$X \sim \mathcal{N}(\overset{\text{mean}}{\mu}, \overset{\text{variance}}{\sigma^2})$$

Match PDF to distribution:

1. $\mathcal{N}(0, 1)$ A $Z \sim \mathcal{N}(0, 1)$
 Φ
2. $\mathcal{N}(-2, 0.5)$ D
3. $\mathcal{N}(0, 5)$ C
4. $\mathcal{N}(0, 0.2)$ B



Computing probabilities with Normal RVs: Old school

Standard Normal Table *

Note: An entry in the table is the area under the curve to the left of z , $P(Z \leq z) = \Phi(z)$



| Z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |

$\Phi(z)$ for non-negative z



*particularly useful when we have closed book exams with no calculator**

**we have open book exams with calculators this quarter

Knowing how to use a Standard Normal Table will still be useful in our understanding of Normal RVs.

Computing probabilities with Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. What is $P(X \leq x) = F(x)$?
↑

1. Rewrite in terms of standard normal CDF Φ by computing $z = \frac{(x-\mu)}{\sigma}$.

Linear transforms of Normals are Normal:

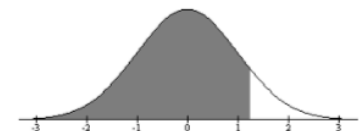
$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) \quad Z = \frac{(X-\mu)}{\sigma}, \text{ where } Z \sim \mathcal{N}(0,1)$$

2. Then, look up in a Standard Normal Table, where $z \geq 0$.

Symmetry of Normal PDFs implies:

$$\Phi(-z) = 1 - \Phi(z)$$

Standard Normal Table
Note: An entry in the table is the area under the curve to the left of



| Z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 |

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$

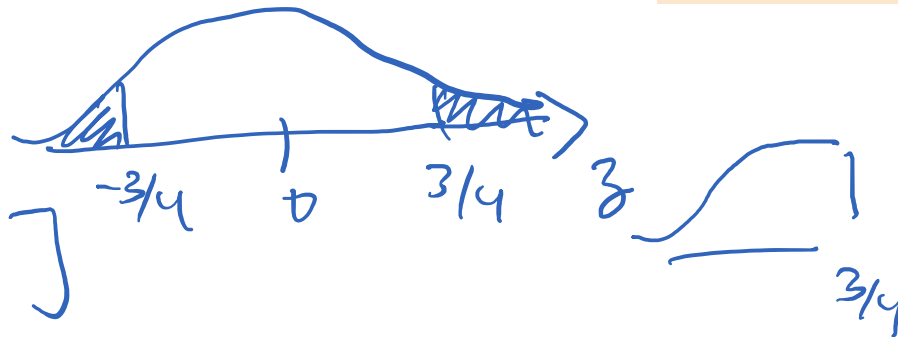
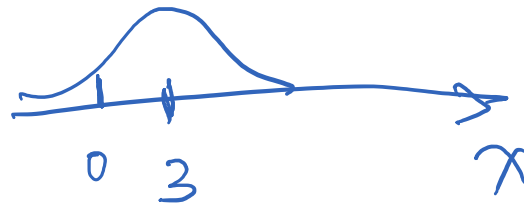
$$= 1 - F(0)$$

$$= 1 - \Phi\left(\frac{0-3}{4}\right)$$

$$= 1 - \Phi\left(-\frac{3}{4}\right)$$

$$= 1 - \left[1 - \Phi\left(\frac{3}{4}\right)\right]$$

$$= \Phi\left(\frac{3}{4}\right) \approx 0.7734$$



- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

Breakout Rooms

Slide 39 has two questions to go over in groups.

Post any clarifications here!

<https://us.edstem.org/courses/2678/discussion/141411>

Breakout rooms: 5 mins



Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$.

Note standard deviation $\sigma = 4$.

How would you write each of the below probabilities as a function of the standard normal CDF, Φ ?

