



# Great Expectations

Chris Piech and Jerry Cain  
CS109, Stanford University





# Quick slide reference

---

- 3 Conditional distributions
- 14 Conditional expectation
- 20 Law of Total Expectation



# Where are we now? A roadmap of CS109

Wed: Statistics of multiple RVs!

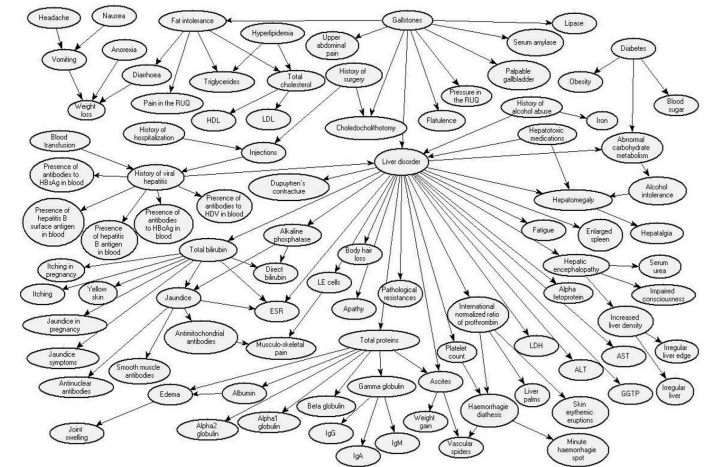
$$\text{Var}(X + Y)$$

$$E[X + Y]$$

$$\text{Cov}(X, Y)$$

$$\rho(X, Y)$$

Next Week: Modeling with Bayesian Networks



Last Fri: Joint distributions

$$p_{X,Y}(x, y)$$



Today:  
Conditional distributions

$$p_{X|Y}(x|y)$$

$$E[X|Y]$$

Time to kick it up a notch!





# Hiring and Engineer

```
DEFINE JOBIINTERVIEWQUICKSORT(LIST):  
  OK SO YOU CHOOSE A PIVOT  
  THEN DIVIDE THE LIST IN HALF  
  FOR EACH HALF:  
    CHECK TO SEE IF IT'S SORTED  
    NO, WAIT, IT DOESN'T MATTER  
    COMPARE EACH ELEMENT TO THE PIVOT  
    THE BIGGER ONES GO IN A NEW LIST  
    THE EQUAL ONES GO INTO, UH  
    THE SECOND LIST FROM BEFORE  
  HANG ON, LET ME NAME THE LISTS  
  THIS IS LIST A  
  THE NEW ONE IS LIST B  
  PUT THE BIG ONES INTO LIST B  
  NOW TAKE THE SECOND LIST  
  CALL IT LIST, UH, A2  
  WHICH ONE WAS THE PIVOT IN?  
  SCRATCH ALL THAT  
  IT JUST RECURSIVELY CALLS ITSELF  
  UNTIL BOTH LISTS ARE EMPTY  
  RIGHT?  
  NOT EMPTY, BUT YOU KNOW WHAT I MEAN  
  AM I ALLOWED TO USE THE STANDARD LIBRARIES?
```

Your company has one job opening for a software engineer.

You have  $n$  candidates. But you have to say yes/no **immediately** after each interview!

What algorithm will maximize the probability that you hire the top candidate?



# Expectation of the Sum

---

$$E[X + Y] = E[X] + E[Y]$$

Generalized:

$$E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i]$$

Holds regardless of dependency between  $X_i$ 's



# Differential Privacy

Aims to provide means to **maximize the accuracy** of probabilistic queries while minimizing the **probability** of identifying its records.



Cynthia Dwork's celebrity lookalike is Cynthia Dwork.

Chris Piech and Jerry Cain, CS109, 2021



# Differential Privacy

100 independent values  $X_1 \dots X_{100}$  where  $X_i \sim \text{Bern}(p)$

```
# Maximize accuracy, while preserving privacy.  
def calculateYi(Xi):  
    obfuscate = random()           random() returns True  
    if obfuscate:                  or False with equal  
        return indicator(random()) likelihood  
    else:  
        return Xi
```

# Differential Privacy

100 independent values  $X_1 \dots X_{100}$  where  $X_i \sim \text{Bern}(p)$

```
# Maximize accuracy, while preserving privacy.  
def calculateYi(Xi):  
    obfuscate = random()           random() returns True  
    if obfuscate:                  or False with equal  
        return indicator(random()) likelihood  
    else:  
        return Xi
```

What is  $E[Y_i]$ ?

$$E[Y_i] = P(Y_i = 1) = \frac{p}{2} + \frac{1}{4}$$



# Differential Privacy

100 independent values  $X_1 \dots X_{100}$  where  $X_i \sim \text{Bern}(p)$

```
# Maximize accuracy, while preserving privacy.  
def calculateYi(Xi):  
    obfuscate = random()           random() returns True  
    if obfuscate:                  or False with equal  
        return indicator(random()) likelihood  
    else:  
        return Xi
```

Let  $Z = \sum_{i=1}^{100} Y_i$

What is the  $E[Z]$ ?

$$E[Z] = E\left[\sum_{i=1}^{100} Y_i\right] = \sum_{i=1}^{100} E[Y_i] = \sum_{i=1}^{100} \left(\frac{p}{2} + \frac{1}{4}\right) = 50p + 25$$

# Differential Privacy

100 independent values  $X_1 \dots X_{100}$  where  $X_i \sim \text{Bern}(p)$

```
# Maximize accuracy, while preserving privacy.  
def calculateYi(Xi):  
    obfuscate = random()           random() returns True  
    if obfuscate:                  or False with equal  
        return indicator(random()) likelihood  
    else:  
        return Xi
```

Let  $Z = \sum_{i=1}^{100} Y_i$     $E[Z] = 50p + 25$    How do you estimate  $p$ ?

$$p \approx \frac{Z - 25}{50}$$

Challenge: What is the probability that our estimate is good?



