

11: Joint (Multivariate) Distributions

Lisa Yan and Jerry Cain
October 7, 2020

Quick slide reference

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11a_normal_approx

Normal Approximation

Normal RVs

$$X \sim \mathcal{N}(\overset{\text{mean}}{\mu}, \overset{\text{variance}}{\sigma^2})$$

- Used to model many real-life situations because it maximizes entropy (i.e., randomness) for a given mean and variance
- Also useful for approximating the Binomial random variable!

Website testing

- 100 people are presented with a new website design.
- $X = \#$ people whose time on site increases
- The design has no effect, so $P(\text{time on site increases}) = 0.5$ independently.
- CEO will endorse the new design if $X \geq 65$.

What is $P(\text{CEO endorses change})$? Give a numerical approximation.

Approach 1: Binomial

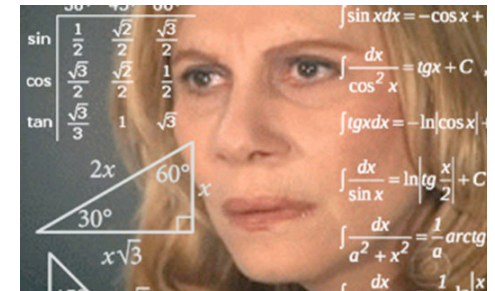
Define

$$X \sim \text{Bin}(n = 100, p = 0.5)$$

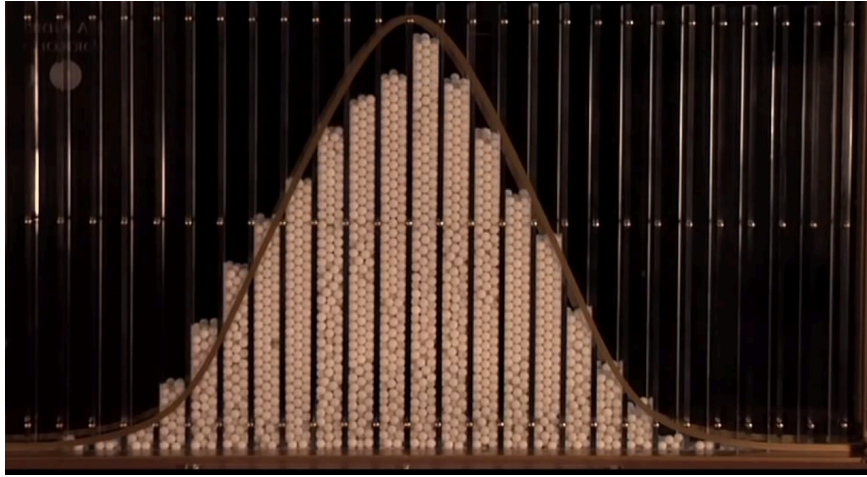
Want: $P(X \geq 65)$

Solve

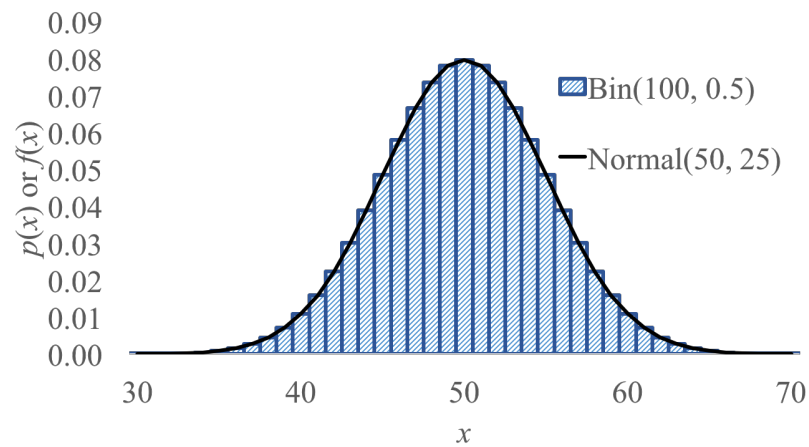
$$P(X \geq 65) = \sum_{i=65}^{100} \binom{100}{i} 0.5^i (1 - 0.5)^{100-i}$$



Don't worry, Normal approximates Binomial



Galton Board



id Jerry Cain, CS109, 2020

(We'll explain *why*
in 2 weeks' time)

Website testing

- 100 people are given a new website design.
- $X = \#$ people whose time on site increases
- The design actually has no effect, so $P(\text{time on site increases}) = 0.5$ independently.
- CEO will endorse the new design if $X \geq 65$.

