24: Naïve Bayes

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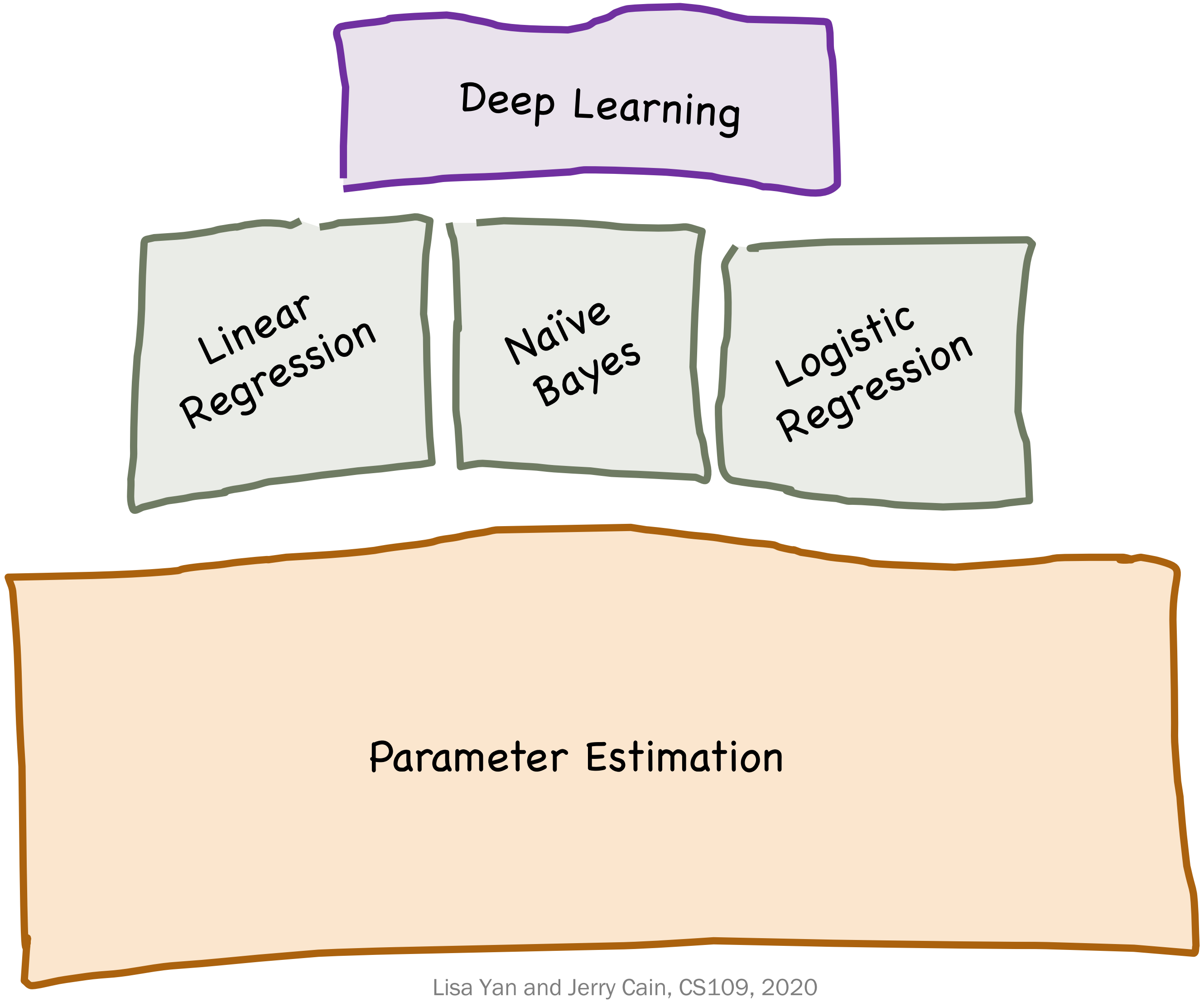
Quick slide reference

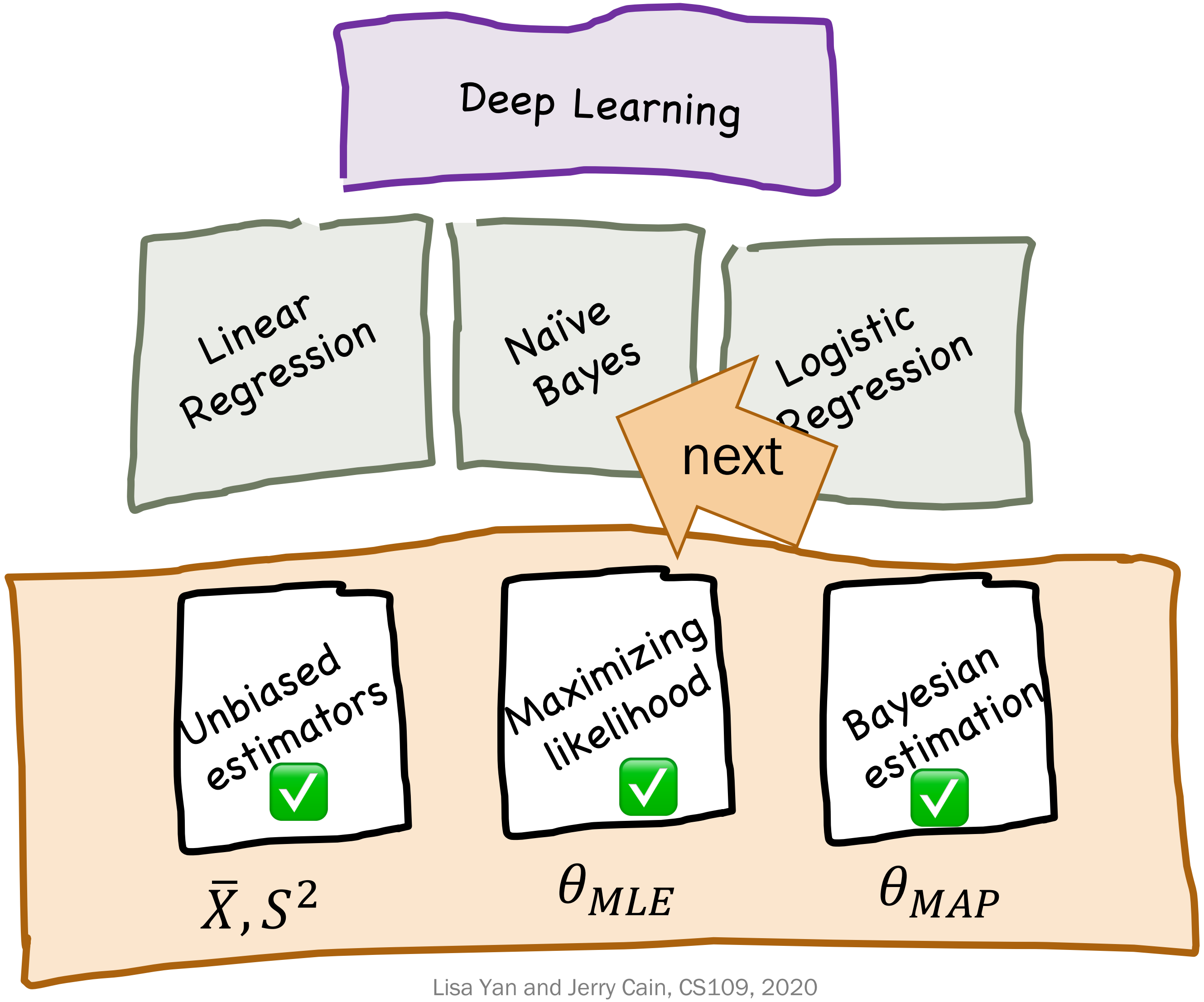
3	Intro: Machine Learning	23a_intro
21	“Brute Force Bayes”	24b_brute_force_bayes
32	Naïve Bayes Classifier	24c_naive_bayes
43	Naïve Bayes: MLE/MAP with TV shows	LIVE
71	Naïve Bayes: MAP with email classification	LIVE

Intro: Machine Learning

Our path from here



Our path from here



Deep Learning

Linear Regression

Naïve Bayes

Logistic Regression

next

Unbiased estimators ✓

Maximizing likelihood ✓

Bayesian estimation ✓

$\bar{X}, S^2$

θ_{MLE}

θ_{MAP}

Machine Learning (formally)

Many different forms of "Machine Learning"

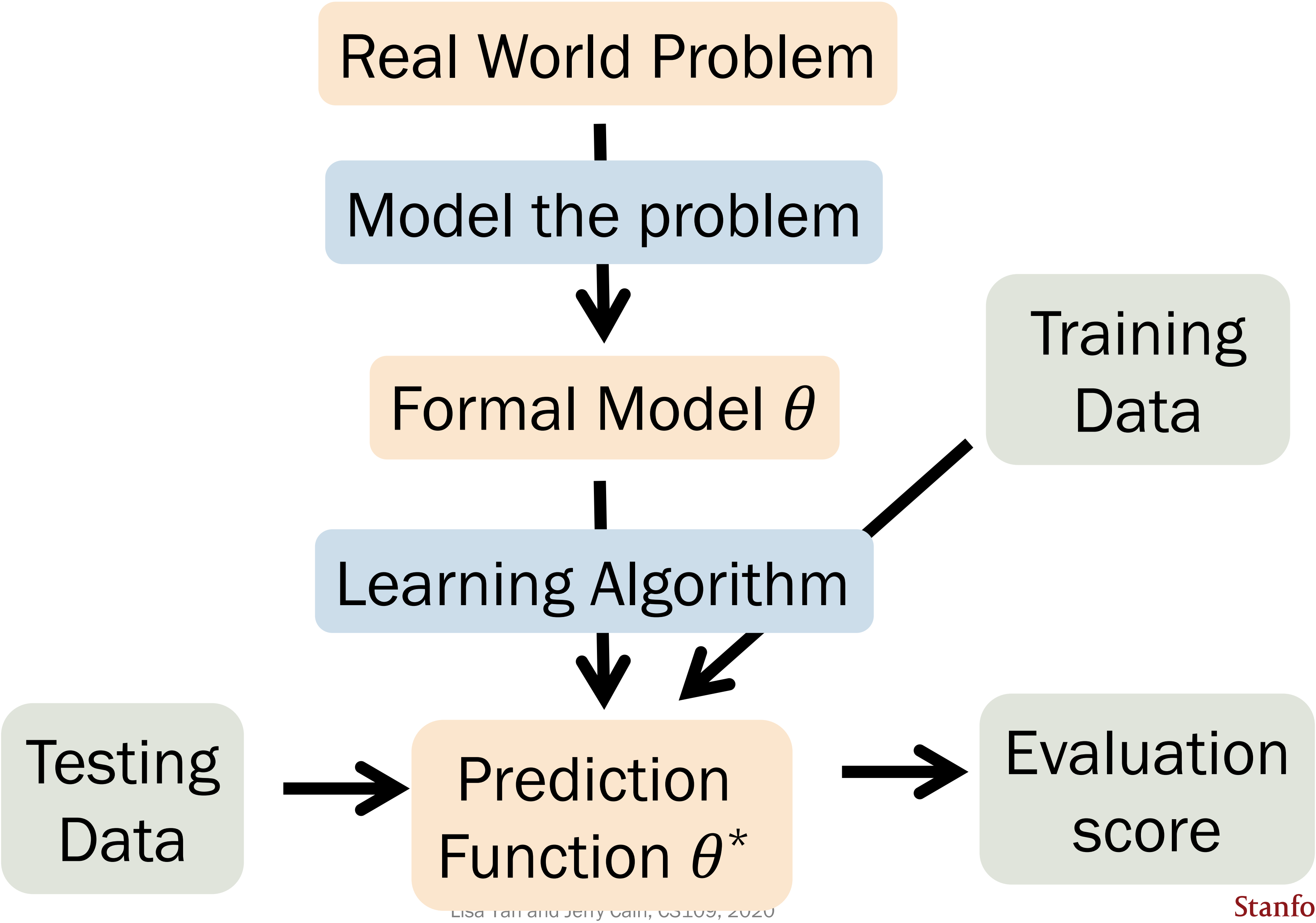
- We focus on the problem of prediction based on observations.