1. $P(X > 0)$ (we just did this)

2. $P(2 < X < 5)$

3. $P(|X - 3| > 6)$

*How standard deviation
away from the mean*

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

*NONNEGATIVE $\Phi(z)$
 $z \geq 0$*



Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

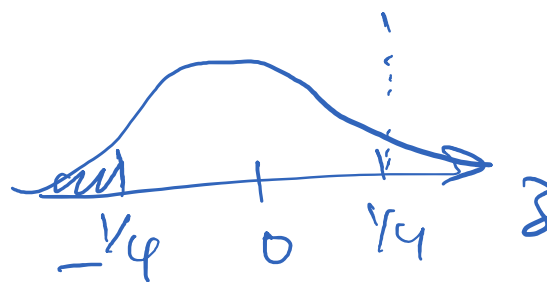
1. $P(X > 0)$

2. $P(2 < X < 5)$



for cont. RV:

$$\begin{aligned} &= F(5) - F(2) \\ &= \Phi\left(\frac{5-3}{4}\right) - \Phi\left(\frac{2-3}{4}\right) \\ &= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{4}\right) \\ &= \Phi\left(\frac{1}{2}\right) - \left[1 - \Phi\left(\frac{1}{4}\right)\right] \\ &\approx 0.2962 \end{aligned}$$



- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

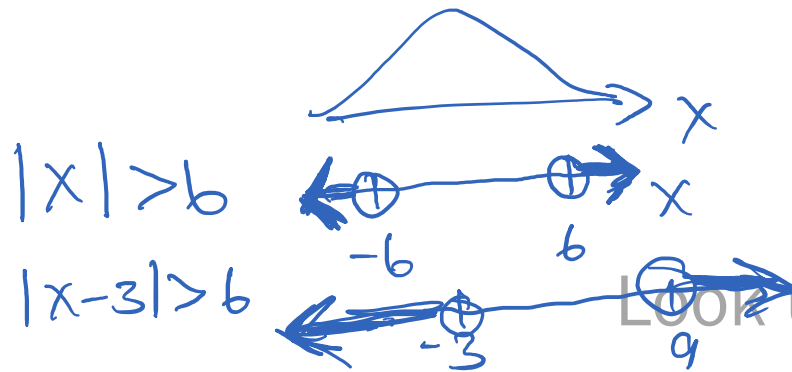
NON NEG $z \geq 0$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

Compute $z = \frac{(x - \mu)}{\sigma}$



- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-x) = 1 - \Phi(x)$

Look up $\Phi(z)$ in table

$$P(X < -3) + P(X > 9)$$

$$= F(-3) + (1 - F(9))$$

$$= \Phi\left(\frac{-3 - 3}{4}\right) + \left(1 - \Phi\left(\frac{9 - 3}{4}\right)\right)$$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$
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- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
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Compute $z = \frac{(x-\mu)}{\sigma}$

$$P(X < -3) + P(X > 9)$$

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$$= \Phi\left(\frac{-3-3}{4}\right) + \left(1 - \Phi\left(\frac{9-3}{4}\right)\right)$$

Look up $\Phi(z)$ in table

$$\begin{aligned} & \stackrel{\text{blue } \Phi\left(\frac{3}{2}\right)}{\leftarrow} = \Phi\left(-\frac{3}{2}\right) + \left(1 - \Phi\left(\frac{3}{2}\right)\right) \\ & = 2\left(1 - \Phi\left(\frac{3}{2}\right)\right) \\ & \approx \mathbf{0.1337} \end{aligned}$$



Interlude for jokes/announcements

Announcements

Quiz: Lectures 1-6 (up to expectation)
PS1, PS2, Sections 1 & 2
private clarifications
post to Ed

Apply to Section Lead!

The application for section leading is now open! Section leaders hired this quarter will start in Winter 2021. The only requirement is that you complete CS106B/X (or equivalent) by the end of this quarter; you don't have to be majoring in CS! We're currently accepting applications for students that have already taken CS106B/X (or equivalent), and the application will be due

- Thursday, October 8th at 11:59pm PT. If you're currently in CS106B, the application will open
- Friday, October 9th and will be due on Friday, October 23rd at 11:59pm PT. You can apply at cs198.stanford.edu. Feel free to email us at cs198@cs.stanford.edu if you have any questions. We hope to read your application soon!