What is $P(\text{CEO endorses change})$? Give a numerical approximation.

Approach 1: Binomial

Define

$$X \sim \text{Bin}(n = 100, p = 0.5)$$

Want: $P(X \geq 65)$

Solve

$$P(X \geq 65) \approx 0.0018$$

Approach 2: approximate with Normal

Define

$$Y \sim \mathcal{N}(\mu, \sigma^2)$$

$$\mu = np = 50$$

$$\sigma^2 = np(1 - p) = 25$$

$$\sigma = \sqrt{25} = 5$$

Solve

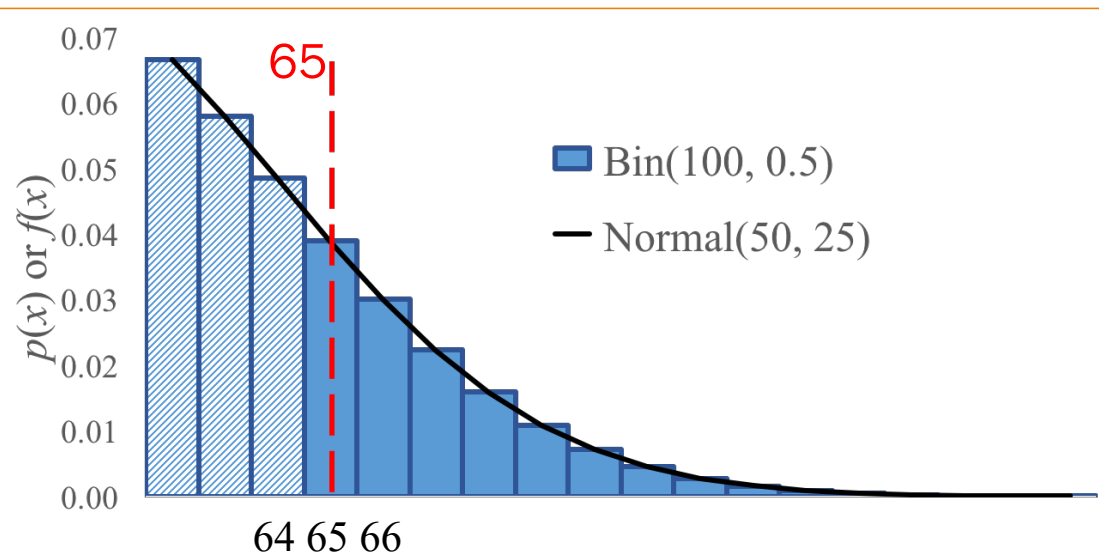
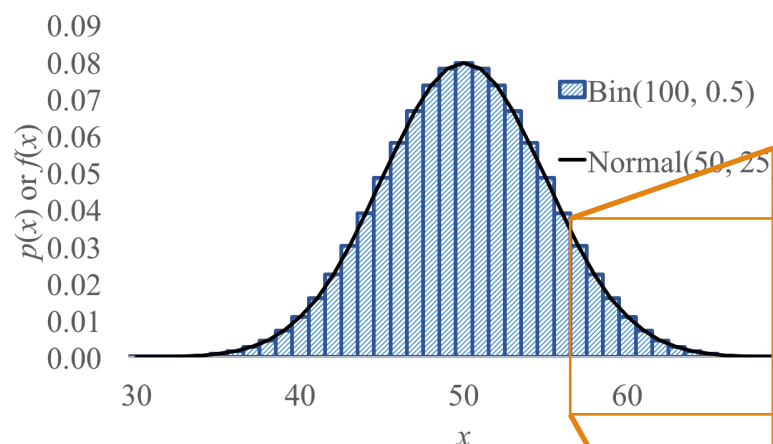
$$\begin{aligned} P(X \geq 65) &\approx P(Y \geq 65) = 1 - F_Y(65) \\ &= 1 - \Phi\left(\frac{65-50}{5}\right) = 1 - \Phi(3) \approx 0.0013? \end{aligned}$$



(this approach is missing something important)

Website testing (with continuity correction)

In our website testing, $Y \sim \mathcal{N}(50, 25)$ approximates $X \sim \text{Bin}(100, 0.5)$.



$$P(X \geq 65) \text{ Binomial}$$

$$\approx P(Y \geq 64.5) \text{ Normal}$$

$$\approx 0.0018 \quad \checkmark \text{ the better Approach 2}$$

You must perform a **continuity correction** when approximating a Binomial RV with a Normal RV.