Machine Learning uses a lot of data.

Task: Identify what a chair is
Data: All the chairs ever

Supervised learning: A category of machine learning where you have labeled data on the problem you are solving.

Supervised learning



Real World Problem

Model the problem

Formal Model θ

Training Data

Learning Algorithm

Testing Data

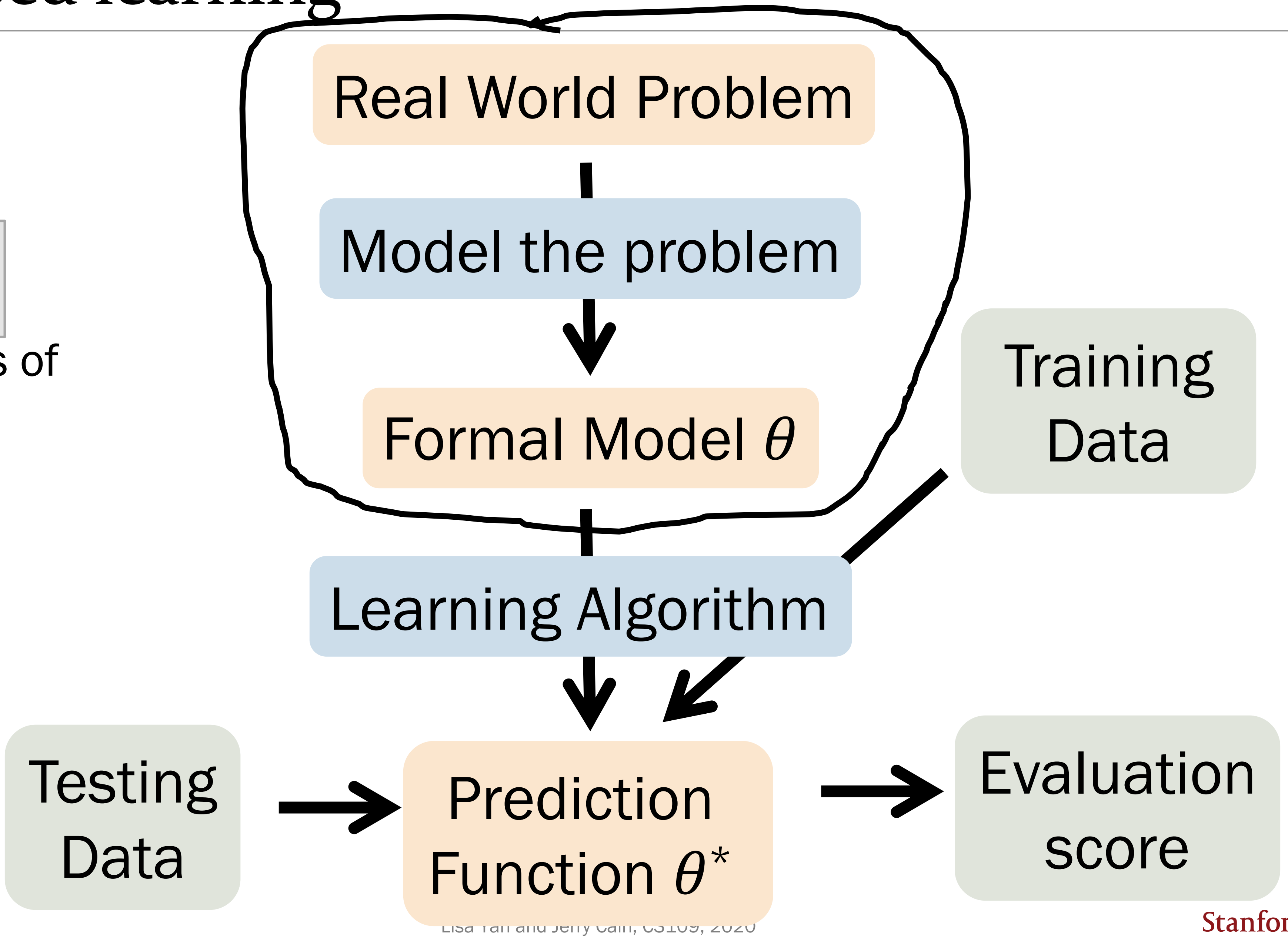
Prediction Function θ^*

Evaluation score

Supervised learning

Modeling

(not the focus of this class)



Real World Problem

Model the problem

Formal Model θ

Training Data

Learning Algorithm

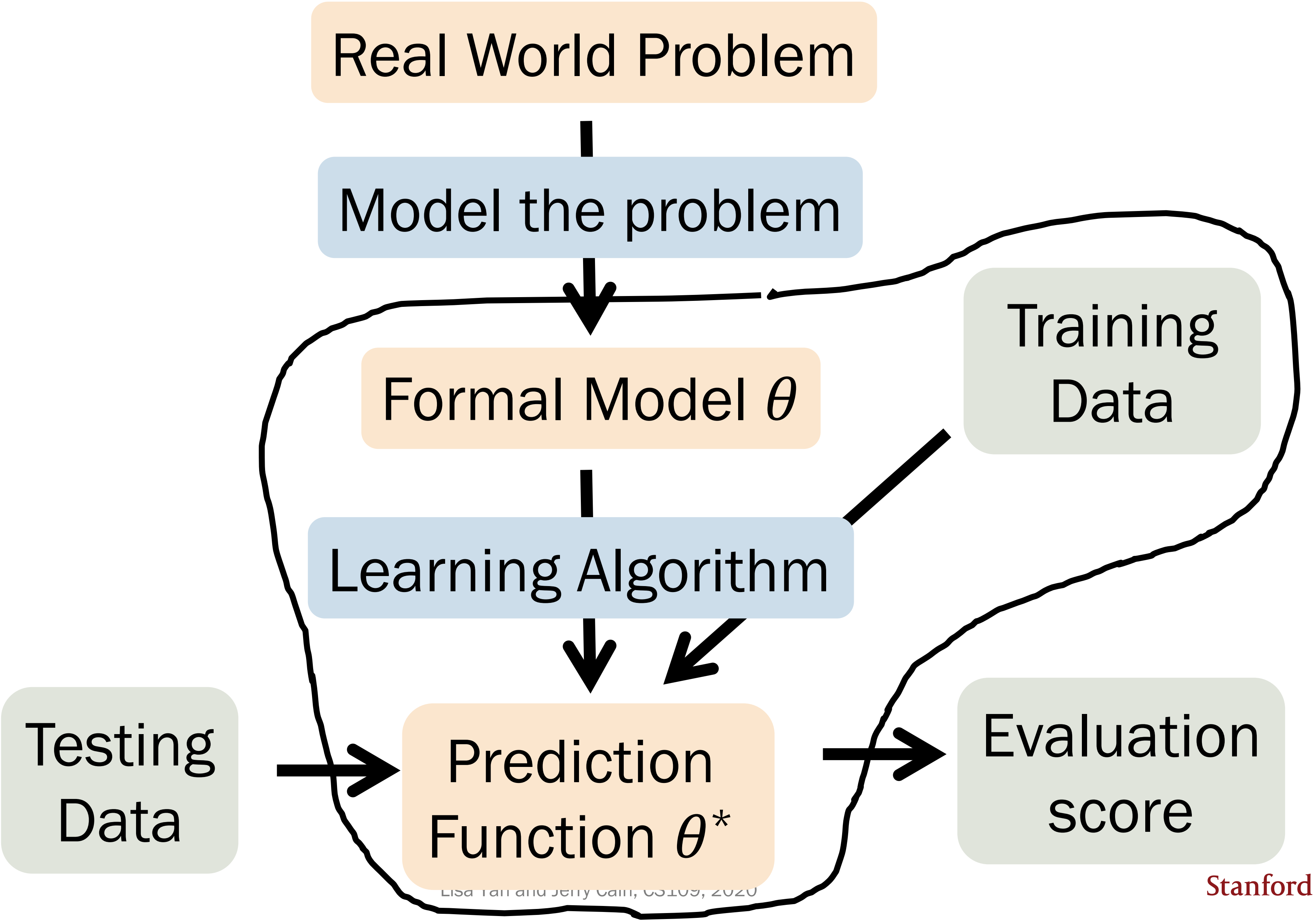
Testing Data

Prediction Function θ^*

Evaluation score

Supervised learning

Training



Real World Problem

Model the problem

Formal Model θ

Learning Algorithm

Training Data

Testing Data

Prediction Function θ^*

Evaluation score

Model and dataset

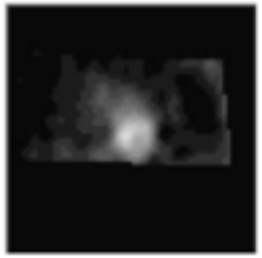
Many different forms of "Machine Learning"

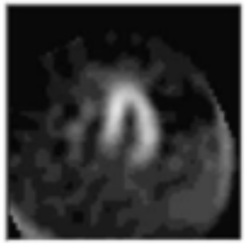
- We focus on the problem of prediction based on observations.

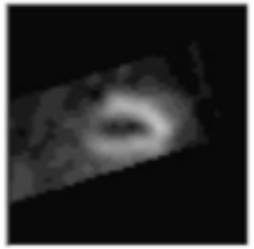
Goal	Based on observed $\boldsymbol{X}$, predict unseen Y
- Features	Vector $\boldsymbol{X}$ of m observed variables $\boldsymbol{X} = (X_1, X_2, \dots, X_m)$
- Output	Variable Y (also called class label if discrete)
Model	$\hat{Y} = g(\boldsymbol{X})$, a function of observations $\boldsymbol{X}$

Training data

$$\boldsymbol{X} = (X_1, X_2, X_3, \ldots, X_{300})$$







	Feature 1	Feature 2	…	Feature 300	Output
Patient 1	1	0	…	1	1
Patient 2	1	1	…	0	0
…			⋮		⋮
Patient n	0	0	…	1	1

Training data notation

$$\left(\boldsymbol{x}^{(1)}, y^{(1)}\right), \left(\boldsymbol{x}^{(2)}, y^{(2)}\right), \ldots, \left(\boldsymbol{x}^{(n)}, y^{(n)}\right)$$

n datapoints, generated i.i.d.