Problem Set 3

not on quiz

Covers: Lecture 11 (Wed 10/7)
Due: next Friday 1pm

Friday's concept check (#12)

Extra credit + feedback on Nooks v
Zoom

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Breakout rooms: 5 mins



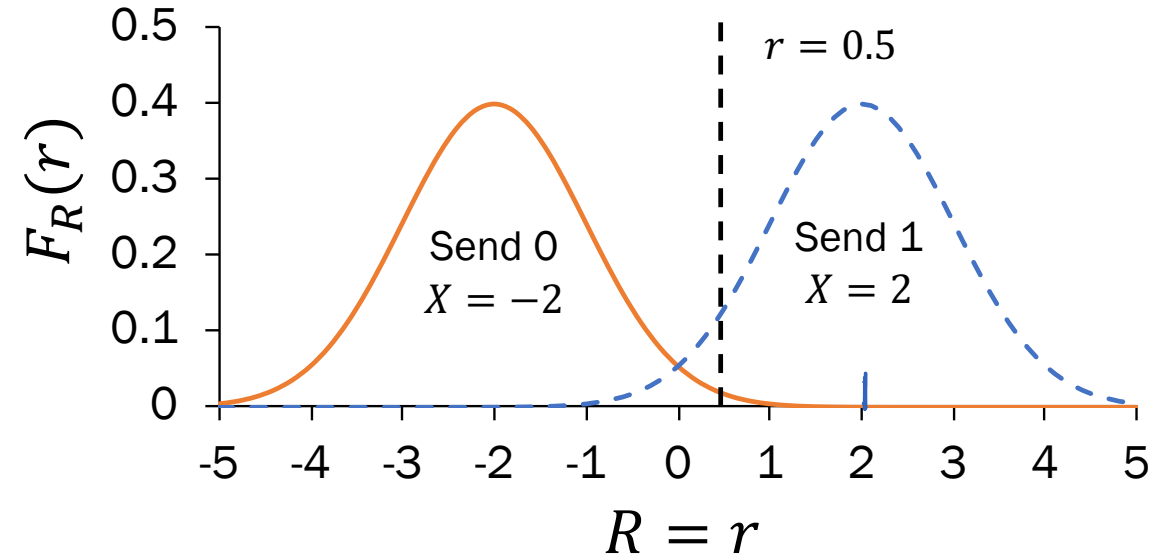
Noisy Wires

~3 min

Send a voltage of 2 V or -2 V on wire (to denote 1 and 0, respectively).

- X = voltage sent (2 or -2)
- Y = noise, $Y \sim \mathcal{N}(0, 1)$
- $R = X + Y$ voltage received.

Decode: 1 if $R \geq 0.5$
 0 otherwise.



1. What is $P(\text{decoding error} \mid \text{original bit is 1})$?
i.e., we sent 1, but we decoded as 0?
2. What is $P(\text{decoding error} \mid \text{original bit is 0})$?

These probabilities are unequal. Why might this be useful?



Noisy Wires

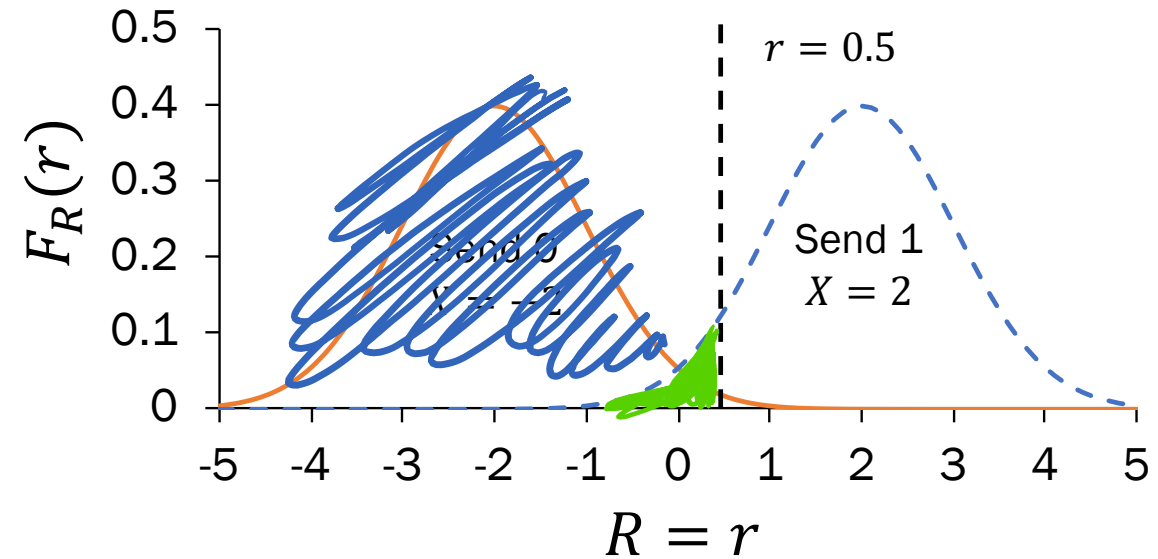
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Decode: 1 if $R \geq 0.5$
 0 otherwise.