Continuity correction

If $Y \sim \mathcal{N}(np, np(1 - p))$ approximates $X \sim \text{Bin}(n, p)$, how do we approximate the following probabilities?

Discrete (e.g., Binomial)
probability question



Continuous (Normal)
probability question

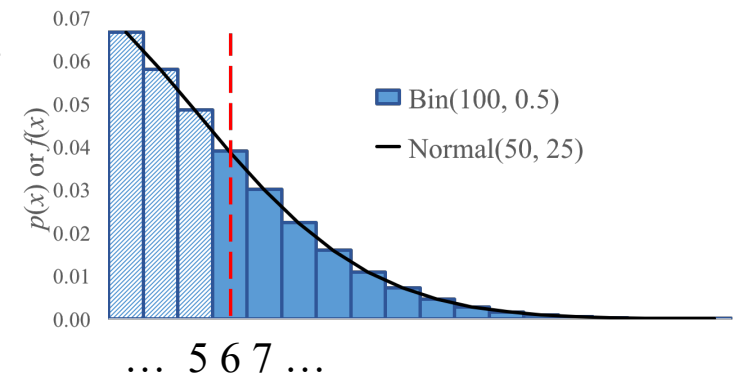
$$P(X = 6)$$

$$P(X \geq 6)$$

$$P(X > 6)$$

$$P(X < 6)$$

$$P(X \leq 6)$$



Continuity correction

If $Y \sim \mathcal{N}(np, np(1 - p))$ approximates $X \sim \text{Bin}(n, p)$, how do we approximate the following probabilities?

Discrete (e.g., Binomial)
probability question



Continuous (Normal)
probability question

$$P(X = 6)$$

$$P(X \geq 6)$$

$$P(X > 6)$$

$$P(X < 6)$$

$$P(X \leq 6)$$

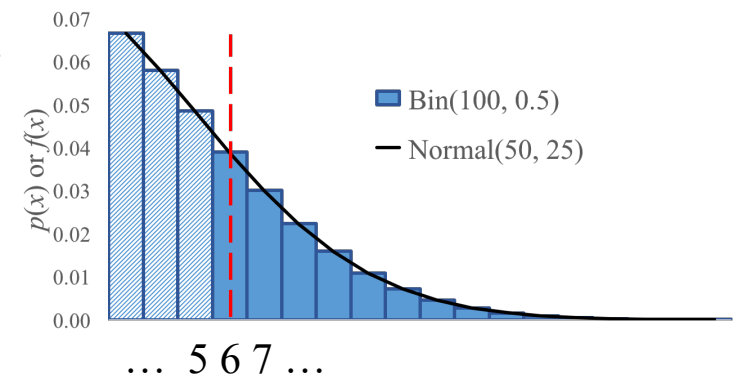
$$P(5.5 \leq Y \leq 6.5)$$

$$P(Y \geq 5.5)$$

$$P(Y \geq 6.5)$$

$$P(Y \leq 5.5)$$

$$P(Y \leq 6.5)$$



Who gets to approximate?

$$X \sim \text{Bin}(n, p)$$
$$E[X] = np$$
$$\text{Var}(X) = np(1 - p)$$



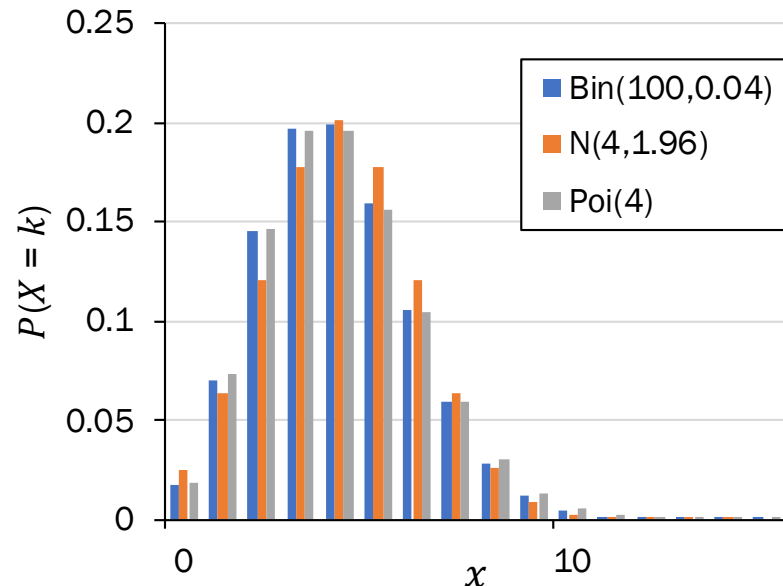
$$Y \sim \text{Poi}(\lambda)$$
$$\lambda = np$$

?