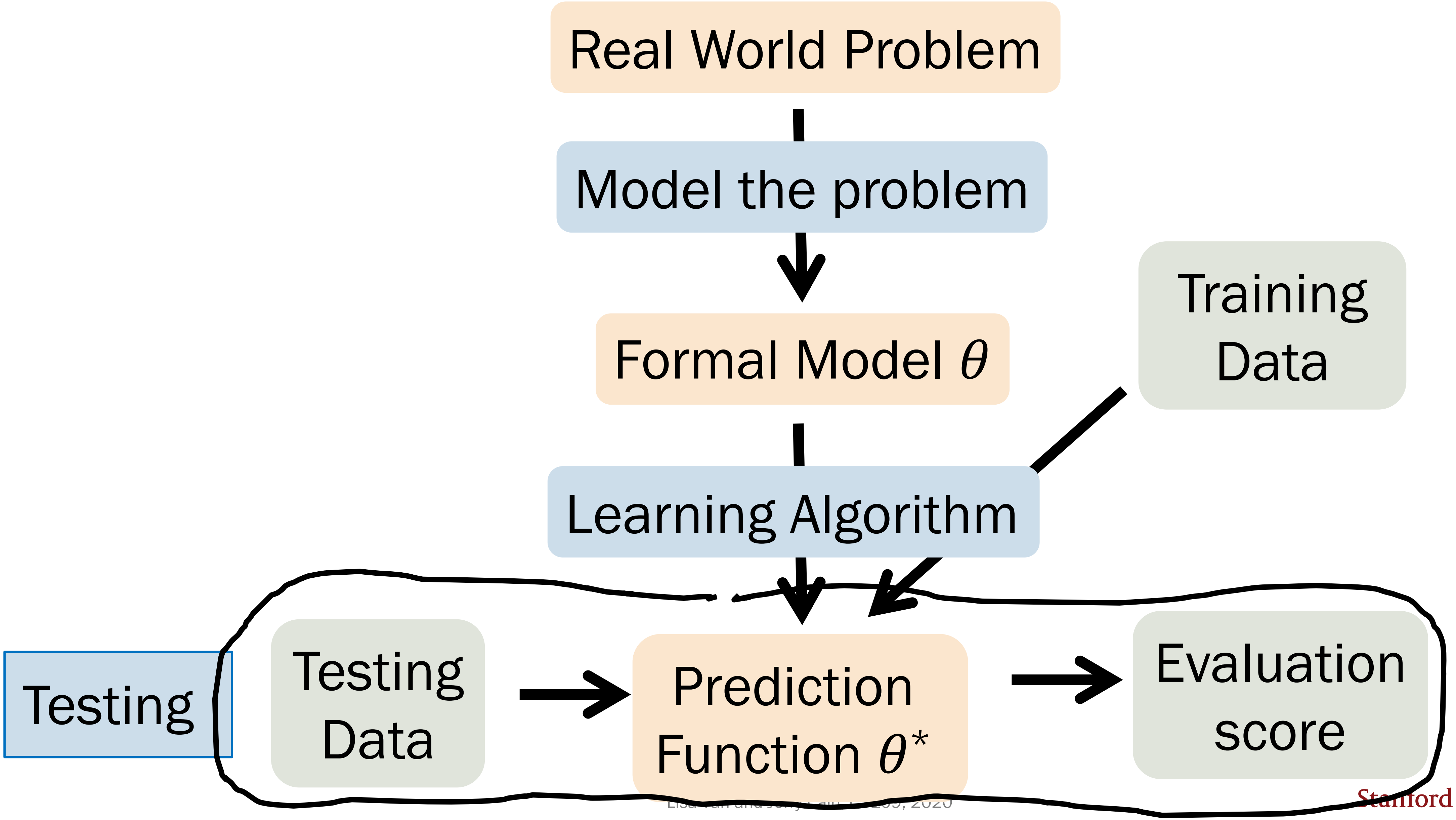
i–th datapoint $\left(\boldsymbol{x}^{(i)}, y^{(i)}\right)$:

- m features: $\boldsymbol{x}^{(i)} = \left(x_1^{(i)}, x_2^{(i)}, \ldots, x_m^{(i)}\right)$
- A single output $y^{(i)}$
- Independent of all other datapoints

Training Goal:

Use these n datapoints to learn a model $\hat{Y} = g(\boldsymbol{X})$ that predicts Y

Supervised learning



Real World Problem

Model the problem

Formal Model θ

Training Data

Learning Algorithm

Testing

Testing Data

Prediction Function θ^*

Evaluation score

Testing data notation

$$\left(\boldsymbol{x}^{(1)}, y^{(1)}\right), \left(\boldsymbol{x}^{(2)}, y^{(2)}\right), \ldots, \left(\boldsymbol{x}^{(n)}, y^{(n)}\right)$$

n_{test} other datapoints, generated i.i.d.

i–th datapoint $\left(\boldsymbol{x}^{(i)}, y^{(i)}\right)$:

- Has the same structure as your training data

Testing Goal:

Using the model $\hat{Y} = g(\boldsymbol{X})$ that you trained, see how well you can predict Y on known data

Two tasks we will focus on

Many different forms of "Machine Learning"

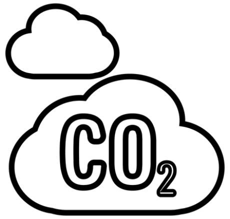
- We focus on the problem of **prediction** based on observations.

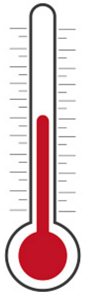
Goal	Based on observed $\boldsymbol{X}$, predict unseen Y
- **Features**	Vector $\boldsymbol{X}$ of m observed variables $\boldsymbol{X} = (X_1, X_2, \dots, X_m)$
- **Output**	Variable Y (also called class label if discrete)

Model	$\hat{Y} = g(\boldsymbol{X})$, a function of observations $\boldsymbol{X}$
- **Regression**	prediction when Y is continuous
- **Classification**	prediction when Y is discrete

Regression: Predicting real numbers

Training data: $(\boldsymbol{x}^{(1)}, y^{(1)}), (\boldsymbol{x}^{(2)}, y^{(2)}), \ldots, (\boldsymbol{x}^{(n)}, y^{(n)})$



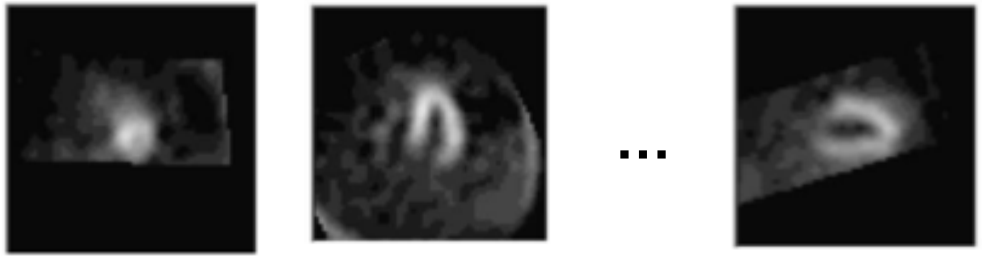


Global Land-Ocean temperature

	CO2 levels	Sea level	…	Feature m	Output
Year 1	338.8	0	…	1	0.26
Year 2	340.0	1	…	0	0.32
…			⋮		⋮
Year n	340.76	0	…	1	0.14

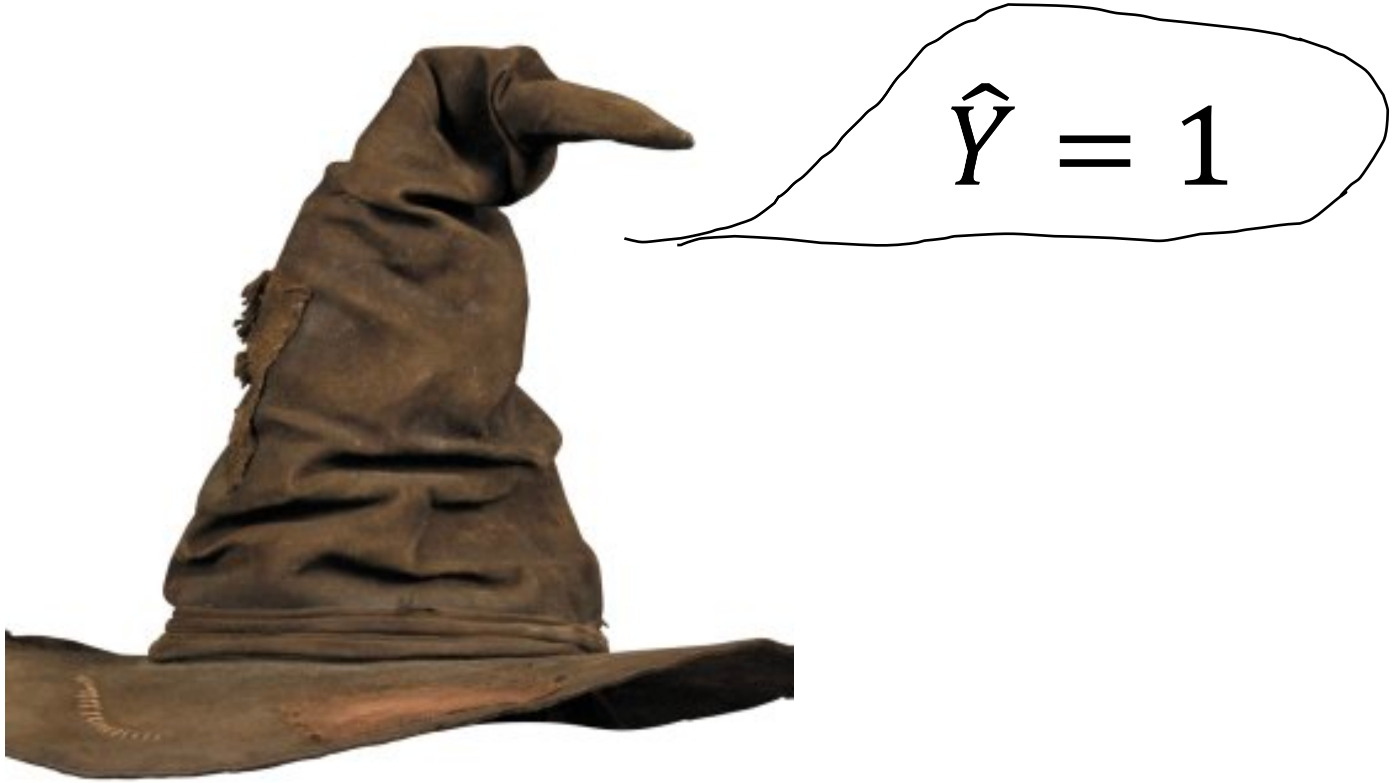
Classification: Predicting class labels

$$\boldsymbol{X} = (X_1, X_2, X_3, \dots, X_{300})$$



	Feature 1	Feature 2	…	Feature 300	Output
Patient 1	1	0	…	1	1
Patient 2	1	1	…	0	0
…			⋮		⋮
Patient n	0	0	…	1	1

Classification: Harry Potter Sorting Hat

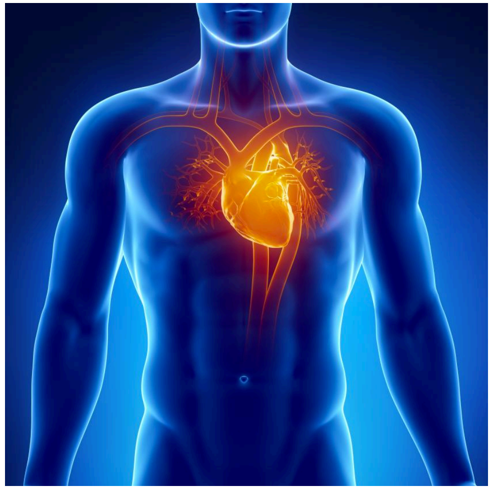


$$\boldsymbol{X} = (1, 1, 1, 0, 0, \dots, 1)$$

Our focus today!

