13: Statistics of Multiple RVs

Lisa Yan and Jerry Cain
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Quick slide reference

13a_expectation_sum

Expectation of Common RVs

Linearity of Expectation is useful

Expectation is a linear mathematical operation. If $X = \sum_{i=1}^{n} X_i$:

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

- Even if you don't know the **distribution** of X (e.g., because the joint distribution of $(X_1, \ldots, X_n)$ is unknown), you can still compute **expectation** of X!!

- Problem-solving key: Define X_i such that $X = \sum_{i=1}^{n} X_i$

Most common use cases:

- $E[X_i]$ easy to calculate
- Or sum of dependent RVs

Expectations of common RVs: Binomial

Review

$X \sim \text{Bin}(n, p)$ $\quad E[X] = np$

# of successes in n independent trials with probability of success p

Recall: $\text{Bin}(1, p) = \text{Ber}(p)$

$$X = \sum_{i=1}^{n} X_i$$

Let $X_i = i$th trial is heads
$X_i \sim \text{Ber}(p), E[X_i] = p$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np$$

Expectations of common RVs: Negative Binomial

$Y \sim \text{NegBin}(r, p)$ $\quad E[Y] = \frac{r}{p}$

\# of independent trials with probability of success p until r successes

Recall: $\text{NegBin}(1, p) = \text{Geo}(p)$

$$Y = \sum_{i=1}^{?} Y_i$$

1. How should we define Y_i?
2. How many terms are in our summation?

Expectations of common RVs: Negative Binomial

$Y \sim \text{NegBin}(r, p) \quad E[Y] = \frac{r}{p}$

# of independent trials with probability of success p until r successes

Recall: $\text{NegBin}(1, p) = \text{Geo}(p)$

$$Y = \sum_{i=1}^{?} Y_i$$

Let Y_i = # trials to get ith success (after $(i-1)$th success)

$Y_i \sim \text{Geo}(p), E[Y_i] = \frac{1}{p}$

$$E[Y] = E\left[\sum_{i=1}^{r} Y_i\right] = \sum_{i=1}^{r} E[Y_i] = \sum_{i=1}^{r} \frac{1}{p} = \frac{r}{p}$$

13b_coupon_collecting

Coupon Collecting Problems

Linearity of Expectation is useful

Expectation is a linear mathematical operation. If $X = \sum_{i=1}^{n} X_i$:

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

- Even if you *don't know* the distribution of X (e.g., because the joint distribution of $(X_1, \dots, X_n)$ is unknown), you can still compute *expectation* of the sum!!

- Problem-solving key: Define X_i such that $X = \sum_{i=1}^{n} X_i$

Most common use cases:
- $E[X_i]$ easy to calculate
- Or sum of dependent RVs

Coupon collecting problems: Server requests

The **coupon collector's problem** in probability theory:

- You buy boxes of cereal.
- There are k different types of coupons
- For each box you buy, you "collect" a coupon of type i.

1. How many coupons do you expect after buying n boxes of cereal?

<u>Servers</u>

requests

k servers

request to server i

What is the expected number of utilized servers after n requests?

* 52% of Amazon profits

** more profitable than Amazon's North America commerce operations

[source]

Computer cluster utilization

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Consider a computer cluster with k servers. We send n requests.

- Requests independently go to server i with probability p_i
- Let X = # servers that receive ≥ 1 request.

What is $E[X]$?

Computer cluster utilization

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Consider a computer cluster with k servers. We send n requests.

- Requests independently go to server i with probability p_i
- Let X = # servers that receive ≥ 1 request.

What is $E[X]$?

1. Define additional random variables.

Let: A_i = event that server i receives ≥ 1 request

X_i = indicator for A_i

$$P(A_i) = 1 - P(\text{no requests to } i) = 1 - (1 - p_i)^n$$

Note: A_i are dependent!

2. Solve.

$$E[X_i] = P(A_i) = 1 - (1 - p_i)^n$$

$$E[X] = E\left[\sum_{i=1}^{k} X_i\right] = \sum_{i=1}^{k} E[X_i] = \sum_{i=1}^{k} (1 - (1 - p_i)^n)$$

$$= \sum_{i=1}^{k} 1 - \sum_{i=1}^{k} (1 - p_i)^n = k - \sum_{i=1}^{k} (1 - p_i)^n$$

Coupon collecting problems: Hash tables

The coupon collector's problem in probability theory:

- You buy boxes of cereal.
- There are k different types of coupons
- For each box you buy, you "collect" a coupon of type i.

Servers	Hash Tables
requests	strings
k servers	k buckets
request to server i	hashed to bucket i

1. How many coupons do you expect after buying n boxes of cereal? → What is the expected number of utilized servers after n requests?

2. How many boxes do you expect to buy until you have one of each coupon? → What is the expected number of strings to hash until each bucket has ≥ 1 string?

Stay tuned for live lecture!

Covariance

Statistics of sums of RVs

For any random variables X and Y,

$$E[X+Y] = E[X] + E[Y]$$

$$\text{Var}(X+Y) = \quad ?$$

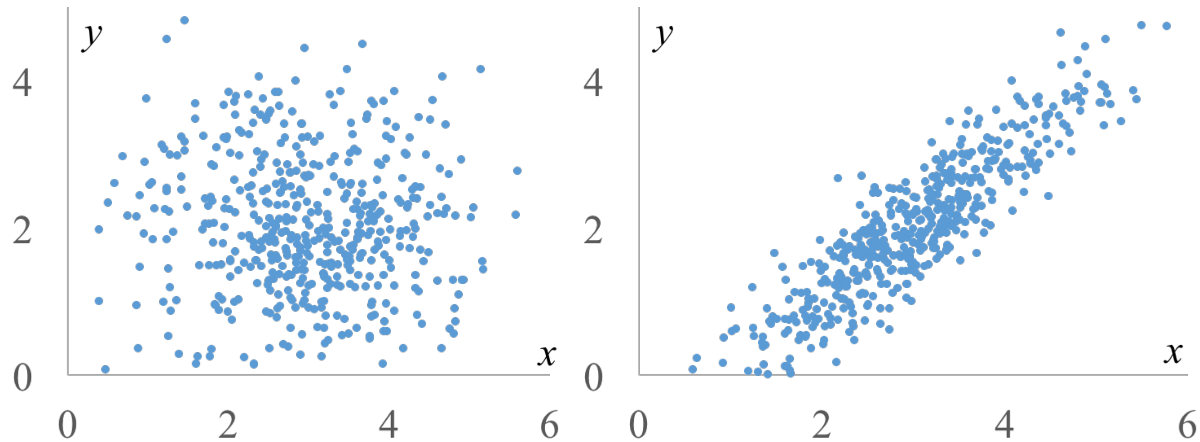
But first...
a new statistic!

Spot the difference

Compare/contrast the following two distributions:

Assume all points are equally likely.

$P(X = x, Y = y) = \frac{1}{N}$



Both distributions have the same $E[X]$, $E[Y]$, Var(X), and Var(Y)

Difference: how the two variables vary with ***each other***.