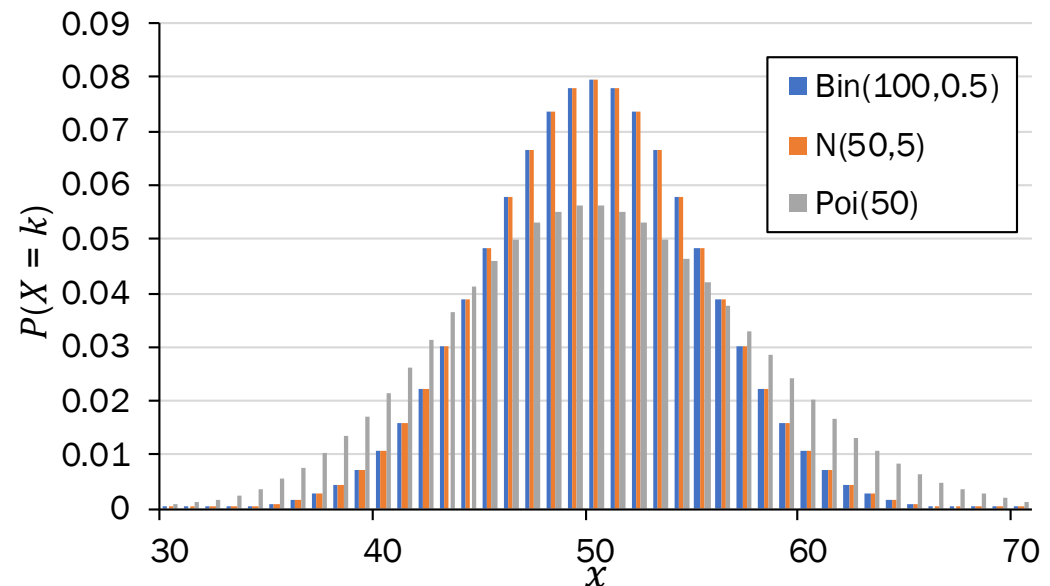


$$Y \sim \mathcal{N}(\mu, \sigma^2)$$
$$\mu = np$$
$$\sigma^2 = np(1 - p)$$

Who gets to approximate?



Poisson approximation
 n large (> 20), p small (< 0.05)
slight dependence okay



Normal approximation
 n large (> 20), p mid-ranged ($np(1 - p) > 10$)
independence

1. If there is a choice, use Normal to approximate.
2. When using Normal to approximate a discrete RV, use a continuity correction.

Discrete Joint RVs

From last time

Review



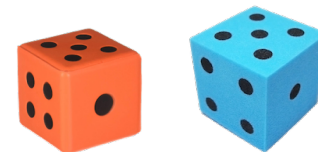
$$P(A_W > A_B)$$

This is a probability of an event involving *two* random variables!

What is the probability that the Warriors win?
How do you model zero-sum games?

Joint probability mass functions

Roll two 6-sided dice, yielding values X and Y .



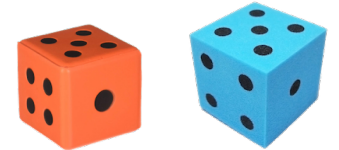
X random variable

$P(X = 1)$
probability of
an event

$P(X = k)$
probability mass function

Joint probability mass functions

Roll two 6-sided dice, yielding values X and Y .

 X

random variable

$$P(X = 1)$$

probability of
an event

$$P(X = k)$$

probability mass function

 X, Y

random variables

$$P(X = 1 \cap Y = 6)$$

$$P(X = 1, Y = 6)$$

new notation: the comma

probability of the intersection
of two events

$$P(X = a, Y = b)$$

joint probability mass function

Discrete joint distributions

For two discrete joint random variables X and Y , the **joint probability mass function** is defined as:

$$p_{X,Y}(a, b) = P(X = a, Y = b)$$

The **marginal distributions** of the joint PMF are defined as:

$$p_X(a) = P(X = a) = \sum_y p_{X,Y}(a, y)$$

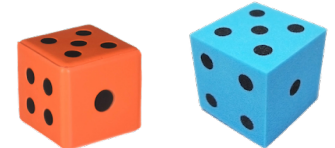
$$p_Y(b) = P(Y = b) = \sum_x p_{X,Y}(x, b)$$

Use marginal distributions to get a 1-D RV from a joint PMF.

Two dice

Roll two 6-sided dice, yielding values X and Y .