Classification: Example datasets

Heart



Ancestry

23andMe

NETFLIX

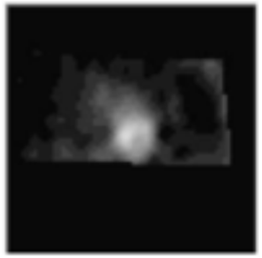
Netflix

24b_brute_force_bayes

“Brute Force Bayes”

Classification: Having a healthy heart

$$\boldsymbol{X} = (X_1)$$



	Feature 1	Output
Patient 1	1	0
Patient 2	1	1
	⋮	⋮
Patient n	0	1

Single feature: Region of Interest (ROI) is healthy (1) or unhealthy (0)

How can we predict the class label

heart is healthy (1) or unhealthy (0)?

The following strategy is **not used in practice** but helps us understand how we approach classification.

Classification: "Brute Force Bayes"

$$\hat{Y} = g(\boldsymbol{X})$$

Our prediction for Y is a function of $\boldsymbol{X}$

$$= \arg\max_{y=\{0,1\}} P(Y \mid \boldsymbol{X})$$

Proposed model: Choose the Y that is most likely given $\boldsymbol{X}$

$$= \arg\max_{y=\{0,1\}} \frac{P(\boldsymbol{X}|Y)P(Y)}{P(\boldsymbol{X})}$$

(Bayes' Theorem)

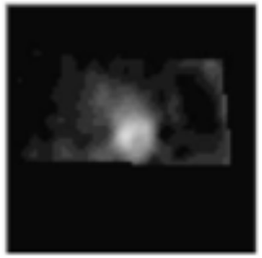
$$= \arg\max_{y=\{0,1\}} P(\boldsymbol{X}|Y)P(Y)$$

($1/P(\boldsymbol{X})$ is constant w.r.t. y)

If we estimate $P(\boldsymbol{X}|Y)$ and $P(Y)$, we can classify datapoints!

Training: Estimate parameters

$\boldsymbol{X} = (X_1)$



	Feature 1	Output
Patient 1	1	0
Patient 2	1	1
	$\vdots$	$\vdots$
Patient n	0	1

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

Conditional probability tables $\hat{P}(\boldsymbol{X}|Y)$

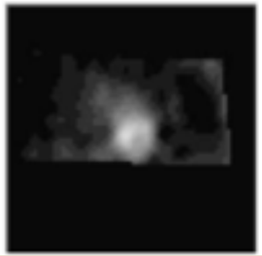
	$\hat{P}(\boldsymbol{X}\|Y=0)$	$\hat{P}(\boldsymbol{X}\|Y=1)$
$X_1 = 0$	θ_1	θ_3
$X_1 = 1$	θ_2	θ_4

Marginal probability table $\hat{P}(Y)$

	$\hat{P}(Y)$
$Y = 0$	θ_5
$Y = 1$	θ_6

Training Goal: Use n datapoints to learn $2 \cdot 2 + 2 = 6$ parameters.

Training: Estimate parameters $\hat{P}(\boldsymbol{X}|Y)$



Count:	# datapoints
X_1 = 0, Y = 0:	4
X_1 = 1, Y = 0:	6
X_1 = 0, Y = 1:	0
X_1 = 1, Y = 1:	100
Total:	110

Pa

Pa

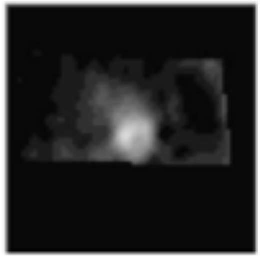
Patient n 0 1

	$\hat{P}(\boldsymbol{X}\|Y=0)$	$\hat{P}(\boldsymbol{X}\|Y=1)$
$X_1 = 0$	θ_1	θ_3
$X_1 = 1$	θ_2	θ_4

$\boldsymbol{X}|Y = 0$ and $\boldsymbol{X}|Y = 1$
are each multinomials with 2 outcomes!

Use MLE or Laplace (MAP) estimate for parameters $\hat{P}(\boldsymbol{X}|Y)$ and $\hat{P}(Y)$

Training: MLE estimates, $\hat{P}(\boldsymbol{X}|Y)$



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X_1 = 0, Y = 0:	4
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X_1 = 0, Y = 1:	0
X_1 = 1, Y = 1:	100
Total:	110

Pa

Pa

Patient n 0 1

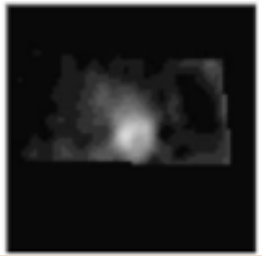
MLE

	$\hat{P}(\boldsymbol{X}\|Y=0)$	$\hat{P}(\boldsymbol{X}\|Y=1)$
$X_1=0$	0.4	0.0
$X_1=1$	0.6	1.0

MLE of $\hat{P}(X_1=x|Y=y) = \dfrac{\#(X_1=x, Y=y)}{\#(Y=y)}$

Just count!

Training: Laplace (MAP) estimates, $\hat{P}(\boldsymbol{X}|Y)$



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X_1 = 0, Y = 0:	4
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X_1 = 0, Y = 1:	0
X_1 = 1, Y = 1:	100
Total:	110

Pa

Pa

Patient n 0 1

MLE

	$\hat{P}(\boldsymbol{X}\vert Y=0)$	$\hat{P}(\boldsymbol{X}\vert Y=1)$
$X_1=0$	0.4	0.0
$X_1=1$	0.6	1.0

MLE of $\hat{P}(X_1 = x|Y = y) = \dfrac{\#(X_1 = x, Y = y)}{\#(Y = y)}$

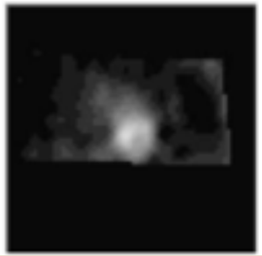
Just count!

MAP

Laplace of $\hat{P}(X_1 = x|Y = y) = \ ?$

Just count + add imaginary trials!

Training: Laplace (MAP) estimates, $\hat{P}(\boldsymbol{X}|Y)$



Count:	# datapoints
X_1 = 0, Y = 0:	4
X_1 = 1, Y = 0:	6
X_1 = 0, Y = 1:	0
X_1 = 1, Y = 1:	100
Total:	110

Pa

Pa

Patient n 0 1

MLE

	$\hat{P}(\boldsymbol{X}\|Y=0)$	$\hat{P}(\boldsymbol{X}\|Y=1)$
$X_1 = 0$	0.4	0.0
$X_1 = 1$	0.6	1.0

MLE of $\hat{P}(X_1 = x|Y = y) = \dfrac{\#(X_1 = x, Y = y)}{\#(Y = y)}$

Just count!

MAP

	$\hat{P}(\boldsymbol{X}\|Y=0)$	$\hat{P}(\boldsymbol{X}\|Y=1)$
$X_1 = 0$	0.42	0.01
$X_1 = 1$	0.58	0.99

Laplace of $\hat{P}(X_1 = x|Y = y) = \dfrac{\#(X_1 = x, Y = y) + 1}{\#(Y = y) + 2}$

Just count + add imaginary trials!

Testing

$$\hat{Y} = \underset{y=\{0,1\}}{\arg\max}\, \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

(MAP)	$\hat{P}(\boldsymbol{X}\|Y=0)$	$\hat{P}(\boldsymbol{X}\|Y=1)$
$X_1 = 0$	0.42	0.01
$X_1 = 1$	0.58	0.99

(MLE)	$\hat{P}(Y)$
$Y = 0$	0.09
$Y = 1$	0.91

New patient has a healthy ROI ($X_1 = 1$). What is your prediction, $\hat{Y}$?

$$\hat{P}(X_1 = 1|Y = 0)\hat{P}(Y = 0) = 0.58 \cdot 0.09 \approx 0.052$$
$$\hat{P}(X_1 = 1|Y = 1)\hat{P}(Y = 1) = 0.99 \cdot 0.91 \approx 0.901$$

A. $0.052 < 0.5 \quad \Rightarrow \quad \hat{Y} = 1$
B. $0.901 > 0.5 \quad \Rightarrow \quad \hat{Y} = 1$
C. $0.052 < 0.901 \Rightarrow \quad \hat{Y} = 1$

Sanity check: Why don't these sum to 1?

Testing

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

(MAP)	$\hat{P}(\boldsymbol{X}\|Y=0)$	$\hat{P}(\boldsymbol{X}\|Y=1)$
$X_1 = 0$	0.42	0.01
$X_1 = 1$	0.58	0.99

(MLE)	$\hat{P}(Y)$
$Y = 0$	0.09
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$$\hat{P}(X_1 = 1|Y = 0)\hat{P}(Y = 0) = 0.58 \cdot 0.09 \approx 0.052$$
$$\hat{P}(X_1 = 1|Y = 1)\hat{P}(Y = 1) = 0.99 \cdot 0.91 \approx 0.901$$

A. $0.052 < 0.5 \Rightarrow \hat{Y} = 1$
B. $0.901 > 0.5 \Rightarrow \hat{Y} = 1$
[✓ C.] $0.052 < 0.901 \Rightarrow \hat{Y} = 1$

Sanity check: Why don't these sum to 1?

“Brute Force Bayes” classifier

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

($\hat{P}(Y)$ is an estimate of $P(Y)$,
$\hat{P}(\boldsymbol{X}|Y)$ is an estimate of $P(\boldsymbol{X}|Y)$)

Training

Estimate these probabilities, i.e., “learn” these parameters using MLE or Laplace (MAP)

$$\hat{P}(X_1, X_2, \dots, X_m|Y = 1)$$
$$\hat{P}(X_1, X_2, \dots, X_m|Y = 0)$$
$$\hat{P}(Y = 1) \qquad \hat{P}(Y = 0)$$

Testing

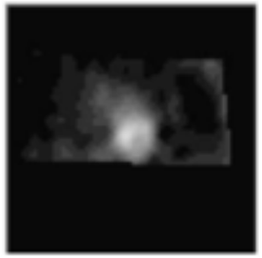
Given an observation $\boldsymbol{X} = (X_1, X_2, \dots, X_m)$, predict

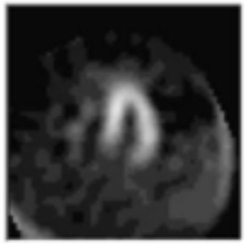
$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\hat{P}(X_1, X_2, \dots, X_m|Y)\hat{P}(Y)\right)$$

Naïve Bayes Classifier

Brute Force Bayes: $m = 300$ (# features)

$$\boldsymbol{X} = (X_1, X_2, X_3, \dots, X_{300})$$





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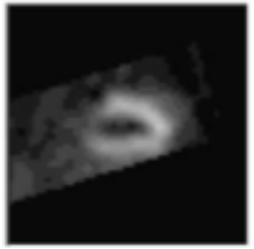
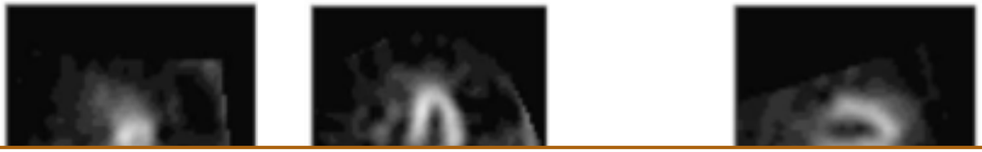


	Feature 1	Feature 2		Feature 300	Output
Patient 1	1	0	…	1	1
Patient 2	1	1	…	0	0
…			⋮		⋮
Patient n	0	0	…	1	1

This won't be too bad, right?