# Discrete conditional distributions

# Discrete conditional distributions

---

Recall the definition of the conditional probability of event  $E$  given event  $F$ :

$$P(E|F) = \frac{P(EF)}{P(F)}$$

For discrete random variables  $X$  and  $Y$ , the **conditional PMF** of  $X$  given  $Y$  is

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

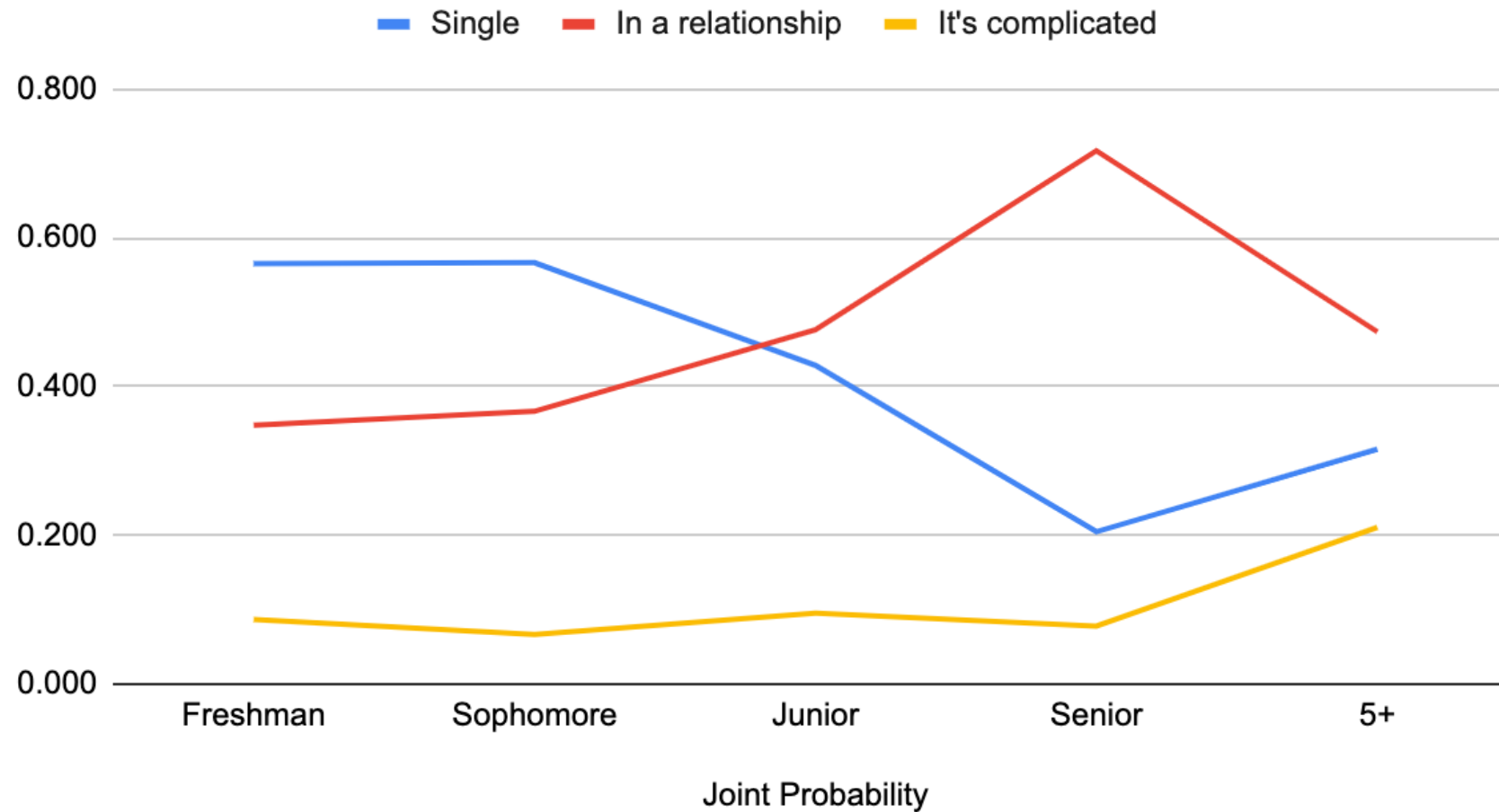
Different notation,  
same idea:

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$



# Dating at Stanford

Single, In a relationship and It's complicated



# Discrete probabilities of CS109

Each student responds with:

Year  $Y$

- 1: Frosh/Soph
- 2: Jr/Sr
- 3: Co-term/grad/NDO

Timezone  $T$  (12pm California time in my timezone is):

- -1: AM
- 0: noon
- 1: PM

	Joint PMF		
	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.06	.01	.01
$T = 0$	.29	.14	.09
$T = 1$	.30	.08	.02

$$P(Y = 3, T = 1)$$

Joint PMFs sum to 1.



# Discrete probabilities of CS109

The below are **conditional probability tables** for conditional PMFs

(A)  $P(Y = y|T = t)$  and (B)  $P(T = t|Y = y)$ .

1. Which is which?
2. What's the missing probability?

	Joint PMF		
	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.06	.01	.01
$T = 0$	.29	.14	.09
$T = 1$	.30	.08	.02

	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.09	.04	.08
$T = 0$	.45	.61	.75
$T = 1$	.46	.35	.17

	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.75	.125	?
$T = 0$	.56	.27	.17
$T = 1$	.75	.2	.05



# Discrete probabilities of CS109

The below are **conditional probability tables** for conditional PMFs

(A)  $P(Y = y|T = t)$  and (B)  $P(T = t|Y = y)$ .

1. Which is which?
2. What's the missing probability?

	Joint PMF		
	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.06	.01	.01
$T = 0$	.29	.14	.09
$T = 1$	.30	.08	.02

(B)  $P(T = t|Y = y)$

	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.09	.04	.08
$T = 0$	.45	.61	.75
$T = 1$	.46	.35	.17

$$.30 / (.06 + .29 + .30)$$

(A)  $P(Y = y|T = t)$

	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.75	.125	.125
$T = 0$	.56	.27	.17
$T = 1$	.75	.2	.05

1-.75-.125

Conditional PMFs also sum to 1 conditioned on different events!

# Extended to Amazon



Roll over image to zoom in

FINEDINE

**Stainless Steel Mixing Bowls by Finedine (Set of 6) Polished Mirror Finish Nesting Bowl, 3/4 - 1.5-3 - 4-5 - 8 Quart - Cooking Supplies**

★★★★☆ 2,366 customer reviews | 75 answered questions

Amazon's Choice for "stainless steel mixing bowls"

Price: **\$24.99 & FREE Shipping** on orders over \$25 shipped by Amazon. [Details](#)

Get \$40 off instantly: Pay \$0.00 upon approval for the Amazon.com Store Card.

✓prime | Try Fast, Free Shipping

- With graduating sizes of 3/4, 1.5, 3, 4, 5 and 8 quart, the bowl set allows users to be well equipped for serving fruit salads, marinating for the grill, and adding last ingredients for dessert.
- Stainless steel bowls with commercial grade metal that can be used as both baking mixing bowls and serving bowls. These metal bowls won't stain or absorb odors and resist rust for years of durability.
- An easy to grip rounded-lip on the stainless steel bowl set makes handling easier while a generous wide rim allows contents to flow evenly when pouring; flat base stabilizes the silver bowls making mixing all the easier.
- A space saving stackable design helps de-clutter kitchen cupboards while the attractive polished mirror finish on the large mixing bowls adds a luxurious aesthetic.
- This incredible stainless steel mixing bowl set is refrigerator, freezer, and dishwasher safe for quick and easy meal prep and clean up. They'd also make a great gift!