1. What is the joint PMF of X and Y ?



$$p_{X,Y}(a,b) = 1/36 \quad (a,b) \in \{(1,1), \dots, (6,6)\}$$

		X					
		1	2	3	4	5	6
Y	1	1/36	1/36
	2
	3
	4
	5
	6	1/36	1/36

An orange arrow points from the text $P(X = 4, Y = 2)$ to the cell at the intersection of $X=4$ and $Y=2$ in the table.

Probability table

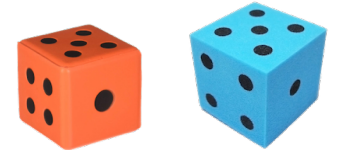
- All possible outcomes for several discrete RVs
- Not parametric (e.g., parameter p in $\text{Ber}(p)$)

Two dice

Roll two 6-sided dice, yielding values X and Y .

1. What is the joint PMF of X and Y ?

$$p_{X,Y}(a,b) = 1/36 \quad (a,b) \in \{(1,1), \dots, (6,6)\}$$



2. What is the marginal PMF of X ?

$$p_X(a) = P(X = a) = \sum_y p_{X,Y}(a,y) = \sum_{y=1}^6 \frac{1}{36} = \frac{1}{6} \quad a \in \{1, \dots, 6\}$$

A computer (or three) in every house.

Consider households in Silicon Valley.

- A household has X Macs and Y PCs.
- Each house has a maximum of 3 computers (Macs + PCs) in the house.

1. What is $P(X = 1, Y = 0)$, the missing entry in the probability table?

		X (# Macs)			
		0	1	2	3
Y (# PCs)	0	.16	?	.07	.04
	1	.12	.14	.12	0
	2	.07	.12	0	0
	3	.04	0	0	0



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	1	.12	.14	.12	0
	2	.07	.12	0	0
	3	.04	0	0	0

A joint PMF must sum to 1:

$$\sum_x \sum_y p_{X,Y}(x, y) = 1$$

A computer (or three) in every house.

Consider households in Silicon Valley.

- A household has X Macs and Y PCs.
- Each house has a maximum of 3 computers (Macs + PCs) in the house.

2. How do you compute the marginal PMF of X ?

		X (# Macs)				
		0	1	2	3	
Y (# PCs)	0	.16	.12	.07	.04	.39
	1	.12	.14	.12	0	.38
	2	.07	.12	0	0	.19
	3	.04	0	0	0	.04
B		.39	.38	.19	.04	sum rows here



A computer (or three) in every house.

Consider households in Silicon Valley.

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	1	.12	.14	.12	0	.38
	2	.07	.12	0	0	.19
	3	.04	0	0	0	.04
		sum cols here				
		sum rows here				

A. $p_{X,Y}(x, 0) = P(X = x, Y = 0)$

B. Marginal PMF of X $p_X(x) = \sum_y p_{X,Y}(x, y)$

C. Marginal PMF of Y $p_Y(y) = \sum_x p_{X,Y}(x, y)$

To find a marginal distribution over one variable, sum over all other variables in the joint PMF.

A computer (or three) in every house.

Consider households in Silicon Valley.

- A household has X Macs and Y PCs.
- Each house has a maximum of 3 computers (Macs + PCs) in the house.

3. Let $C = X + Y$. What is $P(C = 3)$?

		X (# Macs)			
		0	1	2	3
Y (# PCs)	0	.16	.12	.07	.04
	1	.12	.14	.12	0
	2	.07	.12	0	0
	3	.04	0	0	0



A computer (or three) in every house.

Consider households in Silicon Valley.

- A household has X Macs and Y PCs.
- Each house has a maximum of 3 computers (Macs + PCs) in the house.

3. Let $C = X + Y$. What is $P(C = 3)$?

		X (# Macs)			
		0	1	2	3
Y (# PCs)	0	.16	.12	.07	.04
	1	.12	.14	.12	0
	2	.07	.12	0	0
	3	.04	0	0	0