Brute Force Bayes: $m = 300$ (# features)

$$\boldsymbol{X} = (X_1, X_2, X_3, \dots, X_{300})$$



Count:	# datapoints
$X_1 = 0, X_2 = 0, \dots, X_{299} = 0, X_{300} = 0, Y = 0$:	0
$X_1 = 0, X_2 = 0, \dots, X_{299} = 0, X_{300} = 1, Y = 0$:	0
$X_1 = 0, X_2 = 0, \dots, X_{299} = 1, X_{300} = 0, Y = 0$:	1
…	
$X_1 = 0, X_2 = 0, \dots, X_{299} = 0, X_{300} = 0, Y = 1$:	2
$X_1 = 0, X_2 = 0, \dots, X_{299} = 0, X_{300} = 1, Y = 1$:	1
$X_1 = 0, X_2 = 0, \dots, X_{299} = 1, X_{300} = 0, Y = 1$:	1

Pat

Pat

Patient n 0 0 … 1 1

This won't be too bad, right?

Brute Force Bayes

Review

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(Y \mid \boldsymbol{X})$$

Choose the Y that is most likely given $\boldsymbol{X}$

$$= \arg\max_{y=\{0,1\}} \frac{\hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)}{\hat{P}(\boldsymbol{X})}$$

(Bayes' Theorem)

$$= \arg\max_{y=\{0,1\}} \underbrace{\hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)}_{}$$

($1/P(\boldsymbol{X})$ is constant w.r.t. y)

Learn parameters through MLE or MAP

Brute Force Bayes: $m = 300$ (# features)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(Y \mid \boldsymbol{X})$$

$$= \arg\max_{y=\{0,1\}} \frac{\hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)}{\hat{P}(\boldsymbol{X})}$$

$$= \arg\max_{y=\{0,1\}} \underbrace{\hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)}$$

Learn parameters through MLE or MAP

- $\hat{P}(Y = 1 \mid \boldsymbol{x})$: estimated probability a heart is healthy given $\boldsymbol{x}$
- $\boldsymbol{X} = (X_1, X_2, \dots, X_{300})$: whether 300 regions of interest (ROI) are healthy (1) or unhealthy (0)

How many parameters do we have to learn?

	$\hat{P}(\boldsymbol{X} \mid Y)$	$\hat{P}(Y)$	
A.	$2 \cdot 2$	$+\ 2$	$= 6$
B.	$2 \cdot 300$	$+\ 2$	$= 602$
C.	$2 \cdot 2^{300}$	$+\ 2$	$=$ a lot

Brute Force Bayes: $m = 300$ (# features)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(Y \mid \boldsymbol{X})$$

$$= \arg\max_{y=\{0,1\}} \frac{\hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)}{\hat{P}(\boldsymbol{X})}$$

$$= \arg\max_{y=\{0,1\}} \underbrace{\hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)}_{}$$

Learn parameters through MLE or MAP

This approach requires you to learn $O(2^m)$ parameters.

- $\hat{P}(Y = 1 \mid \boldsymbol{x})$: estimated probability a heart is healthy given $\boldsymbol{x}$
- $\boldsymbol{X} = (X_1, X_2, \dots, X_{300})$: whether 300 regions of interest (ROI) are healthy (1) or unhealthy (0)

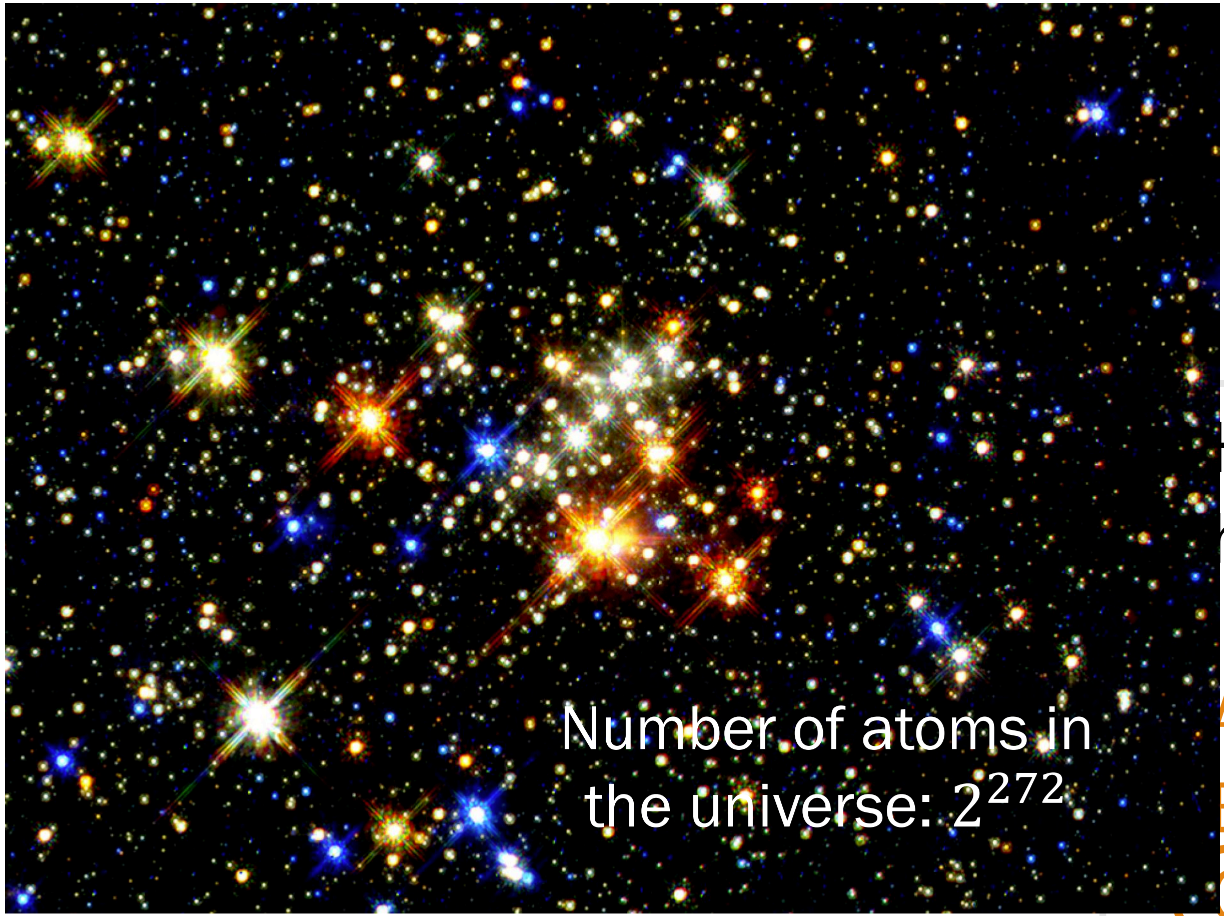
How many parameters do we have to learn?

	$\hat{P}(\boldsymbol{X} \mid Y)$	$\hat{P}(Y)$	
A.	$2 \cdot 2$	$+ 2$	$= 6$
B.	$2 \cdot 300$	$+ 2$	$= 602$
[✓ C.]	$2 \cdot 2^{300}$	$+ 2$	$=$ a lot

Brute Force Bayes: $m = 300$ (# features)

$\hat{P}(Y = 1 \mid \boldsymbol{x})$: estimated probability a heart is healthy given $\boldsymbol{x}$

$\boldsymbol{X} = (X_1, X_2, \dots, X_{300})$: whether 300 regions of interest (ROI) are healthy (1) or unhealthy (0)



How many parameters do we have to learn?

	$\hat{P}(\boldsymbol{X} \mid Y)$	$\hat{P}(Y)$	
A.	$2 \cdot 2$	$+\,2$	$= 6$
B.	$2 \cdot 300$	$+\,2$	$= 602$
[✓ C.]	$2 \cdot 2^{300}$	$+\,2$	$=$ a lot

This approach requires you to learn $O(2^m)$ parameters.

The problem with our current classifier

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(Y \mid \boldsymbol{X})$$

Choose the Y that is most likely given $\boldsymbol{X}$

$$= \arg\max_{y=\{0,1\}} \frac{\hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)}{\hat{P}(\boldsymbol{X})}$$

(Bayes' Theorem)

$$= \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

(1/$P(\boldsymbol{X})$ is constant w.r.t. y)

$$\hat{P}(X_1, X_2, \ldots, X_m|Y)$$

Estimating this joint conditional distribution is often intractable.

What if we could make a simplifying (but naïve) assumption to make estimation easier?

The Naïve Bayes assumption

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(Y \mid \boldsymbol{X})$$

$$= \arg\max_{y=\{0,1\}} \frac{\hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)}{\hat{P}(\boldsymbol{X})}$$

$$= \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

$$= \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y)\right)\hat{P}(Y)$$

Assumption:

$X_1, \ldots, X_m$ are conditionally independent given Y.

Naïve Bayes Assumption

Naïve Bayes Classifier

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Training

What is the Big-O of # of parameters we need to learn?