[Compare with similar items](#)

[Used & new \(7\) from \\$20.62 & FREE shipping on orders over \\$25.00. Details](#)

[Report incorrect product information.](#)

**Packaging may reveal contents.** Choose **Conceal Package** at checkout.

KELIWA

Easy home baking

Shop now

Ad feedback



Keliwa 12 Cup Silicone Muffin - Cupcake Baking Pan / Silicone Mold Non - Stick ...

★★★★☆ 1,803

\$9.99 ✓prime

Customers who



ExcelSteel Stainless Steel Colanders, Set of 3  
★★★★☆ 301  
\$15.83 ✓prime



1Easylife 18/8 Stainless Steel Measuring Spoons, Set of 6 for Measuring Dry and Liquid Ingredients  
★★★★☆ 1,854  
#1 Best Seller in Measuring Spoons  
\$9.99 ✓prime



New Star Foodservice 42917 Stainless Steel 4pcs Measuring Cups and Spoons Combo Set  
★★★★☆ 1,042  
#1 Best Seller in Specialty Spoons  
\$9.95 ✓prime



Rubbermaid Easy Find Lids Food Storage Containers, Racer Red, 42-Piece Set 1880801  
★★★★☆ 10,319  
\$19.99 ✓prime



Mlusco 5 Piece Silicone Cooking Utensil Set with Natural Acacia Hard Wood Handle  
★★★★☆ 461  
\$20.99 ✓prime



Bellemain Micro-perforated Stainless Steel 5-quart Colander-Dishwasher Safe  
★★★★☆ 2,797  
#1 Best Seller in Colanders  
\$19.95 ✓prime



AmazonBasics 6-Piece Nonstick Bakeware Set  
★★★★☆ 67  
\$19.99 ✓prime



HOMWE Kitchen Cutting Board (3-Piece Set) | Juice Grooves w/ Easy-Grip Handles | BPA-Free, ...  
★★★★☆ 240  
\$14.97 ✓prime

P(bought item X | bought item Y)

# Quick check

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Number or function?

1.  $P(X = 2|Y = 5)$

2.  $P(X = x|Y = 5)$

3.  $P(X = 2|Y = y)$

4.  $P(X = x|Y = y)$

True or false?

5.  $\sum_x P(X = x|Y = 5) = 1$

6.  $\sum_y P(X = 2|Y = y) = 1$

7.  $\sum_x \sum_y P(X = x|Y = y) = 1$

8.  $\sum_x \left( \sum_y P(X = x|Y = y)P(Y = y) \right) = 1$





# Quick check

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Number or function?

1.  $P(X = 2|Y = 5)$

number

2.  $P(X = x|Y = 5)$

1-D function

3.  $P(X = 2|Y = y)$

1-D function

4.  $P(X = x|Y = y)$

2-D function

True or false?

5.  $\sum_x P(X = x|Y = 5) = 1$  true

6.  $\sum_y P(X = 2|Y = y) = 1$  false

7.  $\sum_x \sum_y P(X = x|Y = y) = 1$  false

8.  $\sum_x \left( \sum_y P(X = x|Y = y)P(Y = y) \right) = 1$  true

# Independent RVs, defined another way

If  $X$  and  $Y$  are independent discrete random variables, then  $\forall x, y$ :

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{P(X = x)P(Y = y)}{P(Y = y)} = P(X = x)$$

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x, y)}{p_Y(y)} = \frac{p_X(x)p_Y(y)}{p_Y(y)} = p_X(x)$$

Note for conditional expectation, independent  $X$  and  $Y$  implies

$$E[X|Y = y] = \sum_x xp_{X|Y}(x|y) = \sum_x xp_X(x) = E[X]$$

# Conditional Expectation

# Conditional Expectation

---



Conditional Distributions

Expectation



# Conditional expectation

---

Recall the the conditional PMF of  $X$  given  $Y = y$ :

$$p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

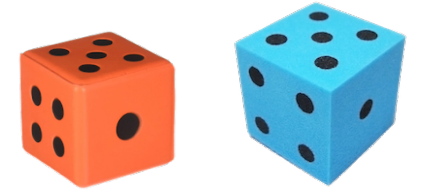
The **conditional expectation** of  $X$  given  $Y = y$  is

$$E[X|Y = y] = \sum_x xP(X = x|Y = y) = \sum_x xp_{X|Y}(x|y)$$

# It's been so long, our dice friends

$$E[X|Y = y] = \sum_x x p_{X|Y}(x|y)$$

- Roll two 6-sided dice.
- Let roll 1 be  $D_1$ , roll 2 be  $D_2$ .
- Let  $S = \text{value of } D_1 + D_2$ .



1. What is  $E[S|D_2 = 6]$ ?  $E[S|D_2 = 6] = \sum_{x=7}^{12} x P(S = x|D_2 = 6)$

$$= \left(\frac{1}{6}\right) (7 + 8 + 9 + 10 + 11 + 12)$$
$$= \frac{57}{6} = 9.5$$

Intuitively:  $6 + E[D_1] = 6 + 3.5 = 9.5$

Let's prove this!