Covariance

The covariance of two variables X and Y is:

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

Proof of second part:

$$\begin{aligned}
\text{Cov}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\
&= E\left[XY - XE[Y] - E[X]Y + E[X]E[Y]\right] \\
&= E[XY] - E\left[XE[Y]\right] - E[E[X]Y] + E\left[E[X]E[Y]\right] \\
&= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\
&= E[XY] - E[X]E[Y]
\end{aligned}$$

(linearity of expectation)

($E[X]$, $E[Y]$ are scalars)

Covarying humans

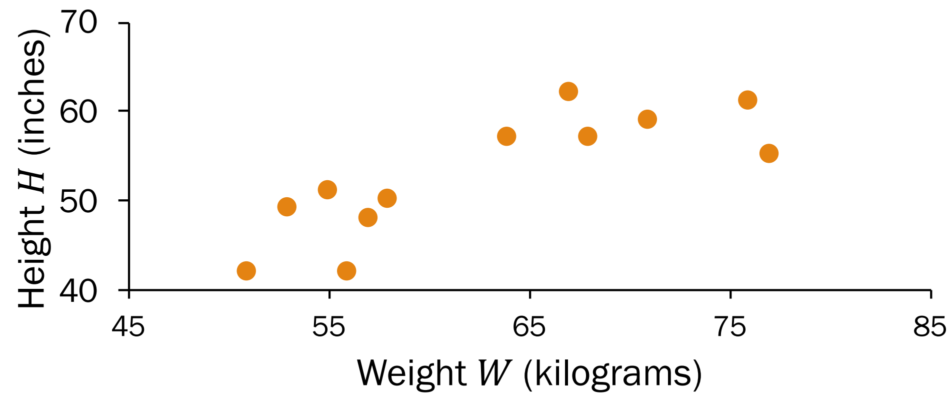
$\text{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$
$= E[XY] - E[X]E[Y]$

Weight (kg)	Height (in)	W · H
64	57	3648
71	59	4189
53	49	2597
67	62	4154
55	51	2805
58	50	2900
77	55	4235
57	48	2736
56	42	2352
51	42	2142
76	61	4636
68	57	3876

$E[W] = 62.75$ $E[H] = 52.75$ $E[WH] = 3355.83$

What is the covariance of weight W and height H?

$$\text{Cov}(W,H) = E[WH] - E[W]E[H]$$
$$= 3355.83 - (62.75)(52.75)$$
(positive) $= 45.77$



Covariance > 0: one variable ↑, other variable ↑

Properties of Covariance

The **covariance** of two variables X and Y is:

$$\text{Cov}(X,Y) = E[(X-E[X])(Y-E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

Properties:

1. $\text{Cov}(X,Y) = \text{Cov}(Y,X)$
2. $\text{Var}(X) = E[X^2] - (E[X])^2 = \text{Cov}(X,X)$
3. Covariance of sums = sum of all pairwise covariances (proof left to you)
   $\text{Cov}(X_1 + X_2, Y_1 + Y_2) = \text{Cov}(X_1,Y_1) + \text{Cov}(X_2,Y_1) + \text{Cov}(X_1,Y_2) + \text{Cov}(X_2,Y_2)$
4. Non-linearity (to be discussed in live lecture)

Variance of sums of RVs

Statistics of sums of RVs

For any random variables X and Y,

$$E[X + Y] = E[X] + E[Y]$$

$$\text{Var}(X + Y) = \text{Var}(X) + 2 \cdot \text{Cov}(X, Y) + \text{Var}(Y)$$

Variance of general sum of RVs

For any random variables X and Y,

$$\text{Var}(X+Y) = \text{Var}(X) + 2 \cdot \text{Cov}(X,Y) + \text{Var}(Y)$$

Proof:

$$\begin{aligned}\text{Var}(X+Y) &= \text{Cov}(X+Y, X+Y) \\ &= \text{Cov}(X,X) + \text{Cov}(X,Y) + \text{Cov}(Y,X) + \text{Cov}(Y,Y) \\ &= \text{Var}(X) + 2 \cdot \text{Cov}(X,Y) + \text{Var}(Y)\end{aligned}$$

$\text{Var}(X) = \text{Cov}(X,X)$

covariance of all pairs

Symmetry of covariance + $\text{Cov}(X,X) = \text{Var}(X)$

More generally:

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i=1}^{n}\sum_{j=\text{i}+1}^{n} \text{Cov}\left(X_i, X_j\right)$$

(proof in extra slides)

Statistics of sums of RVs

For any random variables X and Y,

$$E[X + Y] = E[X] + E[Y]$$

$$\text{Var}(X + Y) = \text{Var}(X) + 2 \cdot \text{Cov}(X, Y) + \text{Var}(Y)$$

For independent X and Y,

$$E[XY] = E[X]E[Y]$$

(Lemma: proof in extra slides)

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

Variance of sum of independent RVs

For independent X and Y,

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

Proof:

1. $\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$ — def. of covariance

   $= E[X]E[Y] - E[X]E[Y]$ — X and Y are independent

   $= 0$

2. $\text{Var}(X+Y) = \text{Var}(X) + 2 \cdot \text{Cov}(X,Y) + \text{Var}(Y)$

   $= \text{Var}(X) + \text{Var}(Y)$

NOT bidirectional:
$\text{Cov}(X,Y) = 0$ does NOT imply independence of X and Y!

Proving Variance of the Binomial

$X \sim \text{Bin}(n, p) \quad \text{Var}(X) = np(1-p)$

To simplify the algebra a bit, let $q = 1 - p$, so $p + q = 1$.

So:

$$\begin{aligned}
E(X^2) &= \sum_{k \geq 0}^{n} k^2 \binom{n}{k} p^k q^{n-k} && \text{Definition of Binomial Distribution: } p + q = 1 \\
&= \sum_{k=0}^{n} kn \binom{n-1}{k-1} p^k q^{n-k} && \text{Factors of Binomial Coefficient: } k\binom{n}{k} = n\binom{n-1}{k-1} \\
&= np \sum_{k=1}^{n} k \binom{n-1}{k-1} p^{k-1} q^{(n-1)-(k-1)} && \text{Change of limit: term is zero when } k-1=0 \\
&= np \sum_{j=0}^{m} (j+1) \binom{m}{j} p^j q^{m-j} && \text{putting } j = k-1,\ m = n-1 \\
&= np \left(\sum_{j=0}^{m} j \binom{m}{j} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right) && \text{splitting sum up into two} \\
&= np \left(\sum_{j=0}^{m} m \binom{m-1}{j-1} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right) && \text{Factors of Binomial Coefficient: } j\binom{m}{j} = m\binom{m-1}{j-1} \\
&= np \left((n-1)p \sum_{j=1}^{m} \binom{m-1}{j-1} p^{j-1} q^{(m-1)-(j-1)} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right) && \text{Change of limit: term is zero when } j-1=0 \\
&= np((n-1)p(p+q)^{m-1} + (p+q)^m) && \text{Binomial Theorem} \\
&= np((n-1)p+1) && \text{as } p+q=1 \\
&= n^2p^2 + np(1-p) && \text{by algebra}
\end{aligned}$$

Then:

$$\begin{aligned}
\text{var}(X) &= E(X^2) - (E(X))^2 \\
&= np(1-p) + n^2p^2 - (np)^2 && \text{Expectation of Binomial Distribution: } E(X) = np \\
&= np(1-p)
\end{aligned}$$

as required.

proofwiki.org

Let's instead prove this using independence and variance!