[✓ A.] $O(m)$

B. $O(2^m)$

C. other

Naïve Bayes Classifier

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Training

for $j = 1, \ldots, m$: $\hat{P}(X_j = 1|Y = 0)$, $\hat{P}(X_j = 1|Y = 1)$

$\hat{P}(Y = 1)$

Use MLE or Laplace (MAP)

Testing

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\log \hat{P}(Y) + \sum_{j=1}^{m} \log \hat{P}(X_j|Y) \right)$$

(for numeric stability)

24: Naïve Bayes (live)

Lisa Yan and Jerry Cain
November 6, 2020

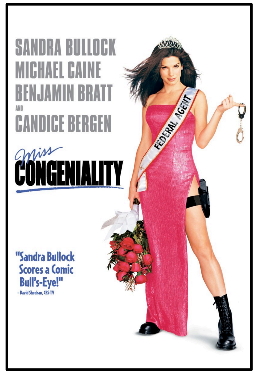
Classification terminology check

A. $\boldsymbol{x}^{(i)}$
B. $y^{(i)}$
C. $\left(\boldsymbol{x}^{(i)}, y^{(i)}\right)$
D. $x_j^{(i)}$

1: like movie
0: dislike movie

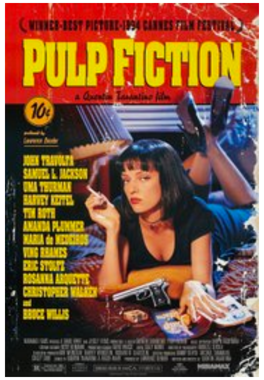
Training data: $\left(\boldsymbol{x}^{(1)}, y^{(1)}\right), \left(\boldsymbol{x}^{(2)}, y^{(2)}\right), \ldots, \left(\boldsymbol{x}^{(n)}, y^{(n)}\right)$

Movie 1



Movie 2

…



Movie m

Output

User 1	1.	1	0	…	1	2. 1
User 2	3.	1	1	…	0	0
…				⋮		⋮
User n		0	4. 0	…	1	1

(by yourself)

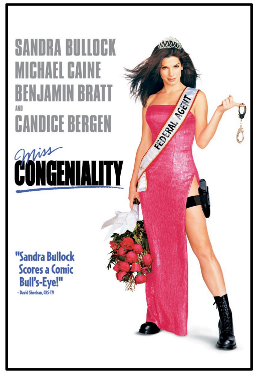
Classification terminology check

A. $\boldsymbol{x}^{(i)}$
B. $y^{(i)}$
C. $\left(\boldsymbol{x}^{(i)}, y^{(i)}\right)$
D. $x_j^{(i)}$

Training data: $\left(\boldsymbol{x}^{(1)}, y^{(1)}\right), \left(\boldsymbol{x}^{(2)}, y^{(2)}\right), \ldots, \left(\boldsymbol{x}^{(n)}, y^{(n)}\right)$

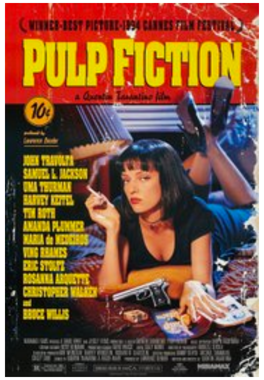
1: like movie
0: dislike movie

Movie 1



Movie 2

…



Movie m

Output

User 1	1.	1	0	…	1	2. 1
User 2	3.	1	1	…	0	0
…				⋮		⋮
User n		0	4. 0	…	1	1

i: i-th user
j: movie j

NETFLIX

and Learn

Predicting user TV preferences

Will a user like the Pokémon TV series?

Observe indicator variables $\boldsymbol{X} = (X_1, X_2)$:

$X_1 = 1$:
"likes Star Wars"

$X_2 = 1$:
"likes Harry Potter"

Output Y indicator:

$Y = 1$:
"likes Pokémon"

Predict $\hat{Y} = \underset{y=\{0,1\}}{\arg\max}\, \hat{P}(Y \mid \boldsymbol{X})$

The model, and the Naïve Bayes assumption

Review

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(Y \mid \boldsymbol{X})$$

$$= \arg\max_{y=\{0,1\}} \frac{\hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)}{\hat{P}(\boldsymbol{X})}$$

$$= \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

$$= \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y)\right)\hat{P}(Y)$$

Naïve Bayes Assumption:

$X_1, \ldots, X_m$ are conditionally independent given Y.

Breakout Rooms

Check out the questions on the next slide (Slide 50). Post any clarifications here!

https://us.edstem.org/courses/2678/discussion/169796

Breakout rooms: 3 min

🤔

Predicting user TV preferences

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

Naïve Bayes Assumption $$P(\boldsymbol{X}|Y) = \prod_{j=1}^{m} P(X_j|Y)$$

Which probabilities do you need to estimate? How many are there?

- Brute Force Bayes (strawman, without NB assumption)
- Naïve Bayes

During training, how to estimate the prob $\hat{P}(X_1 = 1, X_2 = 1|Y = 0)$ with MLE? with Laplace?

- Brute Force Bayes
- Naïve Bayes

🤔

Predicting user TV preferences

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

Naïve Bayes Assumption

$$P(\boldsymbol{X}|Y) = \prod_{j=1}^{m} P(X_j|Y)$$

Which probabilities do you need to estimate? How many are there?

- Brute Force Bayes (strawman, without NB assumption)
- Naïve Bayes

During training, how to estimate the prob $\hat{P}(X_1 = 1, X_2 = 1|Y = 0)$ with MLE? with Laplace?

- Brute Force Bayes
- Naïve Bayes

(strawman brute force) Multinomial MLE and MAP

Model: Multinomial, m outcomes: p_j probability of outcome j

Observe: n_j = # of trials with outcome j
Total of $\sum_{j=1}^{m} n_j$ trials

MLE
$$\hat{p}_j = \frac{n_j}{\sum_{j=1}^{m} n_j}$$

Laplace estimate (MAP w/Laplace smoothing)
$$\hat{p}_j = \frac{n_j + 1}{\sum_{j=1}^{m} n_j + m}$$

X_1	X_2	Y
0	1	1
1	1	0
1	0	1
…	…	…
1	1	1

training data

$$\hat{P}(X_1 = 1\ X_2 = 1 | Y = 0)$$

(Naïve Bayes) Multinomial MLE and MAP

Model: Multinomial, m outcomes:
p_j probability of outcome j

Observe: $n_j = \#$ of trials with outcome j
Total of $\sum_{j=1}^{m} n_j$ trials

MLE

$$\hat{p}_j = \frac{n_j}{\sum_{j=1}^{m} n_j}$$

Laplace estimate (MAP w/Laplace smoothing)

$$\hat{p}_j = \frac{n_j + 1}{\sum_{j=1}^{m} n_j + m}$$

X_1	X_2	Y
0	1	1
1	1	0
1	0	1
…	…	…
1	1	1

training data

$$\hat{P}(X_1 = 1\, X_2 = 1 | Y = 0)$$

NETFLIX

and Learn
naively

Ex 1. Naïve Bayes Classifier (MLE)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Training

$\forall i$: $\hat{P}(X_j = 1|Y = 0), \hat{P}(X_j = 0|Y = 0),$
$\hat{P}(X_j = 1|Y = 1), \hat{P}(X_j = 0|Y = 0),$
$\hat{P}(Y = 1), \hat{P}(Y = 0)$

Use MLE or Laplace (MAP)

Testing

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Think

Slide 59 has two questions to go over by yourself.

Post any clarifications here!

https://us.edstem.org/courses/2678/discussion/153773

Think by yourself: 1 min

(by yourself)

Training: Naïve Bayes for TV shows (**MLE**)

Observe indicator vars. $\boldsymbol{X} = (X_1, X_2)$:

- X_1: “likes Star Wars”
- X_2: “likes Harry Potter”

Predict Y: “likes Pokémon”

X_1 Y	0	1
0	3	10
1	4	13

X_2 Y	0	1
0	5	8
1	7	10

Training data counts

1. How many datapoints (n) are in our train data?
2. Compute MLE estimates for $\hat{P}(X_1|Y)$:

X_1 Y	0	1
0	$\hat{P}(X_1 = 0 \mid Y = 0)$	$\hat{P}(X_1 = 1 \mid Y = 0)$
1	$\hat{P}(X_1 = 0 \mid Y = 1)$	$\hat{P}(X_1 = 1 \mid Y = 1)$

🤔 (by yourself)

Training: Naïve Bayes for TV shows (**MLE**)

Observe indicator vars. $\boldsymbol{X} = (X_1, X_2)$:

- X_1: “likes Star Wars”
- X_2: “likes Harry Potter”

Predict Y: “likes Pokémon”

Y \ X_1	0	1
0	3	10
1	4	13

Y \ X_2	0	1
0	5	8
1	7	10

Training data counts

1. How many datapoints (n) are in our train data?
2. Compute MLE estimates for $\hat{P}(X_1|Y)$:

Y \ X_1	0	1
0		
1		

Training: Naïve Bayes for TV shows (**MLE**)

Observe indicator vars. $\boldsymbol{X} = (X_1, X_2)$:

- X_1: “likes Star Wars”
- X_2: “likes Harry Potter”

Predict Y: “likes Pokémon”

Y \ X_1	0	1
0	3	10
1	4	13

Y \ X_2	0	1
0	5	8
1	7	10

Y	
0	13
1	17

Training data counts

Y \ X_1	0	1
0	0.23	0.77
1	0.24	0.76

(from last slide)

Y \ X_2	0	1
0	$5/13 \approx 0.38$	$8/13 \approx 0.62$
1	$7/17 \approx 0.41$	$10/17 \approx 0.59$

Y	
0	$13/30 \approx 0.43$
1	$17/30 \approx 0.57$

Training : Naïve Bayes for TV shows (**MLE**)

Observe indicator vars. $\boldsymbol{X} = (X_1, X_2)$:

- X_1: “likes Star Wars”
- X_2: “likes Harry Potter”

Predict Y: “likes Pokémon”

X_1 Y	0	1
0	0.23	0.77
1	0.24	0.76

X_2 Y	0	1
0	0.38	0.62
1	0.41	0.59

Y	
0	0.43
1	0.57

Now that we’ve trained and found parameters,
It’s time to classify new users!

Ex 1. Naïve Bayes Classifier (MLE)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Training

$$\forall i:\ \hat{P}(X_j = 1|Y = 0), \hat{P}(X_j = 0|Y = 0),$$
$$\hat{P}(X_j = 1|Y = 1), \hat{P}(X_j = 0|Y = 0),$$
$$\hat{P}(Y = 1), \hat{P}(Y = 0)$$

Use MLE or Laplace (MAP)

Testing

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Testing: Naïve Bayes for TV shows (**MLE**)

Observe indicator vars. $\boldsymbol{X} = (X_1, X_2)$:

- X_1: "likes Star Wars"
- X_2: "likes Harry Potter"

Predict Y: "likes Pokémon"

X_1 Y	0	1
0	0.23	0.77
1	0.24	0.76

X_2 Y	0	1
0	0.38	0.62
1	0.41	0.59

Y	
0	0.43
1	0.57

Suppose a new person "likes Star Wars" ($X_1 = 1$) but "dislikes Harry Potter" ($X_2 = 0$).