# Properties of conditional expectation

---

## 1. LOTUS:

$$E[g(X)|Y = y] = \sum_x g(x)p_{X|Y}(x|y)$$

## 2. Sum of conditional expectation:

$$E\left[\sum_{i=1}^n X_i | Y = y\right] = \sum_{i=1}^n E[X_i | Y = y]$$

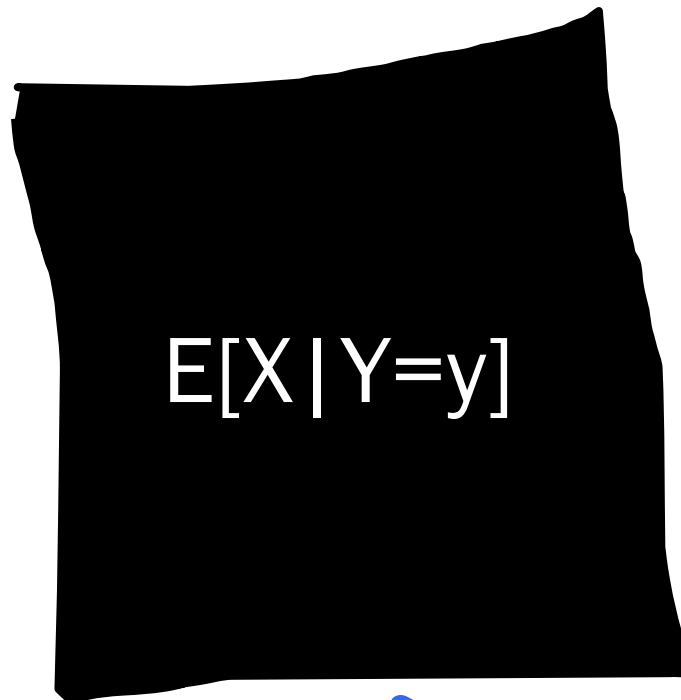
## 3. Law of total expectation

# Conditional Expectation Function

---

Define  $g(Y) = E[X | Y]$

This is just function of  $Y$



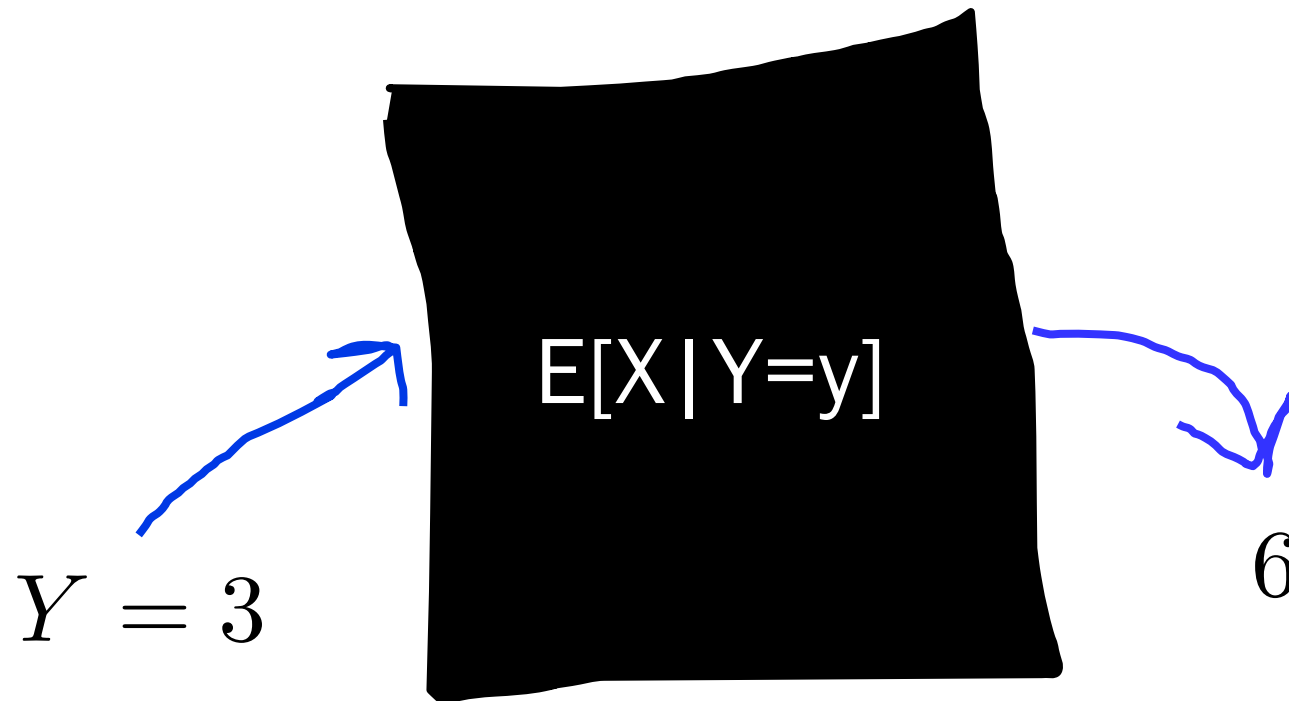
This is a function with  $Y$  as input

# Conditional Expectation Function

---

Define  $g(Y) = E[X | Y]$

This is just function of  $Y$





# Conditional Expectation Function

---

This is a number:

$$E[X]$$



This is a function of  $y$ :

$$E[X|Y = y]$$

$$E[X = 5]$$

Doesn't make sense. Take expectation of random variables, not events

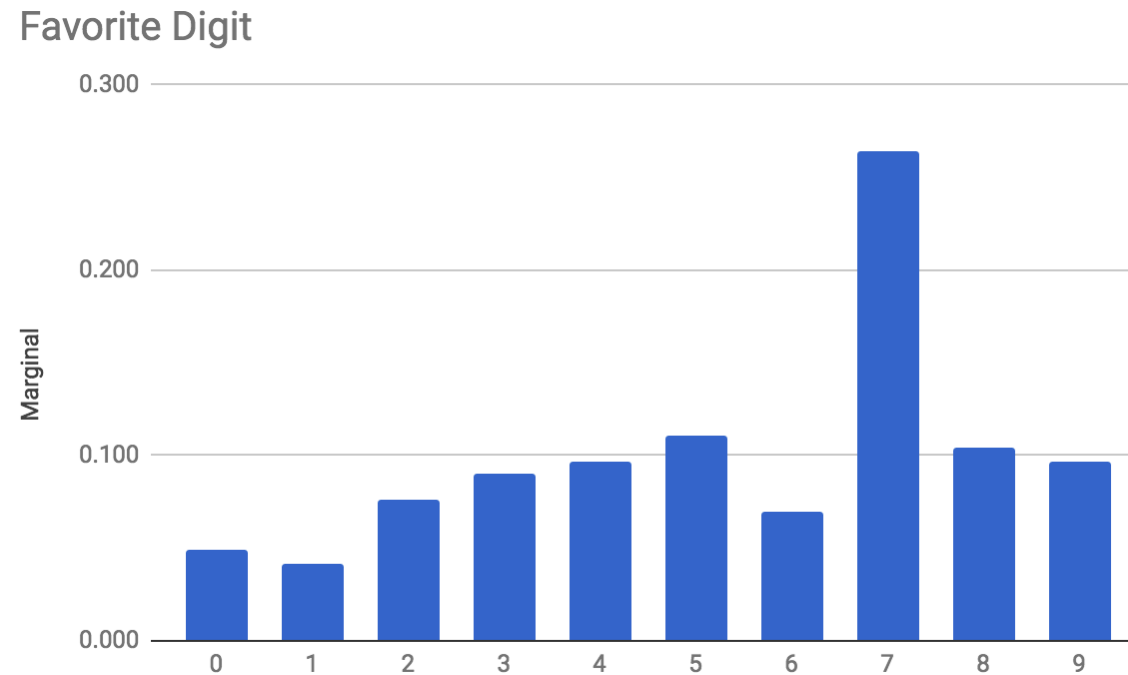
# Conditional Expectation Example

---

X = favorite number

Y = year in school

---



$$E[X] = 0 * 0.05 + \dots + 9 * 0.10 = 5.38$$

# Conditional Expectation Example

---

$X$  = favorite number

$Y$  = year in school

$E[X | Y]$  ?