1. What is $P(\text{decoding error} \mid \text{original bit is 1})$?
i.e., we sent 1, but we decoded as 0?

$$\begin{aligned} P(R < 0.5 \mid X = 2) &= P(\overbrace{2 + Y} < 0.5) = P(Y < -1.5) && Y \text{ is Standard Normal} \\ &= \Phi(-1.5) = 1 - \Phi(1.5) \approx \mathbf{0.0668} \end{aligned}$$

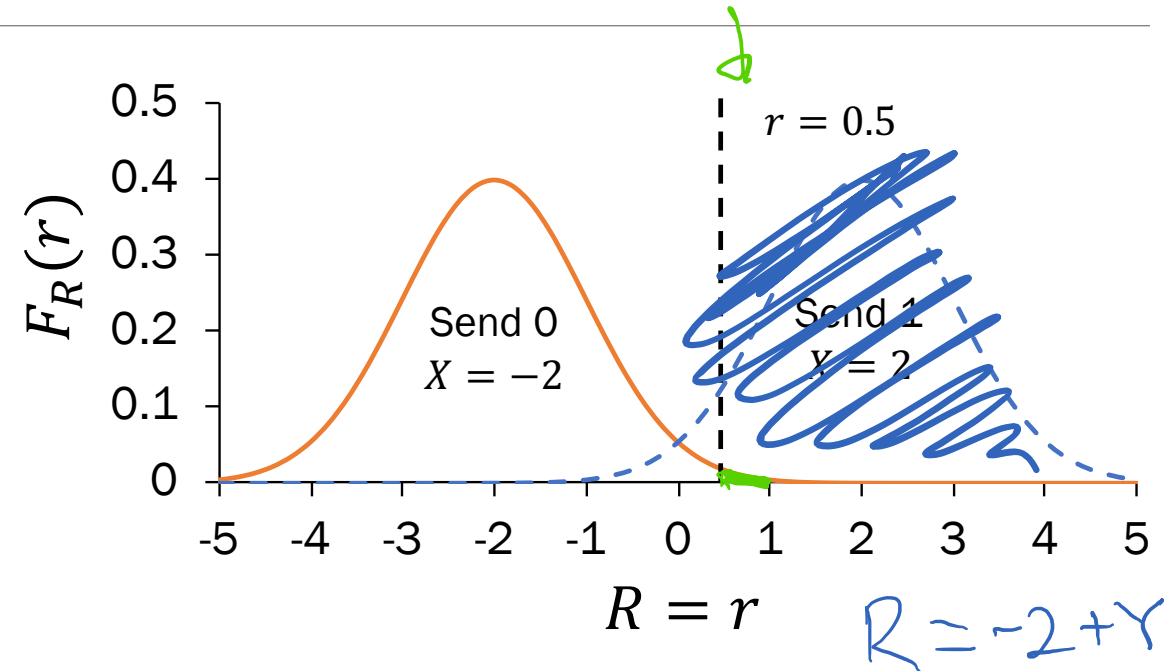


Noisy Wires

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1. What is $P(\text{decoding error} \mid \text{original bit is 1})$?
i.e., we sent 1, but we decoded as 0?

0.0668

2. What is $P(\text{decoding error} \mid \text{original bit is 0})$?

$$P(R \geq 0.5 \mid X = -2) = P(-2 + Y \geq 0.5) = P(Y \geq 2.5) \approx 0.0062$$

Asymmetric decoding probability: We would like to avoid mistaking a 0 for 1. Errors the other way are tolerable.