Proving Variance of the Binomial

$X \sim \text{Bin}(n, p) \quad \text{Var}(X) = np(1-p)$

Let $X = \sum_{i=1}^{n} X_i$

Let X_i = ith trial is heads

$X_i \sim \text{Ber}(p)$

$\text{Var}(X_i) = p(1-p)$

X_i are independent (by definition)

$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^{n} X_i\right)$$

$$= \sum_{i=1}^{n} \text{Var}(X_i)$$

X_i are independent, therefore variance of sum = sum of variance

$$= \sum_{i=1}^{n} p(1-p)$$

Variance of Bernoulli

$$= np(1-p)$$

(live)

13: Statistics of Multiple RVs

Lisa Yan and Jerry Cain

October 12, 2020

Where are we now? A roadmap of CS109

Last week: Joint distributions

$p_{X,Y}(x,y)$

Today: Statistics of multiple RVs!

$\text{Var}(X+Y)$

$E[X+Y]$

$\text{Cov}(X,Y)$

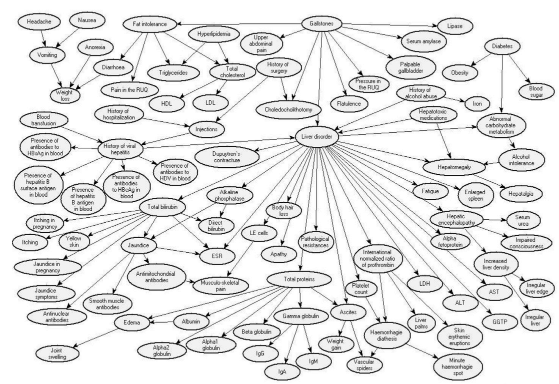
$\rho(X,Y)$

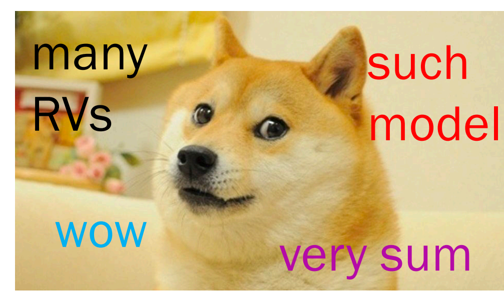
Wednesday:
Conditional distributions

$p_{X|Y}(x|y)$

$E[X|Y]$

Friday: Modeling with Bayesian Networks





Don't we already know linearity of expectation?

Review

Expectation is a linear mathematical operation. If $X = \sum_{i=1}^{n} X_i$:

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

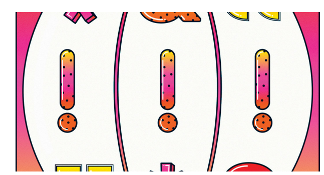
We covered this back in Lecture 6 (when we first learned expectation)!

- Proved binomial: sum of 1s or 0s
- Hat check (section): sum of 1s or 0s
- We ignored (in)dependence of **events**.

Why are we learning this again?

- Well, now we can prove it!
- We can now ignore any **random variables** dependencies!
- Our approach is still the same!

exclamation point jackpot



Proof of expectation of a sum of RVs

$E[X+Y] = E[X] + E[Y]$

$$E[X+Y] = \sum_x \sum_y (x+y) p_{X,Y}(x,y)$$

LOTUS,
$g(X,Y) = X + Y$

$$= \sum_x \sum_y x p_{X,Y}(x,y) + \sum_x \sum_y y p_{X,Y}(x,y)$$

$$= \sum_x x \sum_y p_{X,Y}(x,y) + \sum_y y \sum_x p_{X,Y}(x,y)$$

Linearity of summations (and integrals, btw)

$$= \sum_x x p_X(x) + \sum_y y p_Y(y)$$

Marginal PMFs for X and Y

$$= E[X] + E[Y]$$

☺ yay!

Coupon collecting problems: Hash tables

The coupon collector's problem in probability theory:

- You buy boxes of cereal.
- There are k different types of coupons
- For each box you buy, you "collect" a coupon of type i.

Servers	Hash Tables
requests	strings
k servers	k buckets
request to server i	hashed to bucket i

1. How many coupons do you expect after buying n boxes of cereal? → What is the expected number of utilized servers after n requests?

2. How many boxes do you expect to buy until you have one of each coupon? → What is the expected number of strings to hash until each bucket has ≥ 1 string?

Breakout Rooms

Check out the properties on the next slide (Slide 33). Post any clarifications here!

https://us.edstem.org/courses/2678/discussion/146231

Breakout rooms: 4 min. Introduce yourself!

Hash Tables

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Consider a hash table with k buckets.

- Strings are equally likely to get hashed into any bucket (independently).
- Let Y = # strings to hash until each bucket ≥ 1 string.

What is $E[Y]$?

1. Define additional random variables.

   How should we define Y_i such that $Y = \sum_i Y_i$?

2. Solve.

$p = \frac{1}{k}$

$Y_0 = 1$ $p = k/k \rightarrow$

$Y_1 = \frac{k}{k-1}$ $\frac{1}{p}$ $p = \frac{k-1}{k}$

Hash Tables

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Consider a hash table with k buckets.

- Strings are equally likely to get hashed into any bucket (independently).
- Let Y = # strings to hash until each bucket ≥ 1 string.

What is $E[Y]$?

1. Define additional random variables.

Let: Y_i = # of trials to get success after i-th success

- Success: hash string to previously empty bucket
- If i non-empty buckets: $P(\text{success}) = \frac{k-i}{k}$

2. Solve.

$$P(Y_i = n) = \left(\frac{i}{k}\right)^{n-1}\left(\frac{k-i}{k}\right)$$

Equivalently, $Y_i \sim \text{Geo}\left(p = \frac{k-i}{k}\right)$

$$E[Y_i] = \frac{1}{p} = \frac{k}{k-i}$$

$i = 0$

$i = 1$

Hash Tables

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Consider a hash table with k buckets.

- Strings are equally likely to get hashed into any bucket (independently).
- Let Y = # strings to hash until each bucket ≥ 1 string.

What is $E[Y]$?

1. Define additional random variables.

   Let: Y_i = # of trials to get success after i-th success

   $Y_i \sim \text{Geo}\left(p = \frac{k-i}{k}\right)$, $\quad E[Y_i] = \frac{1}{p} = \frac{k}{k-i}$

2. Solve.

$$Y = Y_0 + Y_1 + \cdots + Y_{k-1}$$

$$E[Y] = E[Y_0] + E[Y_k] + \cdots + E[Y_{k-1}]$$

$$= \frac{k}{k} + \frac{k}{k-1} + \frac{k}{k-2} + \cdots + \frac{k}{1} = k\left[\frac{1}{k} + \frac{1}{k-1} + \cdots + 1\right] = O(k \log k)$$

Covariance

The covariance of two variables X and Y is:

$$\text{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

Covariance measures how one random variable varies with a second.

- Outside temperature and utility bills have a negative covariance.
- Handedness and musical ability have near zero covariance.
- Product demand and price have a positive covariance.

Think

Slide 38 has a question to go over by yourself.