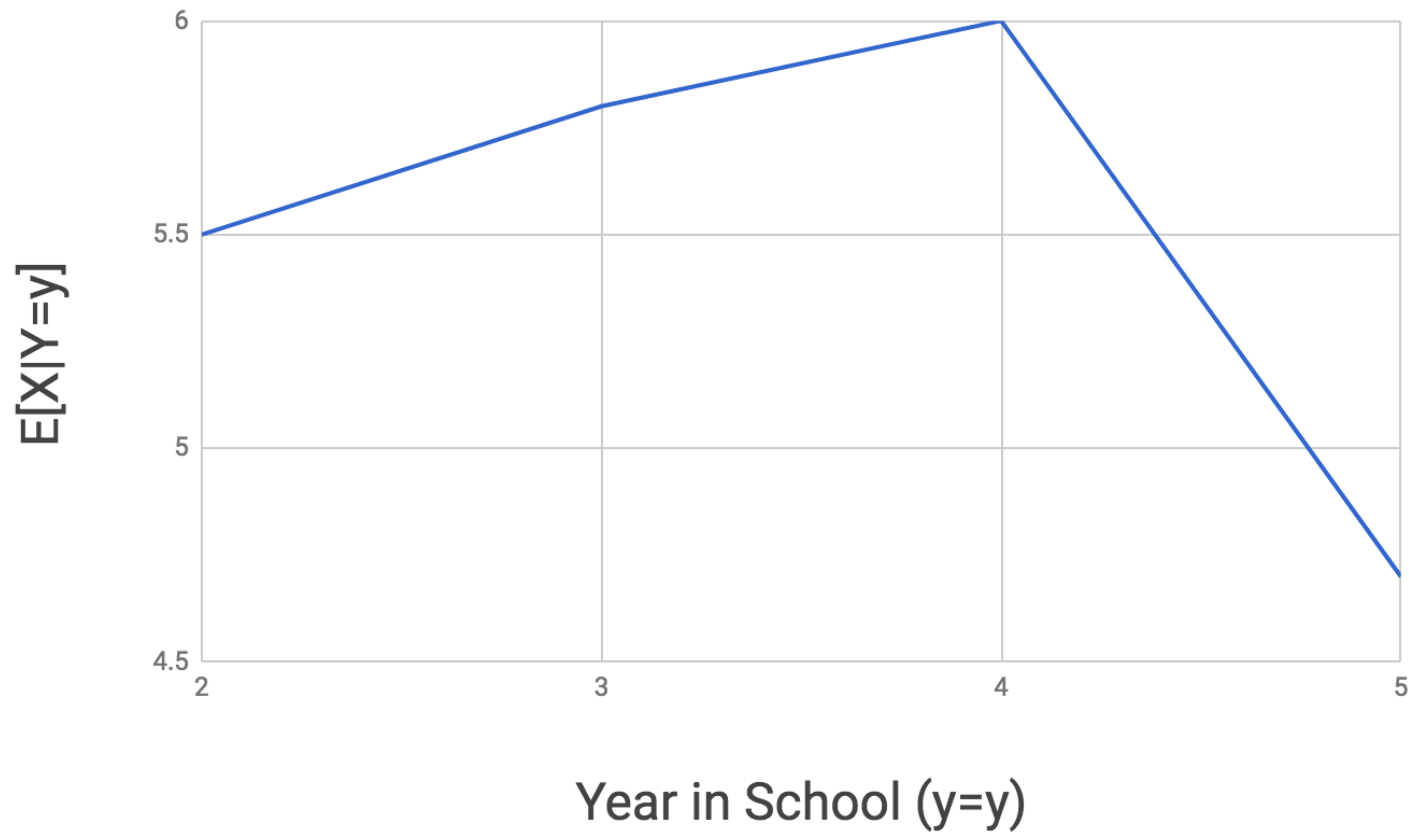
Year in school, $Y = y$	$E[X   Y = y]$
2	5.5
3	5.8
4	6.0
5	4.7

# Conditional Expectation Example

$X$  = favorite number

$Y$  = year in school

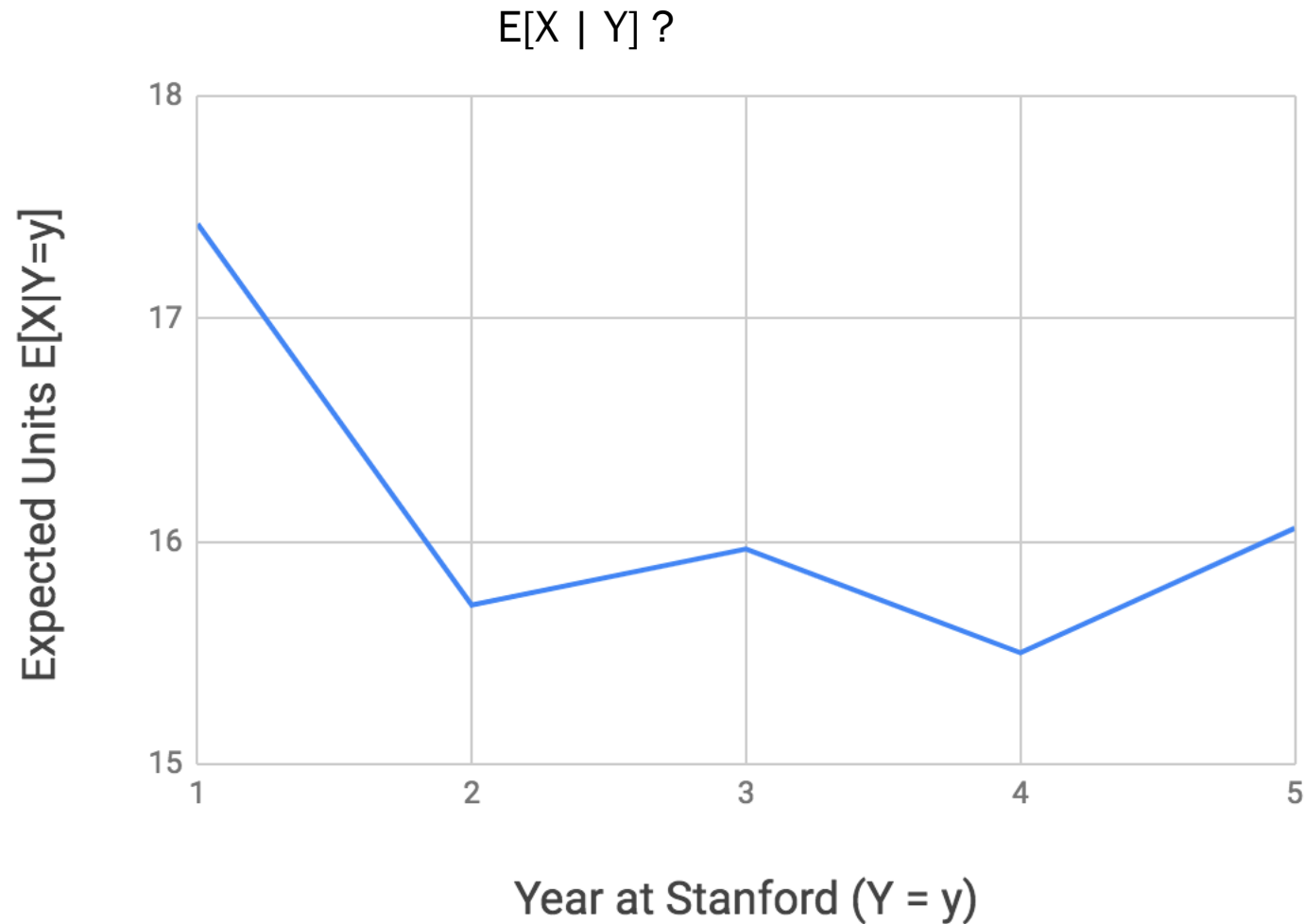
$E[X | Y]$  ?