Challenge: Sampling with the Normal RV

ELO ratings

Basketball == Stats



What is the probability that the Warriors win?
How do you model zero-sum games?

ELO ratings

Each team has an ELO score S , calculated based on their past performance.

- Each game, a team has ability $A \sim \mathcal{N}(S, 200^2)$.
- The team with the higher sampled ability wins.

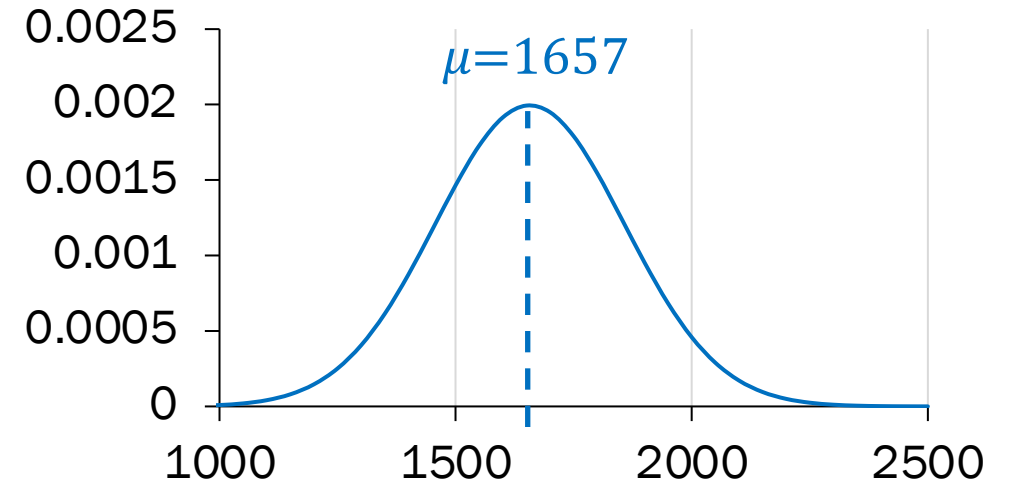


Arpad Elo

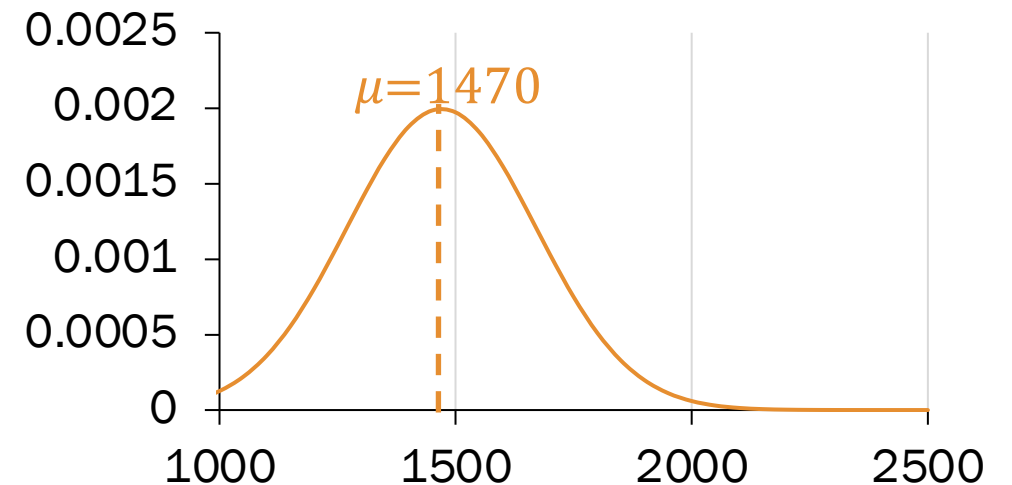
What is the probability that Warriors win this game?