$$P(C = 3) = P(X + Y = 3)$$

Law of Total Probability

$$= \sum_x \sum_y P(X + Y = 3 | X = x, Y = y) P(X = x, Y = y)$$

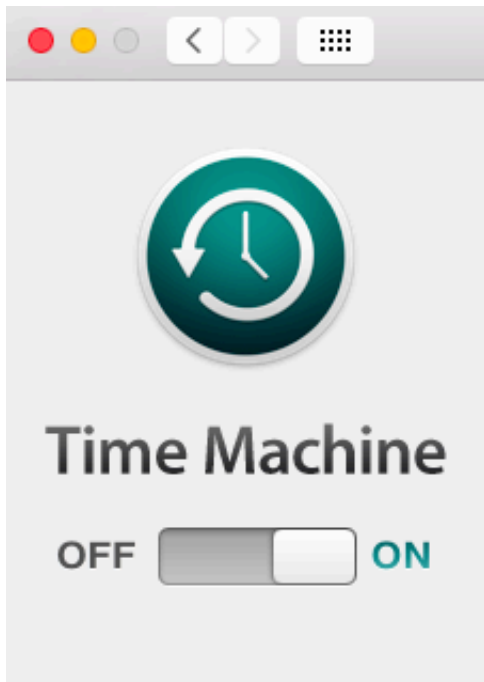
$$= P(X = 0, Y = 3) + P(X = 1, Y = 2) \\ + P(X = 2, Y = 1) + P(X = 3, Y = 0)$$

We'll come back to sums of RVs next lecture!

11c_multinomial

Multinomial RV

Recall the good times



Permutations

$n!$

How many ways are there to order n objects?

Counting unordered objects

Binomial coefficient

How many ways are there to group n objects into **two** groups of size k and $n - k$, respectively?

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Called the binomial coefficient because of something from Algebra

Multinomial coefficient

How many ways are there to group n objects into r groups of sizes n_1, n_2, \dots, n_r respectively?

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

Multinomials generalize Binomials for counting.

Probability

Binomial RV

What is the probability of getting k successes and $n - k$ failures in n trials?

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

Binomial # of ways of ordering the successes

Probability of each ordering of k successes is equal + mutually exclusive

Multinomial RV

What is the probability of getting c_1 of outcome 1, c_2 of outcome 2, ..., and c_m of outcome m in n trials?

Multinomial RVs also generalize Binomial RVs for probability!

Multinomial Random Variable

Consider an experiment of n independent trials:

- Each trial results in one of m outcomes. $P(\text{outcome } i) = p_i$, $\sum_{i=1}^m p_i = 1$
- Let $X_i = \#$ trials with outcome i

Joint PMF

$$P(X_1 = c_1, X_2 = c_2, \dots, X_m = c_m) = \binom{n}{c_1, c_2, \dots, c_m} p_1^{c_1} p_2^{c_2} \dots p_m^{c_m}$$

where $\sum_{i=1}^m c_i = n$ and $\sum_{i=1}^m p_i = 1$

Multinomial # of ways of ordering the outcomes

Probability of each ordering is equal + mutually exclusive

Hello dice rolls, my old friends

A 6-sided die is rolled 7 times.

What is the probability of getting:

- 1 one
- 0 threes
- 0 fives
- 1 two
- 2 fours
- 3 sixes



Hello dice rolls, my old friends

A 6-sided die is rolled 7 times.

What is the probability of getting:

- 1 one
- 1 two
- 0 threes
- 2 fours
- 0 fives
- 3 sixes

$$P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, X_6 = 3)$$

$$= \binom{7}{1,1,0,2,0,3} \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^3 = 420 \left(\frac{1}{6}\right)^7$$

Hello dice rolls, my old friends

A 6-sided die is rolled 7 times.

What is the probability of getting:

- 1 one
- 1 two
- 0 threes
- 2 fours
- 0 fives
- 3 sixes

of times
a six appears

$$P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, X_6 = 3)$$

$$= \binom{7}{1,1,0,2,0,3} \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^3 = 420 \left(\frac{1}{6}\right)^7$$

choose where the sixes appear

probability of rolling a six

this many times

11: Joint (Multivariate) Distributions (live)

Lisa Yan and Jerry Cain
October 7, 2020

Normal RVs

$$X \sim \mathcal{N}(\overset{\text{mean}}{\mu}, \overset{\text{variance}}{\sigma^2})$$

- Used to model many real-life situations because it maximizes entropy (i.e., randomness) for a given mean and variance
- Also useful for approximating the Binomial random variable!

Who gets to approximate?

Review

$$X \sim \text{Bin}(n, p)$$

$$E[X] = np$$

$$\text{Var}(X) = np(1 - p)$$



$$Y \sim \text{Poi}(\lambda)$$

$$\lambda = np$$

n large (> 20)

p small (< 0.05)

slight dependence okay

$$Y \sim \mathcal{N}(\mu, \sigma^2)$$

$$\mu = np$$

$$\sigma^2 = np(1 - p)$$

n large (> 20), p mid-ranged ($np(1 - p) > 10$)

independence

need continuity correction

- Computing probabilities on Binomial RVs is often computationally expensive.
- Two reasonable approximations, but when to use which?