Post any clarifications here!

https://us.edstem.org/courses/2678/discussion/146231

Think by yourself: 1 min

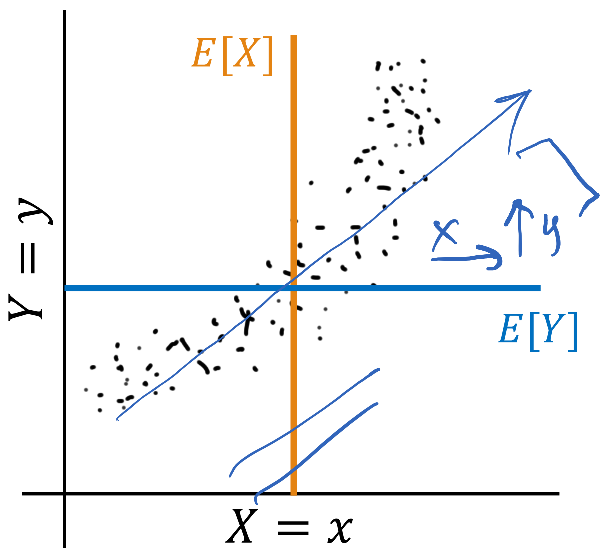
(by yourself)

Feel the covariance

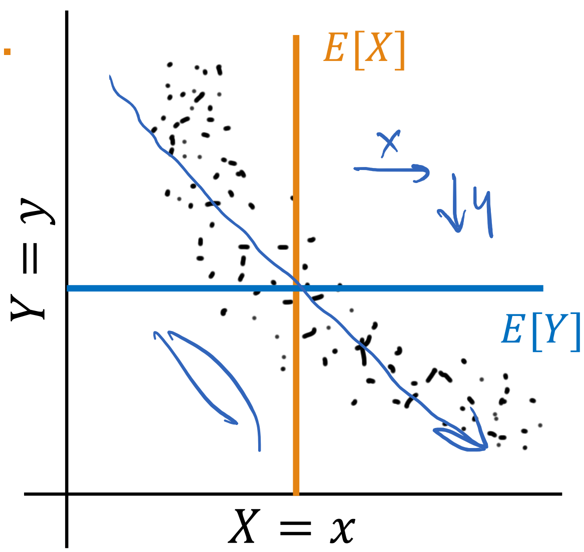
$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

Is the covariance positive, negative, or zero?

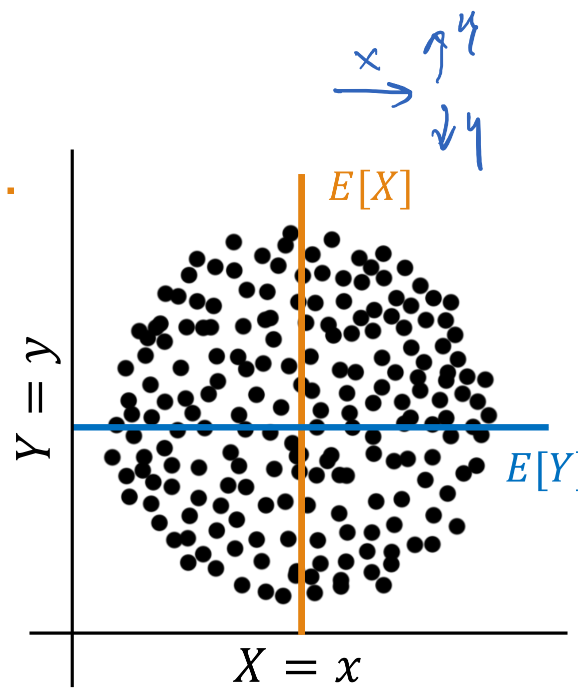
1.



2.



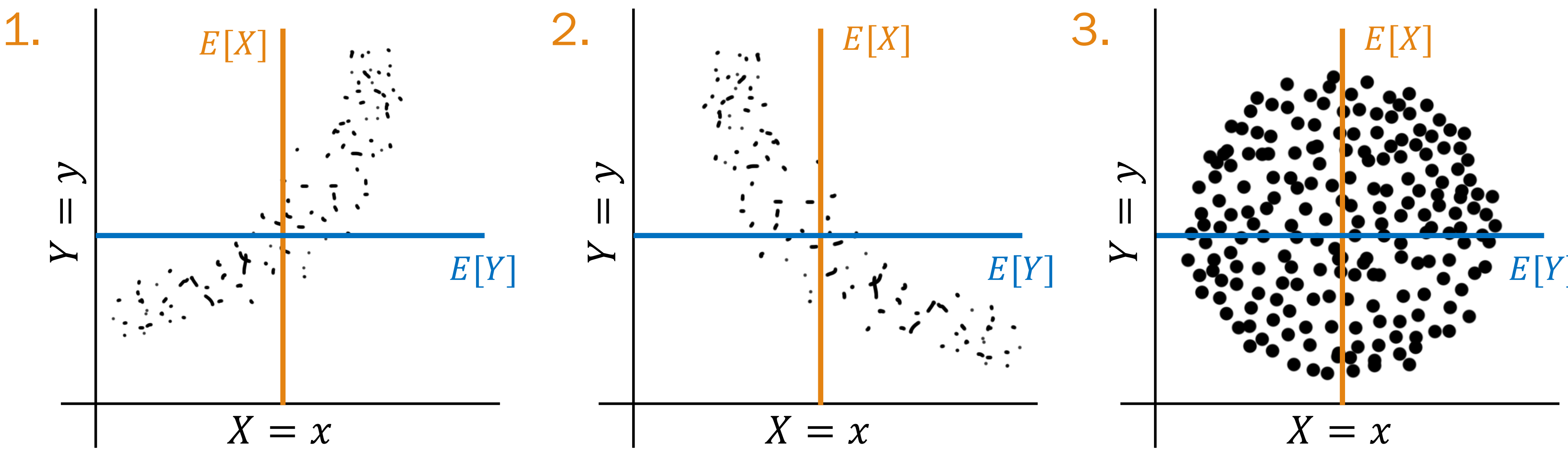
3.



Feel the covariance

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

Is the covariance positive, negative, or zero?



positive ⟷ negative

zero

Properties of Covariance

The **covariance** of two variables X and Y is:

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

Properties:

1. $\text{Cov}(X, Y) = \text{Cov}(Y, X)$
2. $\text{Var}(X) = \text{Cov}(X, X)$
3. $\text{Cov}\left(\sum_i X_i, \sum_j Y_j\right) = \sum_i \sum_j \text{Cov}\left(X_i, Y_j\right)$
4. ~~$\text{Cov}(aX + b, Y) = a\text{Cov}(X, Y) + b$~~ ?

   Covariance is non-linear: $\text{Cov}(aX + b, Y) = a\text{Cov}(X, Y)$

Statistics of sums of RVs

Review

For any random variables X and Y,

$$E[X+Y] = E[X] + E[Y]$$

$$\text{Var}(X+Y) = \text{Var}(X) + 2 \cdot \text{Cov}(X,Y) + \text{Var}(Y)$$

For **independent** X and Y,

$$E[XY] = E[X]E[Y]$$

(Lemma: proof in extra slides)

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

$\text{Cov}(X,Y) = 0$ does NOT imply independence of X and Y!

Zero covariance does not imply independence

Let X take on values $\{-1,0,1\}$
with equal probability $1/3$.

Define $Y = \begin{cases} 1 & \text{if } X = 0 \\ 0 & \text{otherwise} \end{cases}$

What is the joint PMF of X and Y?