Want: $P(\text{Warriors win}) = P(A_W > A_O)$

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponents $A_O \sim \mathcal{N}(S = 1470, 200^2)$



ELO ratings

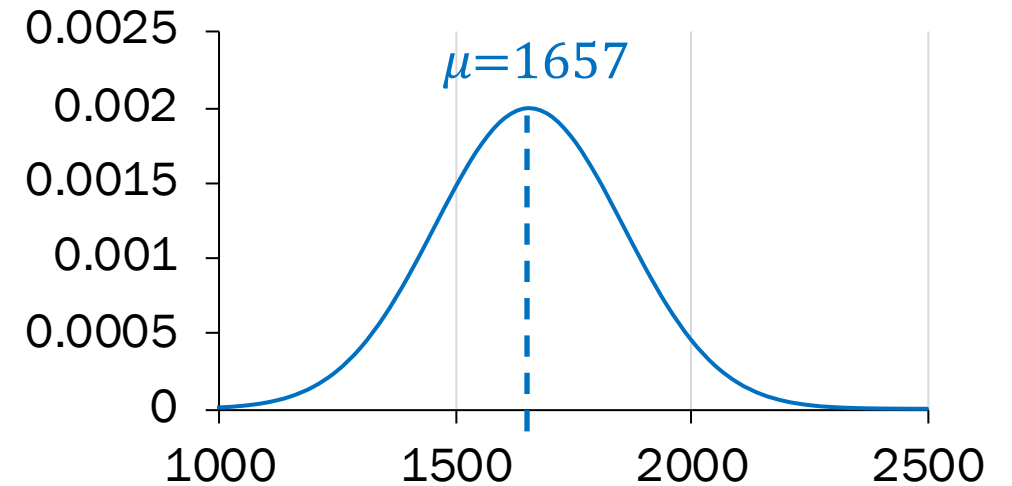
Want: $P(\text{Warriors win}) = P(A_W > A_O)$

```
from scipy import stats
WARRIORS_ELO = 1657
OPPONENT_ELO = 1470
STDEV = 200
NTRIALS = 10000
```

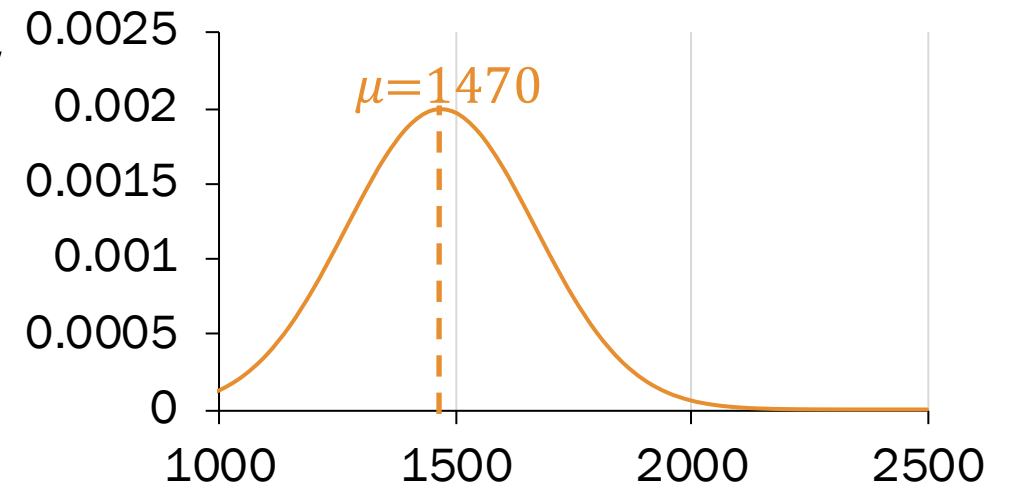
```
nSuccess = 0
for i in range(NTRIALS):
    w = stats.norm.rvs(WARRIORS_ELO, STDEV)
    opp = stats.norm.rvs(OPPONENT_ELO, STDEV)
    if w > opp:
        nSuccess += 1
print("Warriors sampled win fraction",
      float(nSuccess) / NTRIALS)
```

≈ 0.7488, calculated by sampling


Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponents $A_O \sim \mathcal{N}(S = 1470, 200^2)$



Is there a better way?


$$P(A_W > A_O)$$

- This is a probability of an event involving **two continuous random variables!**
- We'll solve this problem analytically in two weeks' time.

Big goal for next time: Events involving **two discrete random variables.**
Stay tuned!