Will they like Pokemon? Need to predict Y:

$$\hat{Y} = \underset{y=\{0,1\}}{\arg\max}\, \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y) \quad = \underset{y=\{0,1\}}{\arg\max}\, \hat{P}(X_1|Y)\hat{P}(X_2|Y)\hat{P}(Y)$$

If $Y = 0$: $\hat{P}(X_1 = 1|Y = 0)\hat{P}(X_2 = 0|Y = 0)\hat{P}(Y = 0) = 0.77 \cdot 0.38 \cdot 0.43 = 0.126$

If $Y = 1$: $\hat{P}(X_1 = 1|Y = 1)\hat{P}(X_2 = 0|Y = 1)\hat{P}(Y = 1) = 0.76 \cdot 0.41 \cdot 0.57 = 0.178$

Since term is greatest when Y = 1, predict $\hat{Y} = 1$

Interlude for jokes/announcements

Announcements

Problem Set 5

Super on-time due (+8%): earlier today
On-time due (+5%): Monday 11/9 1:00pm
Grace period ends (+0%): Tuesday 11/10 1:00pm

Problem Set 6

Out: later today
Due: Monday 11/16
Grace period: Wednesday 11/18
Covers: through Lecture 26

Ex 2. Naïve Bayes Classifier (MAP)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Training

$$\forall i: \ \hat{P}(X_j = 1|Y = 0), \hat{P}(X_j = 0|Y = 0),$$

$$\hat{P}(X_j = 1|Y = 1), \hat{P}(X_j = 0|Y = 0),$$

$$\hat{P}(Y = 1), \hat{P}(Y = 0)$$

Use MLE or
Laplace (MAP)

Testing

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

(note the same as before)

Training: Naïve Bayes for TV shows (**MAP**)

Observe indicator vars. $\boldsymbol{X} = (X_1, X_2)$:

- X_1: “likes Star Wars”
- X_2: “likes Harry Potter”

Predict Y: “likes Pokémon”

Y \ X_1	0	1
0	3	10
1	4	13

Y \ X_2	0	1
0	5	8
1	7	10

Training data counts

What are our MAP estimates using Laplace smoothing for $\hat{P}(X_j|Y)$?

$\hat{P}(X_j = x|Y = y)$:

A. $\frac{\#(X_j=x,Y=y)}{\#(Y=y)}$

B. $\frac{\#(X_j=x,Y=y)+1}{\#(Y=y)+2}$

C. $\frac{\#(X_j=x,Y=y)+1}{\#(Y=y)+4}$

D. other

(by yourself)

Training: Naïve Bayes for TV shows (**MAP**)

Observe indicator vars. $\boldsymbol{X} = (X_1, X_2)$:

- X_1: “likes Star Wars”
- X_2: “likes Harry Potter”

Predict Y: “likes Pokémon”

$Y \backslash X_1$	0	1
0	3	10
1	4	13

$Y \backslash X_2$	0	1
0	5	8
1	7	10

Training data counts

What are our MAP estimates using Laplace smoothing for $\hat{P}(X_j|Y)$ and $\hat{P}(Y)$?

$\hat{P}(X_j = x|Y = y)$:

A. $$\frac{\#(X_j=x, Y=y)}{\#(Y=y)}$$

B. $$\frac{\#(X_j=x, Y=y)+1}{\#(Y=y)+2}$$

C. $$\frac{\#(X_j=x, Y=y)+1}{\#(Y=y)+4}$$

D. other

Training: Naïve Bayes for TV shows (MAP)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{i=1}^{m} \hat{P}(X_i|Y) \right) \hat{P}(Y)$$

Observe indicator vars. $\boldsymbol{X} = (X_1, X_2)$:

- X_1: “likes Star Wars”
- X_2: “likes Harry Potter”

Predict Y: “likes Pokémon”

Y \ X_1	0	1
0	3	10
1	4	13

Y \ X_2	0	1
0	5	8
1	7	10

Y	
0	13
1	17

Training data counts

Y \ X_1	0	1
0	0.27	0.73
1	0.26	0.74

Y \ X_2	0	1
0	0.40	0.60
1	0.42	0.58

In practice:

- We use Laplace for $\hat{P}(X_j|Y)$ in case some events $X_j = x_j$ don’t appear
- We don’t use Laplace for $\hat{P}(Y)$, because all class labels should appear reasonably often

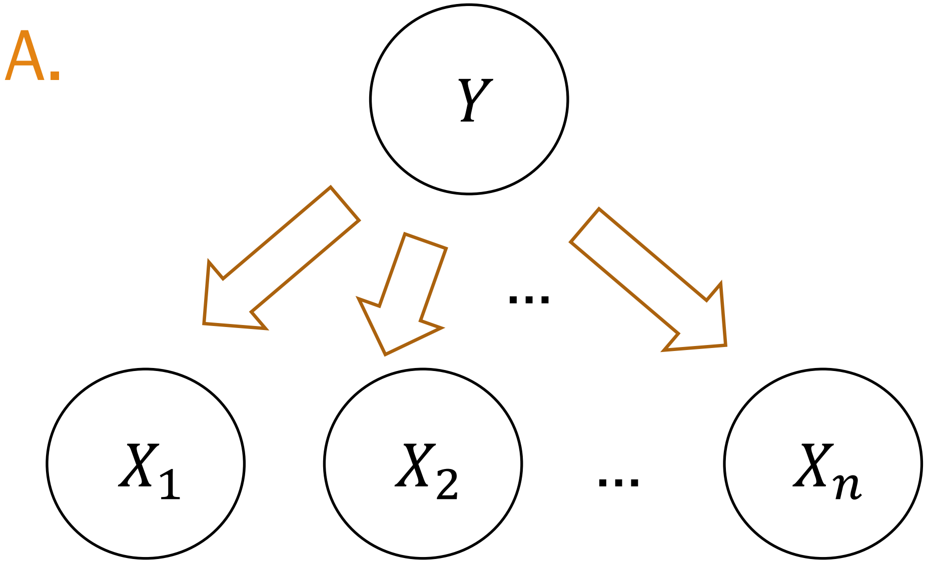
Naïve Bayes Model is a Bayesian Network

Naïve Bayes Assumption

$$P(\boldsymbol{X}|Y) = \prod_{j=1}^{m} P(X_j|Y)$$

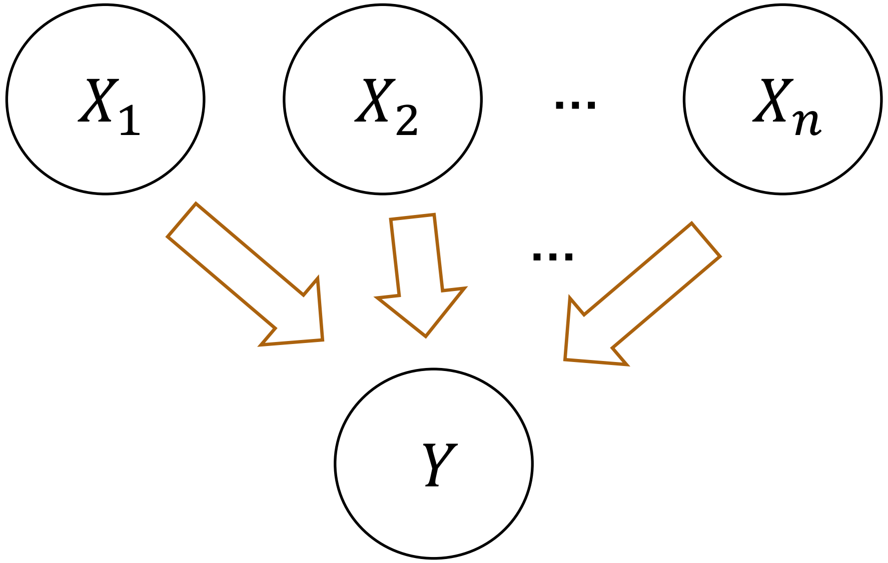
Which Bayesian Network encodes this conditional independence?

A.



X_i are conditionally independent given Y

B.



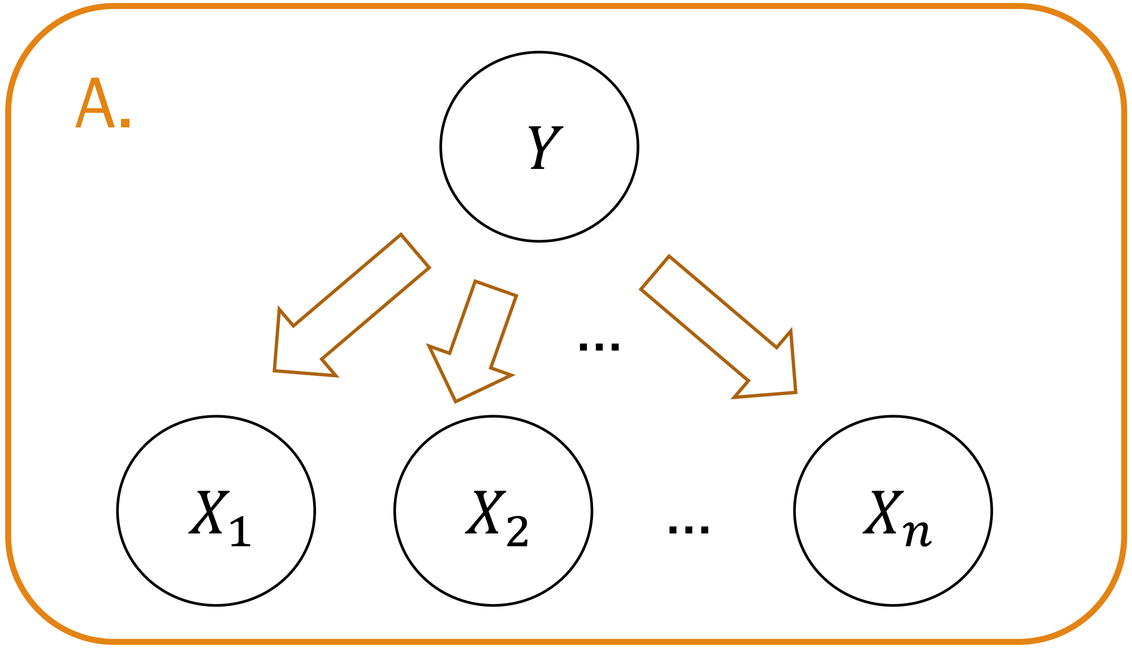
(by yourself)

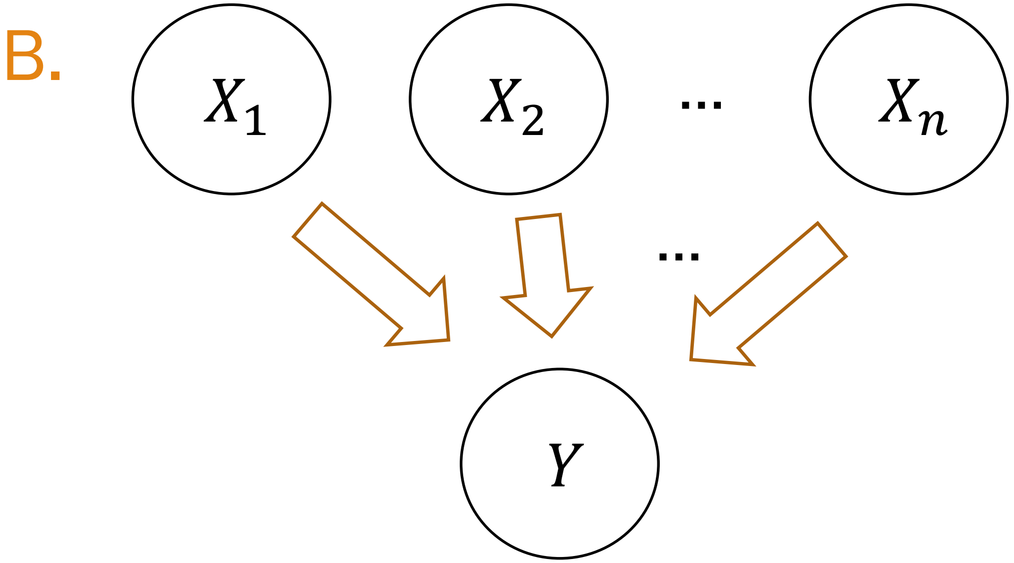
Naïve Bayes Model is a Bayesian Network

Naïve Bayes Assumption

$$P(\boldsymbol{X}|Y) = \prod_{i=1}^{m} P(X_i|Y) \quad \Rightarrow \quad P(\boldsymbol{X}, Y) = P(Y)\prod_{j=1}^{m} P\left(X_j|Y\right)$$

Which Bayesian Network encodes this conditional independence?