Breakout Rooms

Check out the properties on the next slide (Slide 44). Post any clarifications here!

https://us.edstem.org/courses/2678/discussion/146231

Breakout rooms: 4 min. Introduce yourself!

🤔

Zero covariance does not imply independence

Let X take on values $\{-1,0,1\}$ with equal probability $1/3$.

Define $Y = \begin{cases} 1 & \text{if } X = 0 \\ 0 & \text{otherwise} \end{cases}$

Y \ X	-1	0	1	
0	1/3	0	1/3	2/3
1	0	1/3	0	1/3
	1/3	1/3	1/3	

Marginal PMF of Y, $p_Y(y)$

Marginal PMF of X, $p_X(x)$

1. $E[X] =$ $E[Y] =$

2. $E[XY] =$

3. $\text{Cov}(X, Y) =$

4. Are X and Y independent?

Zero covariance does not imply independence

Let X take on values $\{-1,0,1\}$ with equal probability $1/3$.

Define $Y = \begin{cases} 1 & \text{if } X = 0 \\ 0 & \text{otherwise} \end{cases}$

Y \ X	-1	0	1	Marginal PMF of Y, $p_Y(y)$
0	1/3	0	1/3	2/3
1	0	1/3	0	1/3
Marginal PMF of X, $p_X(x)$	1/3	1/3	1/3	

1. $E[X] = -1\left(\frac{1}{3}\right) + 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) = 0$

   $E[Y] = 0\left(\frac{2}{3}\right) + 1\left(\frac{1}{3}\right) = 1/3$

2. $E[XY] = (-1 \cdot 0)\left(\frac{1}{3}\right) + (0 \cdot 1)\left(\frac{1}{3}\right) + (1 \cdot 0)\left(\frac{1}{3}\right)$
   $= 0$

3. $\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$
   $= 0 - 0(1/3) = 0$

   does not imply independence!

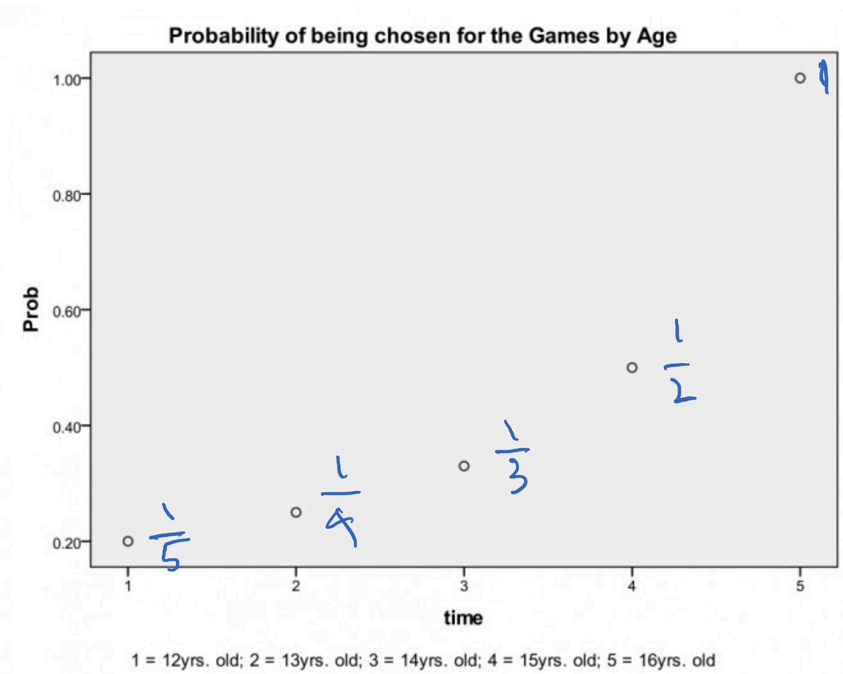
4. Are X and Y independent?

   $P(Y = 0|X = 1) = 1$

   $\neq \quad P(Y = 0) = 2/3$

Interesting probability news

Probability and Game Theory in *The Hunger Games*



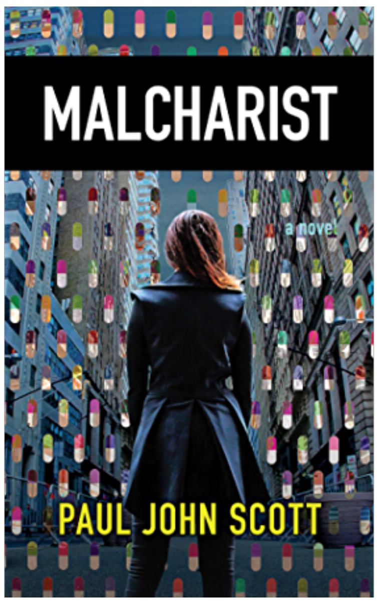
https://www.wired.com/2012/04/probability-and-game-theory-in-the-hunger-games/

"Suppose the parents in a given district gave birth to only…five girls, and that all of these kids were born at the same time."

- Not a probability mass function
- Also duh? (P(you get chosen if you're the only person) = 1)
- You now know enough Python/ probability to write a better simulation to model the Reaping!!!!
- (game theory part of the article is good)

Topical book review! Fiction is brain food.

Rochester author takes scary look at Big Pharma in debut novel



- "Called 'Malcharist,' it is a completely made-up story about a potentially dangerous drug being put on the market – with outsourced drug trial research, ghostwritten studies, lack of access to raw drug-trial data, and doctors essentially paid to champion new drugs."
- "[Paul John] Scott's novel is actually a thriller, with not-quite-believable villains who need to be exposed. Yet it's too wonky to be a beach read. There's even a conversation over the [😍] probability concept of p-values [😍]."
- "Scott takes his writer into one of those medical meetings he once found so cool, and his book reproduces enough of the numbers – yes, [😍] number tables [😍] in a thriller – that the reader can see the fictional speaker's good point that the data really do give up their secrets."

https://www.startribune.com/schafer-debut-novel-by-rochester-author-takes-on-big-pharma-issues/572679992

LIVE

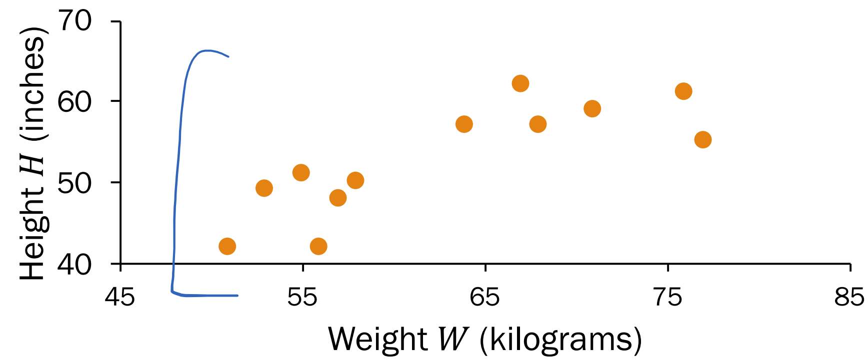
Correlation

Covarying humans

$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$
$= E[XY] - E[X]E[Y]$

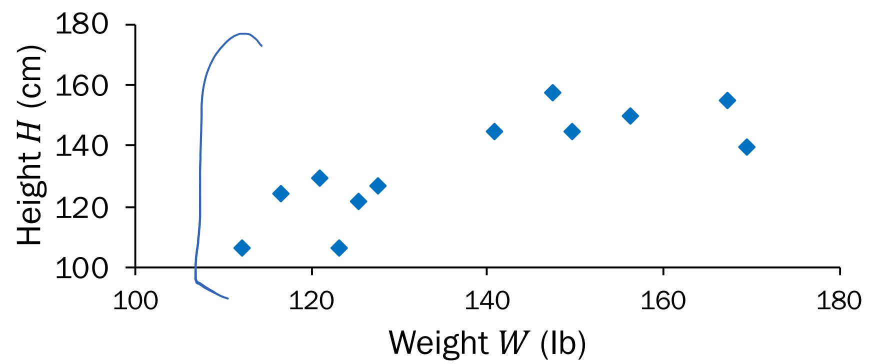
What is the covariance of weight W and height H?