Think

Check out the question on the next slide (Slide 38).

Post any clarifications here!

<https://us.edstem.org/courses/2678/discussion/141412>

Breakout rooms: 5 mins



Stanford Admissions (a while back)

Stanford accepts 2480 students.

- Each admitted student matriculates w.p. 0.68 (independent trials)
- Let $X = \#$ of students who will attend

What is $P(X > 1745)$? *Give a numerical approximation.*

- Strategy:
- A. Just Binomial
 - B. Poisson
 - C. Normal
 - D. None/other



Stanford Admissions (a while back)

Stanford accepts 2480 students.

- Each admitted student matriculates w.p. 0.68 (independent trials)
- Let $X = \#$ of students who will attend

What is $P(X > 1745)$? Give a numerical approximation.

- Strategy:
- A. Just Binomial not an approximation (also computationally expensive)
 - B. Poisson $p = 0.68$, not small enough
 - C. Normal** ✓ Variance $np(1 - p) = 540 > 10$
 - D. None/other

Define an approximation

Let $Y \sim \mathcal{N}(E[X], \text{Var}(X))$

$$E[X] = np = 1686$$

$$\text{Var}(X) = np(1 - p) \approx 540 \rightarrow \sigma = 23.3$$

$$P(X > 1745) \approx P(Y \geq 1745.5) \quad \text{! Continuity correction}$$

Solve

$$\begin{aligned} P(Y \geq 1745.5) &= 1 - F(1745.5) \\ &= 1 - \Phi\left(\frac{1745.5 - 1686}{23.3}\right) \end{aligned}$$

$$= 1 - \Phi(2.54) \approx \mathbf{0.0055}$$

Changes in Stanford Admissions

Stanford accepts 2480 students.

- Each admitted student matriculates w.p. 0.68 (independent trials)
- Let $X = \#$ of students who will attend

Yield rate 20
years ago

What is $P(X > 1745)$? Give a numerical approximation.

The Stanford Daily

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Class of 2018 admit rates lowest in University history

March 28, 2014 16 Comments [Tweet](#) [Like 901](#)

Alex Zivkovic
Desk Editor

Stanford admitted 2,138 students to the Class of 2018 in this year's admissions cycle, producing – at 5.07 percent – the lowest admit rate in University history.

The [University](#) received a total of 42,167 applications this year, a record total and a 8.6 percent increase over [last year's figure of 38,828](#). Stanford [accepted 748 students](#)



Overview for the Class of 2022

- Total Applicants: 47,451 Admit rate: 4.3%
- Total Admits: 2,071 Yield rate: 81.9%
- Total Enrolled: 1,706

People love coming to Stanford!

Multinomial Random Variable

Review

Consider an experiment of n independent trials:

- Each trial results in one of m outcomes. $P(\text{outcome } i) = p_i$, $\sum_{i=1}^m p_i = 1$
- Let $X_i = \#$ trials with outcome i

Joint PMF

$$P(X_1 = c_1, X_2 = c_2, \dots, X_m = c_m) = \binom{n}{c_1, c_2, \dots, c_m} p_1^{c_1} p_2^{c_2} \dots p_m^{c_m}$$

where $\sum_{i=1}^m c_i = n$ and $\sum_{i=1}^m p_i = 1$

Example:

- Rolling 2 twos, 3 threes, and 5 fives on 10 rolls of a fair-sided die
- Generating a random 5-word phrase with 1 “the”, 2 “bacon”, 1 “put”, 1 “on”

Hello dice rolls, my old friends

Review

A 6-sided die is rolled 7 times.

What is the probability of getting:

- 1 one
- 1 two
- 0 threes
- 2 fours
- 0 fives
- 3 sixes

of times
a six appears

$$P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, X_6 = 3)$$

$$= \binom{7}{1,1,0,2,0,3} \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^3 = 420 \left(\frac{1}{6}\right)^7$$

choose where the sixes appear

probability of rolling a six

this many times

Parameters of a Multinomial RV?

$X \sim \text{Bin}(n, p)$ has parameters $n, p \dots$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

p : probability of success outcome on a single trial

A Multinomial RV has parameters n, p_1, p_2, \dots, p_m (Note $p_m = 1 - \sum_{i=1}^{m-1} p_i$)

$$P(X_1 = c_1, X_2 = c_2, \dots, X_m = c_m) = \binom{n}{c_1, c_2, \dots, c_m} p_1^{c_1} p_2^{c_2} \dots p_m^{c_m}$$

p_i : probability of outcome i on a single trial

Where do we get p_i from?