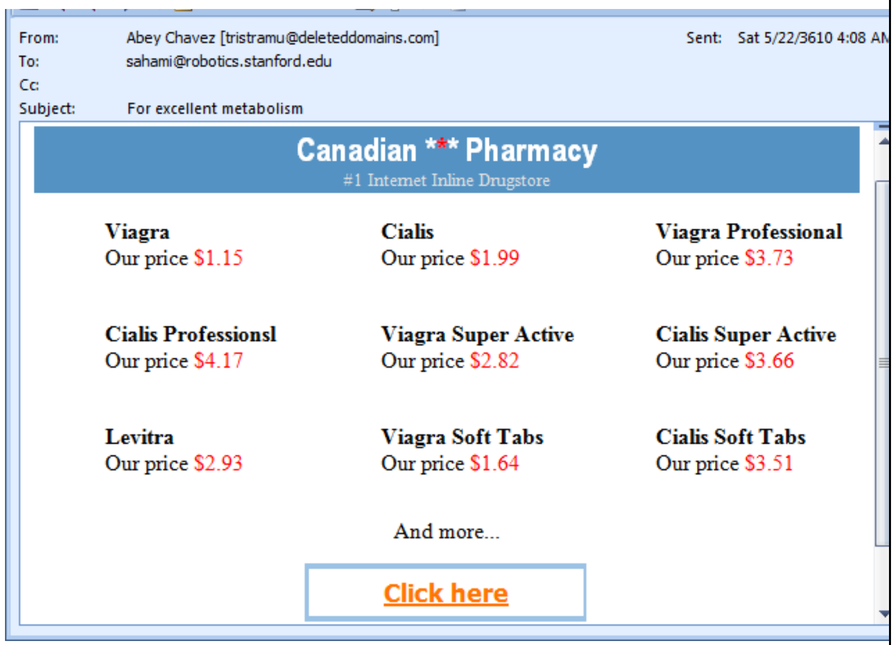


X_i are conditionally independent given parent Y

and Learn

naively

What is Bayes doing in my mail server?



From: Abey Chavez [tristramu@deleteddomains.com]
Sent: Sat 5/22/3610 4:08 AM
To: sahami@robotics.stanford.edu
Cc:
Subject: For excellent metabolism

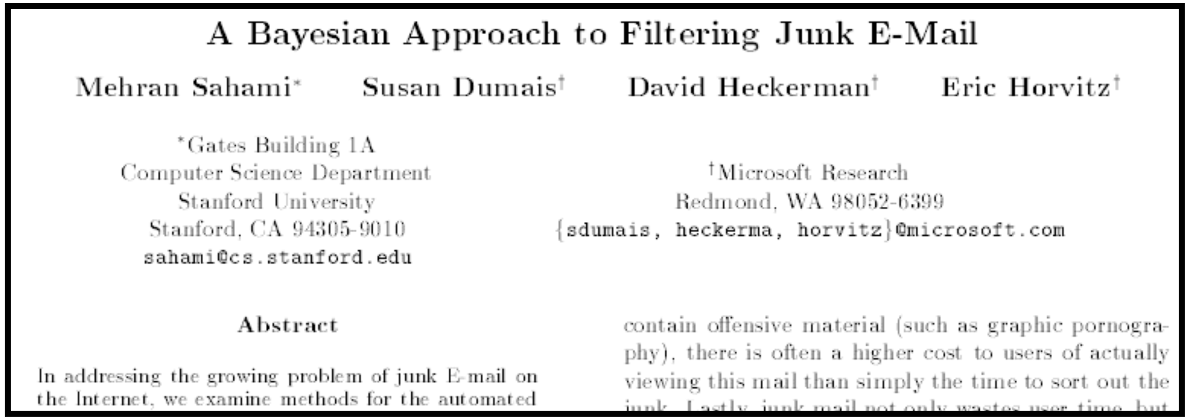
Canadian ** Pharmacy
#1 Internet Inline Drugstore

Viagra Our price $1.15
Cialis Our price $1.99
Viagra Professional Our price $3.73
Cialis Professionsl Our price $4.17
Viagra Super Active Our price $2.82
Cialis Super Active Our price $3.66
Levitra Our price $2.93
Viagra Soft Tabs Our price $1.64
Cialis Soft Tabs Our price $3.51

And more...

Let’s get Bayesian on your spam:

```
Content analysis details:   (49.5 hits, 7.0 required)
 0.9 RCVD_IN_PBL            RBL: Received via a relay in Spamhaus PBL
                            [93.40.189.29 listed in zen.spamhaus.org]
 1.5 URIBL_WS_SURBL         Contains an URL listed in the WS SURBL blocklist
                            [URIs: recragas.cn]
 5.0 URIBL_JP_SURBL         Contains an URL listed in the JP SURBL blocklist
                            [URIs: recragas.cn]
 5.0 URIBL_OB_SURBL         Contains an URL listed in the OB SURBL blocklist
                            [URIs: recragas.cn]
 5.0 URIBL_SC_SURBL         Contains an URL listed in the SC SURBL blocklist
                            [URIs: recragas.cn]
 2.0 URIBL_BLACK            Contains an URL listed in the URIBL blacklist
                            [URIs: recragas.cn]
 8.0 BAYES_99               BODY: Bayesian spam probability is 99 to 100%
                            [score: 1.0000]
```



A Bayesian Approach to Filtering Junk E-Mail

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Abstract

In addressing the growing problem of junk E-mail on the Internet, we examine methods for the automated

contain offensive material (such as graphic pornography), there is often a higher cost to users of actually viewing this mail than simply the time to sort out the

Ex 3. Naïve Bayes Classifier (m, n large)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{i=1}^{m} \hat{P}(X_i|Y)\right)\hat{P}(Y)$$

Training

$\forall i$: $\hat{P}(X$ $= 0)$, Use MLE or

$0|Y = 0)$, Laplace (MAP)

$\hat{P}$

What changes are necessary?

Testing

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{i=1}^{m} \hat{P}(X_i|Y)\right)\hat{P}(Y)$$

Email classification

Goal	Based on email content $\boldsymbol{X}$, predict if email is spam or not.
Features	Consider a lexicon m words (for English: $m \approx 100{,}000$). $\boldsymbol{X} = (X_1, X_2, \dots, X_m)$, m indicator variables $X_j = 1$ if word j appeared in document
Output	$Y = 1$ if email is spam

Note: m is huge. Make Naïve Bayes assumption: $P(\boldsymbol{X}|\text{spam}) = \prod_{j=1}^{m} P\left(X_j|\text{spam}\right)$

Appearances of words in email are conditionally independent given the email is spam or not

Training: Naïve Bayes Email classification

Train set n previous emails $\left(\boldsymbol{x}^{(1)}, y^{(1)}\right), \left(\boldsymbol{x}^{(2)}, y^{(2)}\right), \ldots, \left(\boldsymbol{x}^{(n)}, y^{(n)}\right)$

$\boldsymbol{x}^{(i)} = \left(x_1^{(i)}, x_2^{(i)}, \ldots, x_m^{(i)}\right)$ for each word, whether it appears in email i

$y^{(i)} = 1$ if spam, 0 if not spam

Note: m is huge.

Which estimator should we use for $\hat{P}\left(X_j | Y\right)$?

A. MLE
B. Laplace estimate (MAP)
C. Other MAP estimate
D. Both A and B

(by yourself)

Training: Naïve Bayes Email classification

Train set n previous emails $\left(\boldsymbol{x}^{(1)}, y^{(1)}\right), \left(\boldsymbol{x}^{(2)}, y^{(2)}\right), \ldots, \left(\boldsymbol{x}^{(n)}, y^{(n)}\right)$

$\boldsymbol{x}^{(i)} = \left(x_1^{(i)}, x_2^{(i)}, \ldots, x_m^{(i)}\right)$ for each word, whether it appears in email i

$y^{(i)} = 1$ if spam, 0 if not spam

Note: m is huge.

Which estimator should we use for $\hat{P}\left(X_j | Y\right)$?

A. MLE
[✓ B.] Laplace estimate (MAP)
C. Other MAP estimate
D. Both A and B

Many words are likely to not appear at all in the training set!

Ex 3. Naïve Bayes Classifier (m, n large)

$$\hat{Y} = \underset{y=\{0,1\}}{\arg\max} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Training

$$\forall i: \ \hat{P}(X_j = 1|Y = 0), \hat{P}(X_j = 0|Y = 0),$$
$$\hat{P}(X_j = 1|Y = 1), \hat{P}(X_j = 0|Y = 0),$$
$$\hat{P}(Y = 1), \hat{P}(Y = 0)$$

Use MLE or Laplace (MAP)

Testing

$$\hat{Y} = \underset{y=\{0,1\}}{\arg\max} \left(\prod_{j=}^{m} \right.$$

Laplace (MAP) estimates avoid estimating 0 probabilities for events that don't occur in your training data.

Testing: Naïve Bayes Email classification

For a new email:

- Generate $\boldsymbol{X} = (X_1, X_2, \dots, X_m)$
- Classify as spam or not using Naïve Bayes assumption

Note: m is huge.

Suppose train set size n also huge (many labeled emails).

Can we still use the below prediction?

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

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Will probably lead to underflow!

Ex 3. Naïve Bayes Classifier (m, n large)

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$\hat{P}(Y = 1), \hat{P}(Y = 0)$

Use sums of log-probabilities for numerical stability.

Testing

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\log \hat{P}(Y) + \sum_{j=1}^{m} \log \hat{P}(X_j|Y) \right)$$

How well does Naïve Bayes perform?

After training, you can test with another set of data, called the test set.

- Test set also has known values for Y so we can see how often we were right/wrong in our predictions $\hat{Y}$.

Typical workflow:

- Have a dataset of 1789 emails (1578 spam, 211 ham)
- Train set: First 1538 emails (by time)
- Test set: Next 251 messages

Evaluation criteria on test set:

$$\textbf{precision} = \frac{(\#\text{ correctly predicted class } Y)}{(\#\text{ predicted class } Y)}$$

$$\textbf{recall} = \frac{(\#\text{ correctly predicted class } Y)}{(\#\text{ real class } Y\text{ messages})}$$

	Spam		Non-spam	
	Prec.	Recall	Prec.	Recall
Words only	97.1%	94.3%	87.7%	93.4%
Words + addtl features	100%	98.3%	96.2%	100%

What are precision and recall?

Accuracy (# correct)/(# total) sometimes just doesn't cut it.

Precision:	Of the emails you predicted as spam, how many are *actually* spam?	Measure of false positives
Recall:	Of the emails that are actually spam, how many did you predict?	Measure of false negatives

More on Wikipedia (https://en.wikipedia.org/wiki/Precision_and_recall)
and Problem Set 6!