$\text{Cov}(W, H) = E[WH] - E[W]E[H]$
$= 3355.83 - (62.75)(52.75)$
$= 45.77$ (positive)



What about weight (lb) and height (cm)?

$\text{Cov}(2.20W, 2.54H)$
$= E[2.20W \cdot 2.54H] - E[2.20W]E[2.54H]$
$= 18752.38 - (138.05)(133.99)$
$= 255.06$ (positive)



⚠ Covariance depends on units!

Sign of covariance (+/–) more meaningful than magnitude

Correlation

The **correlation** of two variables X and Y is:

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \, \sigma_Y}$$

$\sigma_X^2 = \text{Var}(X)$,
$\sigma_Y^2 = \text{Var}(Y)$

- Note: $-1 \le \rho(X,Y) \le 1$ [why?
- Correlation measures the **linear relationship** between X and Y:

$\rho(X,Y) = 1 \quad \Longrightarrow Y = aX + b$, where $a = \sigma_Y/\sigma_X$
$\rho(X,Y) = -1 \quad \Longrightarrow Y = aX + b$, where $a = -\sigma_Y/\sigma_X$
$\rho(X,Y) = 0 \quad \Longrightarrow$ "uncorrelated" (absence of linear relationship)

Think

Slide 52 has a question to go over by yourself.

Post any clarifications here!

https://us.edstem.org/courses/2678/discussion/146231

Think by yourself: 1 min

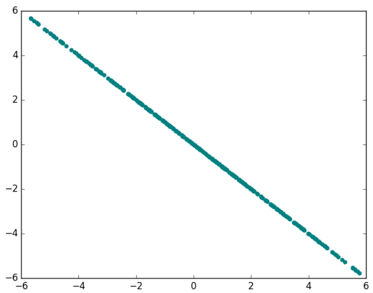
(by yourself)

Correlation reps

What is the correlation coefficient $\rho(X, Y)$?

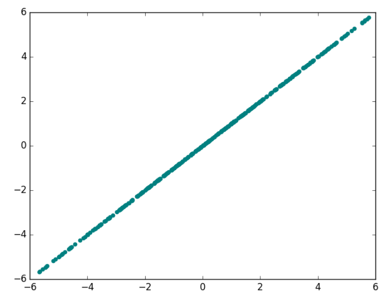
A. $\rho(X, Y) = 1$
B. $\rho(X, Y) = -1$
C. $\rho(X, Y) = 0$
D. Other

1.



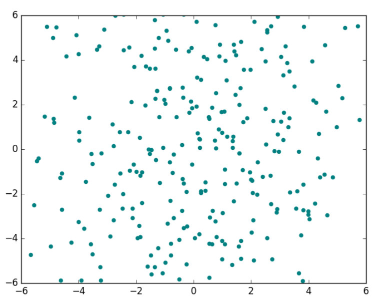
B

2.



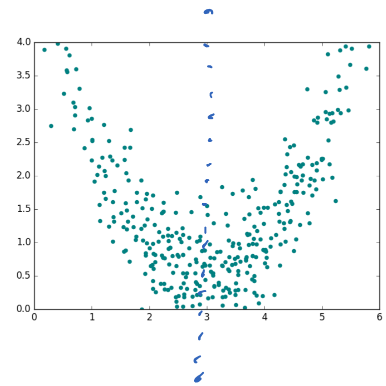
A

3.



C

4.



$Y = X^2$

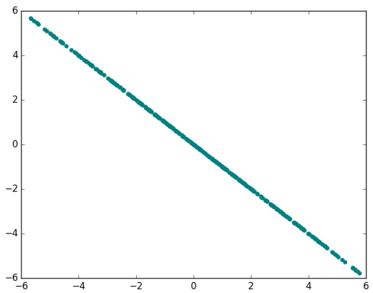
C

Correlation reps

A. $\rho(X,Y) = 1$
B. $\rho(X,Y) = -1$
C. $\rho(X,Y) = 0$
D. Other

What is the correlation coefficient $\rho(X,Y)$?

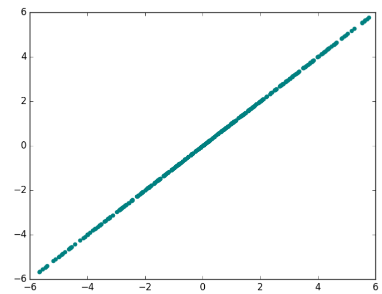
1.



B. $\rho(X,Y) = -1$

$Y = -aX + b$
$a > 0$

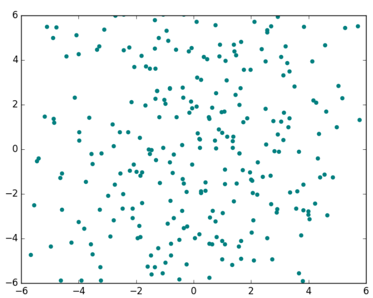
2.



A. $\rho(X,Y) = 1$

$Y = aX + b$
$a > 0$

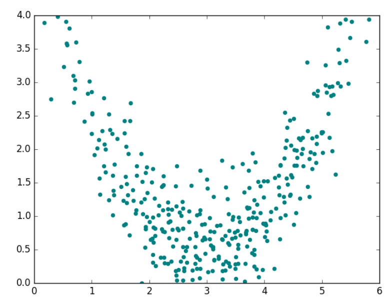
3.



C. $\rho(X,Y) = 0$

"uncorrelated"

4.

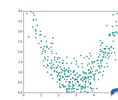


C. $\rho(X,Y) = 0$

$Y = X^2$

X and Y can be nonlinearly related even if $\rho(X,Y) = 0$.