# Conditional Expectation Example

$X$  = units in fall quarter

$Y$  = year in school

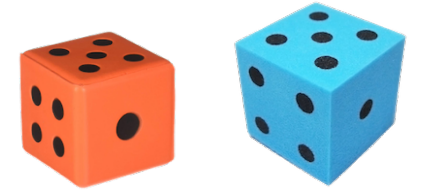




# It's been so long, our dice friends

$$E[X|Y = y] = \sum_x x p_{X|Y}(x|y)$$

- Roll two 6-sided dice.
- Let roll 1 be  $D_1$ , roll 2 be  $D_2$ .
- Let  $S = \text{value of } D_1 + D_2$ .



1. What is  $E[S|D_2 = 6]$ ?

$$\frac{57}{6} = 9.5$$

2. What is  $E[S|D_2]$ ?

- A. A function of  $S$
- B. A function of  $D_2$
- C. A number

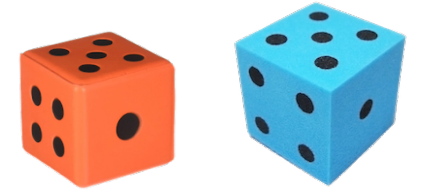
3. Give an expression for  $E[S|D_2]$ .



# It's been so long, our dice friends

$$E[X|Y = y] = \sum_x x p_{X|Y}(x|y)$$

- Roll two 6-sided dice.
- Let roll 1 be  $D_1$ , roll 2 be  $D_2$ .
- Let  $S = \text{value of } D_1 + D_2$ .



1. What is  $E[S|D_2 = 6]$ ?

$$\frac{57}{6} = 9.5$$

2. What is  $E[S|D_2]$ ?

- A. A function of  $S$
  - B.** A function of  $D_2$
  - C. A number
3. Give an expression for  $E[S|D_2]$ .

$$E[S|D_2 = d_2] = E[D_1 + d_2|D_2 = d_2]$$

$$= \sum_{d_1} (d_1 + d_2) P(D_1 = d_1 | D_2 = d_2)$$

$$= \sum_{d_1} d_1 P(D_1 = d_1) + d_2 \sum_{d_1} P(D_1 = d_1)$$

( $D_1 = d_1, D_2 = d_2$   
independent events)

$$= E[D_1] + d_2 = 3.5 + d_2$$

$$E[S|D_2] = 3.5 + D_2$$

# Law of Total Expectation

# Properties of conditional expectation

---

## 1. LOTUS:

$$E[g(X)|Y = y] = \sum_x g(x)p_{X|Y}(x|y)$$

## 2. Linearity of conditional expectation:

$$E\left[\sum_{i=1}^n X_i | Y = y\right] = \sum_{i=1}^n E[X_i | Y = y]$$

## 3. Law of total expectation:

$$E[X] = E[E[X|Y]] \quad \text{what?!}$$

# Law of Total Expectation

---



$E[X]$  can be calculated using  $E[X|Y]$

$$E[X] = \sum_y E[X|Y = y]P(Y = y)$$

# Proof of Law of Total Expectation

$$E[X] = E[E[X|Y]]$$

$$E[E[X|Y]] = \sum_y E[X|Y = y]P(Y = y)$$

$$g(Y) = E[X|Y]$$

$$= \sum_y \sum_x xP(X = x|Y = y)P(Y = y)$$

Def of  $E[X|Y]$

$$= \sum_y \sum_x xP(X = x, Y = y)$$

Chain rule!

$$= \sum_x \sum_y xP(X = x, Y = y)$$

I switch the order of the sums

$$= \sum_x x \sum_y P(X = x, Y = y)$$

Move that  $x$  outside the  $y$  sum

$$= \sum_x xP(X = x)$$

Marginalization

$$= E[X]$$

Def of  $E[X]$



# Another way to compute $E[X]$

$$E[X] = E[E[X|Y]]$$

$$E[E[X|Y]] = \sum_y P(Y = y)E[X|Y = y] = E[X]$$

If we only have a conditional PMF of  $X$  on some discrete variable  $Y$ , we can compute  $E[X]$  as follows:

1. Compute expectation of  $X$  given some value of  $Y = y$
2. Repeat step 1 for all values of  $Y$
3. Compute a weighted sum (where weights are  $P(Y = y)$ )

```
def recurse():  
    if (random.random() < 0.5):  
        return 3  
    else: return (2 + recurse())
```

Useful for analyzing recursive code!!

# Quick check

---

1.  $E[X]$
2.  $E[X, Y]$
3.  $E[X + Y]$
4.  $E[X|Y]$
5.  $E[X|Y = 6]$
6.  $E[X = 1]$
- 7.\*  $E[Y|X = x]$

- A. value
- B. one RV, function on  $Y$
- C. one RV, function on  $X$
- D. two RVs, function on  $X$  and  $Y$
- E. doesn't make sense

# Analyzing recursive code

$$E[X] = E[E[X|Y]] = \sum_y E[X|Y = y]P(Y = y)$$

```
def recurse():  
    # equally likely values 1,2,3  
    x = np.random.choice([1,2,3])  
    if (x == 1): return 3  
    elif (x == 2): return (5 + recurse())  
    else: return (7 + recurse())
```

Let  $Y$  = return value of `recurse()`.  
What is  $E[Y]$ ?