Interlude for jokes/announcements

Announcements

Quiz #1

Time frame: Wednesday, 10/7 2:00pm – Friday, 10/9, 1:00pm PT

Covers: Up to end of Week 2 (including Lecture 6), PSets 1 and 2

Info and practice: <https://web.stanford.edu/class/cs109/exams/quizzes.html>



Thoughts on CS109, pre-quiz:

- A checkpoint for *you*, not other people.
- We are all here to learn. This exam was designed for a range of students.
- Typesetting will take a bit of time.
- Take breaks, stretch, sleep!
- The staff and I are here for *you*.

Other things this week

- Friday's concept check #12 is EC!
- Remember Zoom chat is a family channel

Interesting probability article

Estimating Coronavirus Prevalence by Cross-Checking Countries

<https://medium.com/@jsteinhardt/estimating-coronavirus-prevalence-by-cross-checking-countries-c7e4211f0e18>

We'll make the modeling assumption that N_{ij} is a **Poisson distribution with rate parameter** $A_{ij} * \lambda_i * \alpha_j$. What this means is that the **expected number of cases** should be equal to the total amount of travel, times some source-dependent multiplier α_j ..., times some country-dependent multiplier λ_i (the infection prevalence in country i)."



LIVE

Prelude: The Federalist Papers

Probabilistic text analysis

Probabilities of *counts* of words = Multinomial distribution

Example document:

#words: $n = 48$

“When my late husband was alive he deposited some amount of Money with china Bank in which the amount will be declared to you once you respond to this message indicating your interest in helping to receive the fund and use it for Gods work as my wish.”

$$P \left(\begin{array}{l} \text{bank} = 1 \\ \text{fund} = 1 \\ \text{money} = 1 \\ \text{wish} = 1 \\ \dots \\ \text{to} = 3 \end{array} \middle| \text{spam} \right) = \frac{n!}{1! 1! 1! 1! \dots 3!} p_{\text{bank}}^1 p_{\text{fund}}^1 \dots p_{\text{to}}^3$$

Note: $P(\text{bank} | \text{spam}) \gg P(\text{bank} | \text{writer} = \text{you})$

Old and New Analysis

Authorship of the Federalist Papers

- 85 essays advocating ratification of the US constitution
- Written under the pseudonym “Publius” (really, Alexander Hamilton, James Madison, John Jay)



Who wrote which essays?

- Analyze probability of words in each essay and compare against word distributions from known writings of three authors
- Curious what the analysis is? You'll love PSet 4!

LIVE

Statistics of Two RVs

Expectation and Covariance

In real life, we often have many RVs interacting at once.

- We've seen some simpler cases (e.g., sum of independent Poissons).
- Computing joint PMFs in general is hard!
- But **often you don't need to model** joint RVs completely.

Instead, we'll focus next on reporting **statistics** of multiple RVs:

- **Expectation of sums** (you've seen some of this, more on this today)
- **Covariance**: measure of how two RVs vary with each other (more on this come Monday)

Properties of Expectation, extended to two RVs

1. Linearity:

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

2. Expectation of a sum = sum of expectation:

$$E[X + Y] = E[X] + E[Y]$$



(we've seen this;
we'll prove today!)

3. Unconscious statistician:

$$E[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X, Y}(x, y)$$

True for both independent
and dependent random
variables!

Proof of expectation of a sum of RVs

$$E[X + Y] = E[X] + E[Y]$$

$$E[X + Y] = \sum_x \sum_y (x + y)p_{X,Y}(x, y)$$

LOTUS,
 $g(X, Y) = X + Y$

$$= \sum_x \sum_y xp_{X,Y}(x, y) + \sum_x \sum_y yp_{X,Y}(x, y)$$

$$= \sum_x x \sum_y p_{X,Y}(x, y) + \sum_y y \sum_x p_{X,Y}(x, y)$$

Linearity of summations
(cont. case: linearity of integrals)

$$= \sum_x xp_X(x) + \sum_y yp_Y(y)$$

Marginal PMFs for X and Y

$$= E[X] + E[Y]$$

Expectations of common RVs: Binomial

Review

$$X \sim \text{Bin}(n, p) \quad E[X] = np$$

of successes in n independent trials with probability of success p

Recall: $\text{Bin}(1, p) = \text{Ber}(p)$

$$X = \sum_{i=1}^n X_i$$

Let $X_i = i$ th trial is heads
 $X_i \sim \text{Ber}(p), E[X_i] = p$



$$E[X] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] = \sum_{i=1}^n p = np$$