Throwback to CS103: Conditional statements

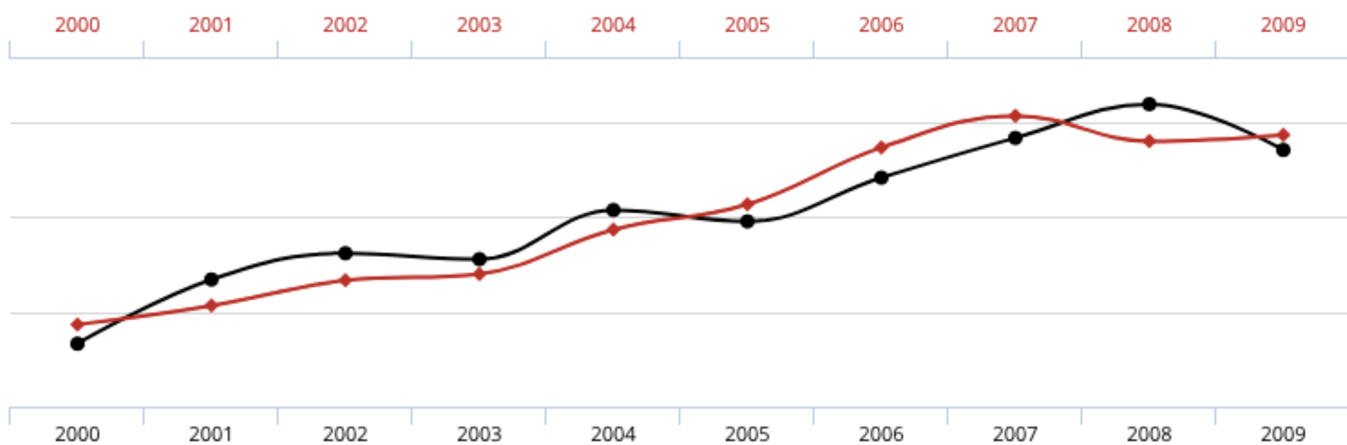
Statement $P \rightarrow Q$:	Independence → No correlation	✅
Contrapositive $\neg Q \rightarrow \neg P$:	Correlation → Dependence	✅ (logically equivalent)
Inverse $\neg P \rightarrow \neg Q$:	Dependence → Correlation	❌ (not always) $Y = X^2$ $\rho(X, Y) = 0$ 
Converse $Q \rightarrow P$:	No correlation → Independence	❌ (not always) Slide 45

"Correlation does not imply causation"

Spurious Correlations

$\rho(X,Y)$ is used a lot to statistically quantify the relationship b/t X and Y.

Correlation:
0.947091



[Spurious correlations]

Spurious Correlations

$\rho(X, Y)$ is used a lot to statistically quantify the relationship b/t X and Y.

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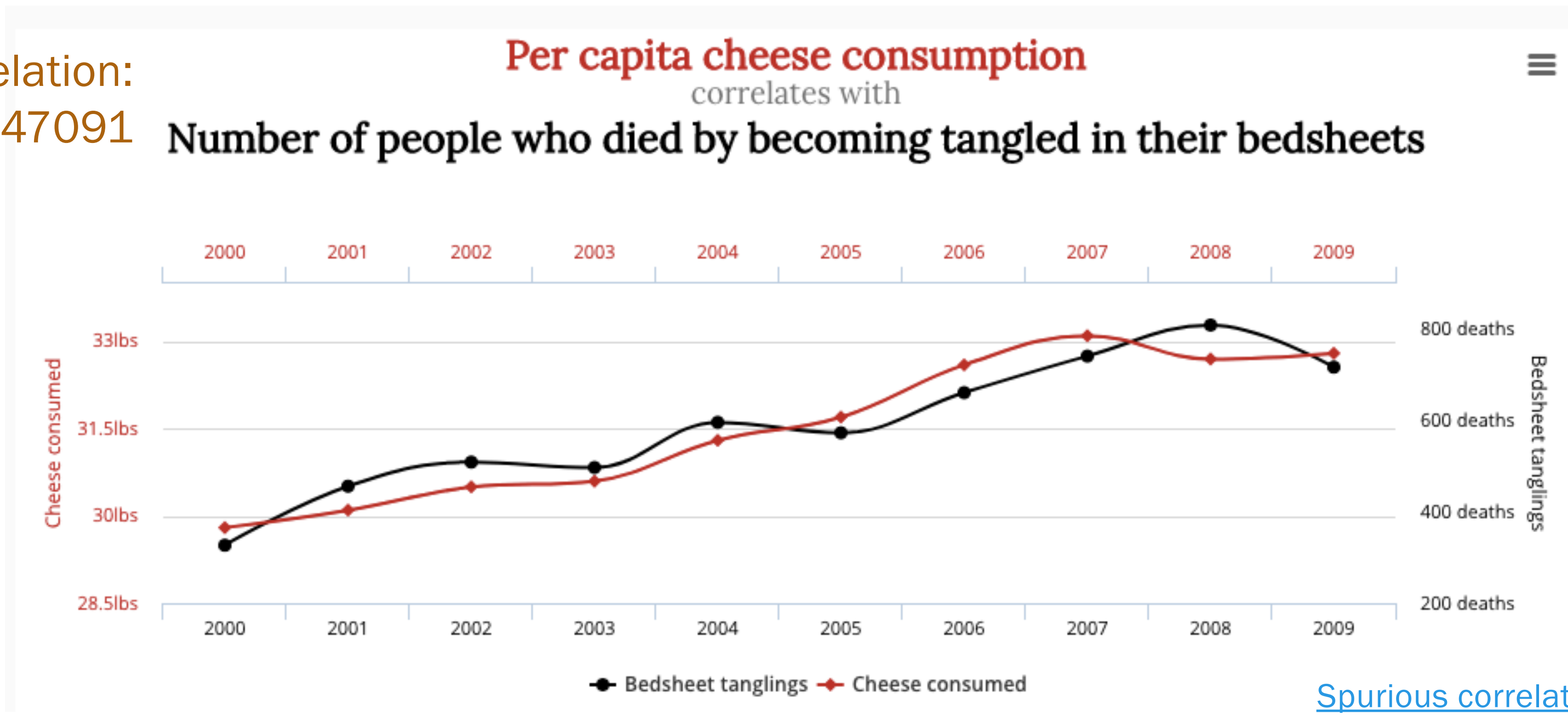
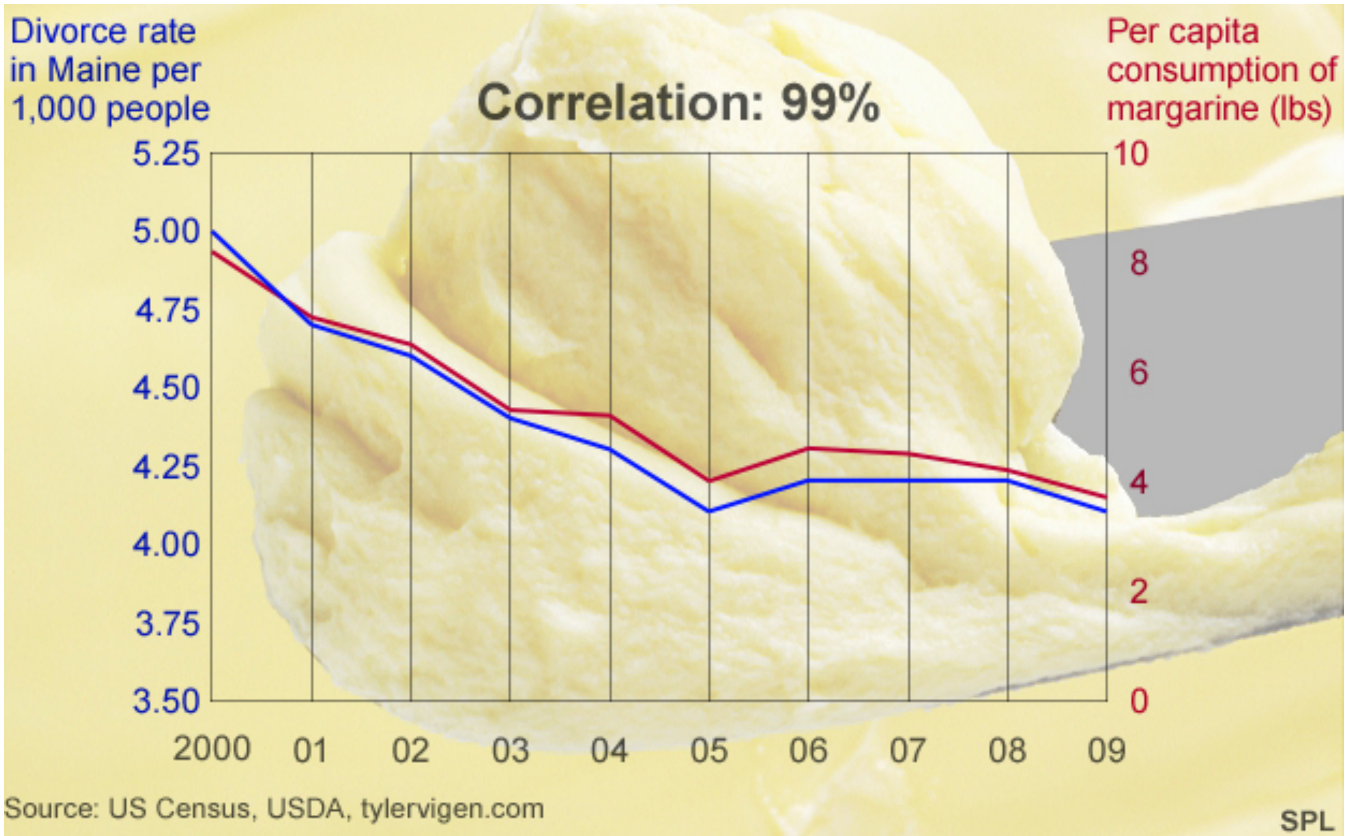