# Analyzing recursive code

$$E[X] = E[E[X|Y]] = \sum_y E[X|Y = y]P(Y = y)$$

```
def recurse():  
    # equally likely values 1,2,3  
    x = np.random.choice([1,2,3])  
    if (x == 1): return 3  
    elif (x == 2): return (5 + recurse())  
    else: return (7 + recurse())
```

Let  $Y$  = return value of `recurse()`.  
What is  $E[Y]$ ?

$$E[Y] = E[Y|X = 1]P(X = 1) + E[Y|X = 2]P(X = 2) + E[Y|X = 3]P(X = 3)$$



$$E[Y|X = 1] = 3$$

When  $X = 1$ , return 3.

# Analyzing recursive code

$$E[X] = E[E[X|Y]] = \sum_y E[X|Y = y]P(Y = y) \quad \text{If } Y \text{ discrete}$$

```
def recurse():  
    # equally likely values 1,2,3  
    x = np.random.choice([1,2,3])  
    if (x == 1): return 3  
    elif (x == 2): return (5 + recurse())  
    else: return (7 + recurse())
```

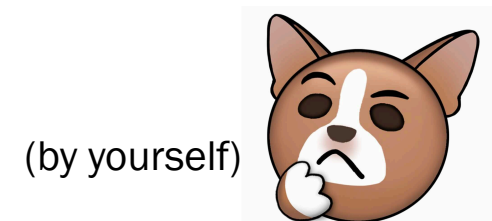
Let  $Y$  = return value of `recurse()`.  
What is  $E[Y]$ ?

$$E[Y] = E[Y|X = 1]P(X = 1) + E[Y|X = 2]P(X = 2) + E[Y|X = 3]P(X = 3)$$

$$E[Y|X = 1] = 3$$

What is  $E[Y|X = 2]$ ?

- A.  $E[5] + Y$
- B.  $E[Y + 5] = 5 + E[Y]$
- C.  $5 + E[Y|X = 2]$



# Analyzing recursive code


$$E[X] = E[E[X|Y]] = \sum_y E[X|Y = y]P(Y = y) \quad \text{If } Y \text{ discrete}$$

```
def recurse():  
    # equally likely values 1,2,3  
    x = np.random.choice([1,2,3])  
    if (x == 1): return 3  
    elif (x == 2): return (5 + recurse())  
    else: return (7 + recurse())
```

Let  $Y =$  return value of `recurse()`.  
What is  $E[Y]$ ?

$$E[Y] = E[Y|X = 1]P(X = 1) + E[Y|X = 2]P(X = 2) + E[Y|X = 3]P(X = 3)$$

  
 $E[Y|X = 1] = 3$

  
When  $X = 2$ , return 5 +  
a future return value of `recurse()`.

What is  $E[Y|X = 2]$ ?

- A.  $E[5] + Y$
- B.  $E[Y + 5] = 5 + E[Y]$
- C.  $5 + E[Y|X = 2]$

# Analyzing recursive code

$$E[X] = E[E[X|Y]] = \sum_y E[X|Y = y]P(Y = y) \quad \text{If } Y \text{ discrete}$$

```
def recurse():  
    # equally likely values 1,2,3  
    x = np.random.choice([1,2,3])  
    if (x == 1): return 3  
    elif (x == 2): return (5 + recurse())  
    else: return (7 + recurse())
```

Let  $Y$  = return value of `recurse()`.  
What is  $E[Y]$ ?

$$E[Y] = E[Y|X = 1]P(X = 1) + E[Y|X = 2]P(X = 2) + E[Y|X = 3]P(X = 3)$$

$$E[Y|X = 1] = 3$$

$$E[Y|X = 2] = E[5 + Y]$$

When  $X = 3$ , return  
7 + a future return value  
of `recurse()`.

$$E[Y|X = 3] = E[7 + Y]$$



# Analyzing recursive code

$$E[X] = E[E[X|Y]] = \sum_y E[X|Y = y]P(Y = y) \quad \text{If } Y \text{ discrete}$$

```
def recurse():  
    # equally likely values 1,2,3  
    x = np.random.choice([1,2,3])  
    if (x == 1): return 3  
    elif (x == 2): return (5 + recurse())  
    else: return (7 + recurse())
```

Let  $Y$  = return value of `recurse()`.  
What is  $E[Y]$ ?

$$E[Y] = E[Y|X = 1]P(X = 1) + E[Y|X = 2]P(X = 2) + E[Y|X = 3]P(X = 3)$$

$$E[Y|X = 1] = 3 \quad E[Y|X = 2] = E[5 + Y] \quad E[Y|X = 3] = E[7 + Y]$$

$$E[Y] = 3(1/3) + (5 + E[Y])(1/3) + (7 + E[Y])(1/3)$$

$$E[Y] = (1/3)(15 + 2E[Y]) = 5 + (2/3)E[Y]$$

$$E[Y] = 15$$

On your own: What is  $\text{Var}(Y)$ ?

# Hiring and Engineer

```
DEFINE JOBINTERVIEWQUICKSORT(LIST):  
  OK SO YOU CHOOSE A PIVOT  
  THEN DIVIDE THE LIST IN HALF  
  FOR EACH HALF:  
    CHECK TO SEE IF IT'S SORTED  
    NO, WAIT, IT DOESN'T MATTER  
    COMPARE EACH ELEMENT TO THE PIVOT  
    THE BIGGER ONES GO IN A NEW LIST  
    THE EQUAL ONES GO INTO, UH  
    THE SECOND LIST FROM BEFORE  
  HANG ON, LET ME NAME THE LISTS  
  THIS IS LIST A  
  THE NEW ONE IS LIST B  
  PUT THE BIG ONES INTO LIST B  
  NOW TAKE THE SECOND LIST  
  CALL IT LIST, UH, A2  
  WHICH ONE WAS THE PIVOT IN?  
  SCRATCH ALL THAT  
  IT JUST RECURSIVELY CALLS ITSELF  
  UNTIL BOTH LISTS ARE EMPTY  
  RIGHT?  
  NOT EMPTY, BUT YOU KNOW WHAT I MEAN  
  AM I ALLOWED TO USE THE STANDARD LIBRARIES?
```

Your company has one job opening for a software engineer.

You have  $n$  candidates. But you have to say yes/no **immediately** after each interview!

Proposed algorithm: reject the first  $k$  and accept the next one who is better than all of them.

What's the best value of  $k$ ?

# Hiring and Engineer

$n$  candidates, must say yes/no **immediately** after each interview.  
Reject the first  $k$ , accept the next who is better than all of them.  
What's the best value of  $k$ ?

**B**: event that you hire the best engineer

**X**: position of the best engineer on the interview schedule

---

What is the  $P(B|X = i)$ ?