[Spurious correlations]

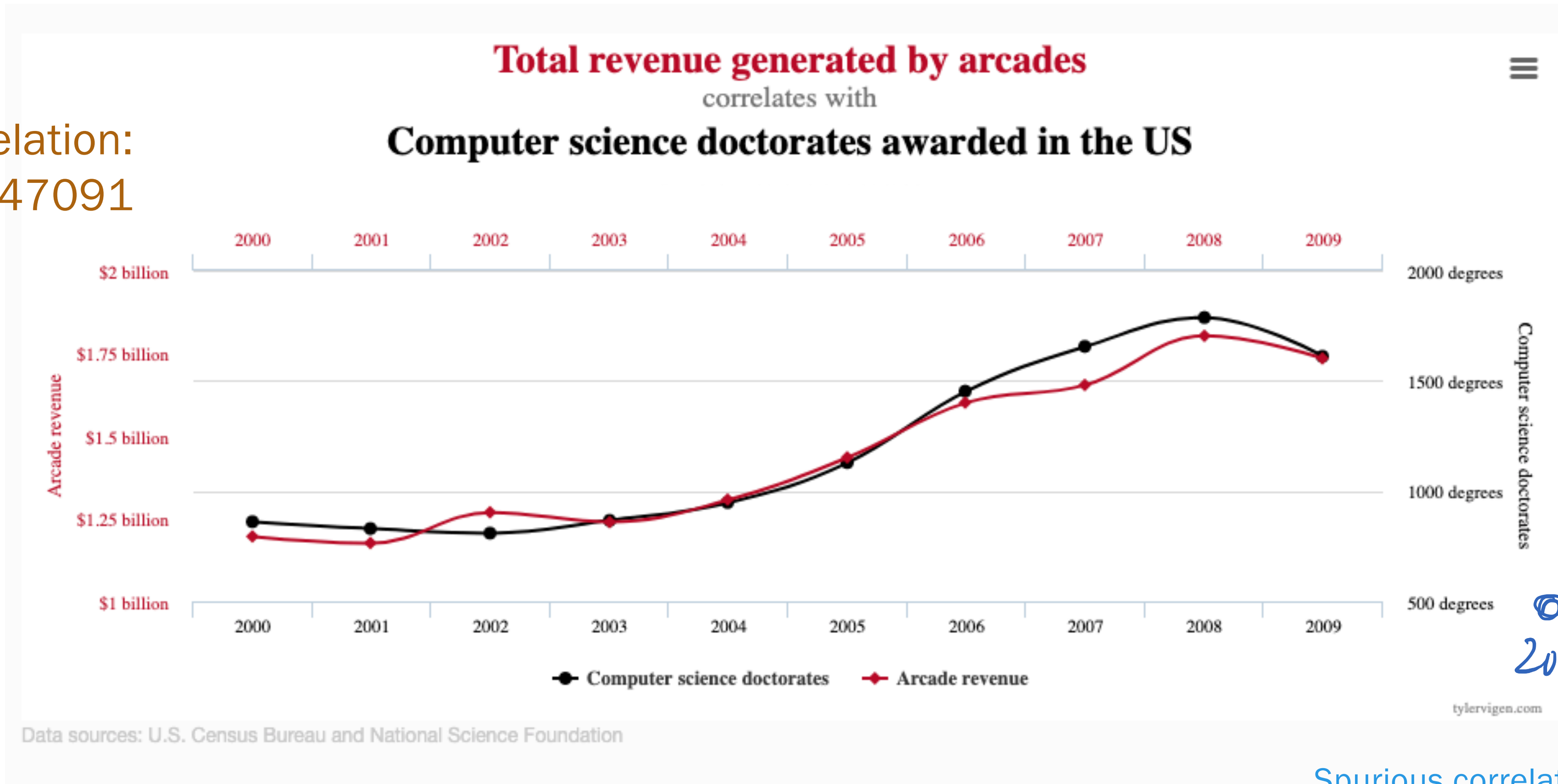
Divorce vs. Margarine



http://www.bbc.com/news/magazine-27537142

Arcade revenue vs. CS PhDs

Correlation:
0.947091



Spurious correlations

Extra

Expectation of product of independent RVs

If X and Y are independent, then

$$E[XY] = E[X]E[Y]$$
$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Proof: $E[g(X)h(Y)] = \sum_y \sum_x g(x)h(y)p_{X,Y}(x,y)$ (for continuous proof, replace summations with integrals)

$= \sum_y \sum_x g(x)h(y)p_X(x)p_Y(y)$ — X and Y are independent

$= \sum_y \left(h(y)p_Y(y) \sum_x g(x)p_X(x) \right)$ — Terms dependent on y are constant in integral of x

$= \left(\sum_x g(x)p_X(x) \right) \left(\sum_y h(y)p_Y(y) \right)$ — Summations separate

$= E[g(X)]E[h(Y)]$

Variance of Sums of Variables

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} \text{Cov}\left(X_i, X_j\right)$$

Proof:

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \text{Cov}(X_i, X_j)$$

Var(X) = Cov(X, X)

covariance of all pairs

$$= \sum_{i=1}^{n} \text{Var}(X_i) + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \text{Cov}\left(X_i, X_j\right)$$

Symmetry of covariance
Cov(X, X) = Var(X)

$$= \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} \text{Cov}\left(X_i, X_j\right)$$

Adjust summation bounds