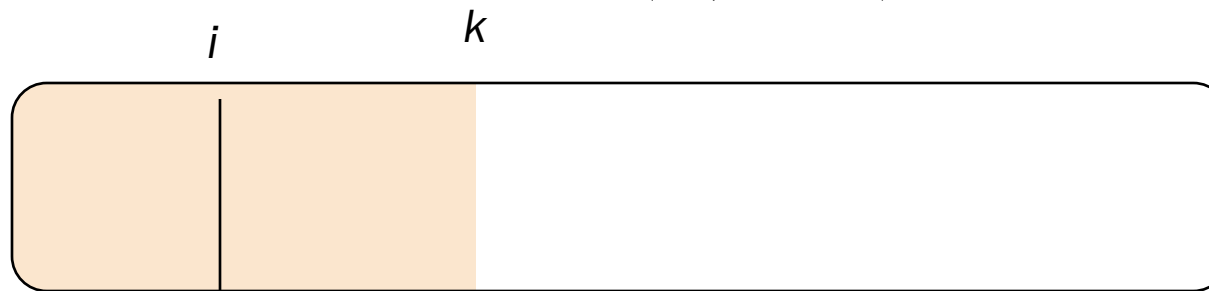
# Hiring and Engineer

$n$  candidates, must say yes/no **immediately** after each interview.  
Reject the first  $k$ , accept the next who is better than all of them.  
What's the best value of  $k$ ?

**B**: event that you hire the best engineer

**X**: position of the best engineer on the interview schedule

What is the  $P(B|X = i)$ ?



# Hiring and Engineer

$n$  candidates, must say yes/no **immediately** after each interview.  
Reject the first  $k$ , accept the next who is better than all of them.  
What's the best value of  $k$ ?

**B**: event that you hire the best engineer

**X**: position of the best engineer on the interview schedule

What is the  $P(B|X = i)$ ?



Hint: where is the best  
among the first  $i - 1$   
candidates?



# Hiring and Engineer

$n$  candidates, must say yes/no **immediately** after each interview.  
Reject the first  $k$ , accept the next who is better than all of them.  
What's the best value of  $k$ ?

**B**: event that you hire the best engineer

**X**: position of the best engineer on the interview schedule

What is the  $P(B|X = i)$ ?

$k$

Here?



Hint: where is the best  
among the first  $i - 1$   
candidates?



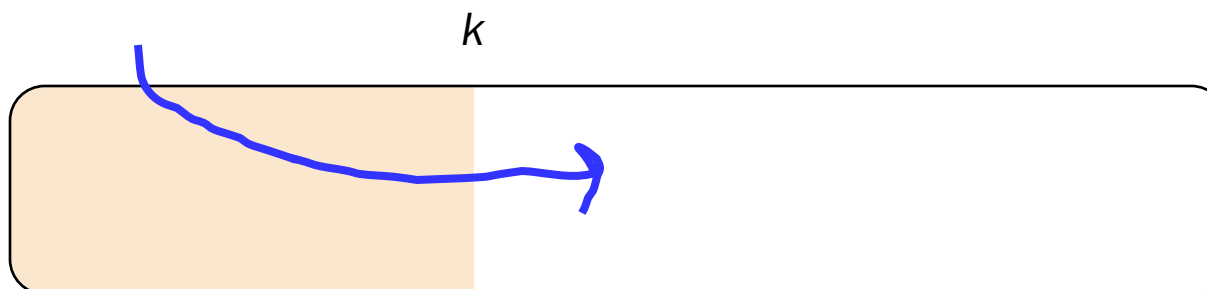
# Hiring and Engineer

$n$  candidates, must say yes/no **immediately** after each interview.  
Reject the first  $k$ , accept the next who is better than all of them.  
What's the best value of  $k$ ?

**B**: event that you hire the best engineer

**X**: position of the best engineer on the interview schedule

Here? What is the  $P(B|X = i)$ ?



Hint: where is the best  
among the first  $i - 1$   
candidates?





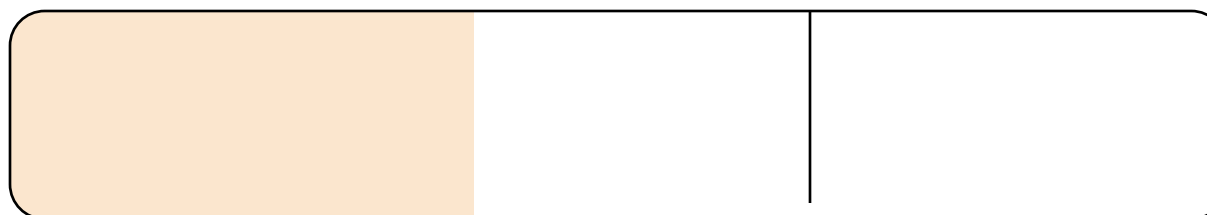
# Hiring and Engineer

$n$  candidates, must say yes/no **immediately** after each interview.  
Reject the first  $k$ , accept the next who is better than all of them.  
What's the best value of  $k$ ?

**B**: event that you hire the best engineer

**X**: position of the best engineer on the interview schedule

$$P(B|X = i) = \frac{k}{i-1} \text{ if } i > k$$



Hint: where is the best  
among the first  $i - 1$   
candidates?



# Hiring and Engineer

$n$  candidates, must say yes/no **immediately** after each interview.  
Reject the first  $k$ , accept the next who is better than all of them.  
What's the best value of  $k$ ?

**B**: event that you hire the best engineer

**X**: position of the best engineer on the interview schedule

$$P_k(B) = \sum_{i=1}^n P_k(B|X = i)P(X = i)$$

By the law of total expectation

$$= \frac{1}{n} \sum_{i=1}^n P_k(B|X = i)$$

$$= \frac{1}{n} \sum_{i=k+1}^n \frac{k}{i-1}$$

since we know  $P_k(\text{Best}|X = i)$

$$\approx \frac{1}{n} \int_{i=k+1}^n \frac{k}{i-1} di$$

By Riemann Sum approximation

$$= \frac{k}{n} \ln(i-1) \Big|_{k+1}^n = \frac{k}{n} \ln \frac{n-1}{k} \approx \frac{k}{n} \ln \frac{n}{k}$$

# Hiring and Engineer

$n$  candidates, must say yes/no **immediately** after each interview.  
Reject the first  $k$ , accept the next who is better than all of them.  
What's the best value of  $k$ ?

**B**: event that you hire the best engineer

**X**: position of the best engineer on the interview schedule

$$P_k(B) = \sum_{i=1}^n P_k(B|X = i)P(X = i)$$
$$\approx \frac{k}{n} \ln \frac{n}{k}$$

By the law of total expectation

Fun fact. Optimized when:

$$k = \frac{n}{e}$$