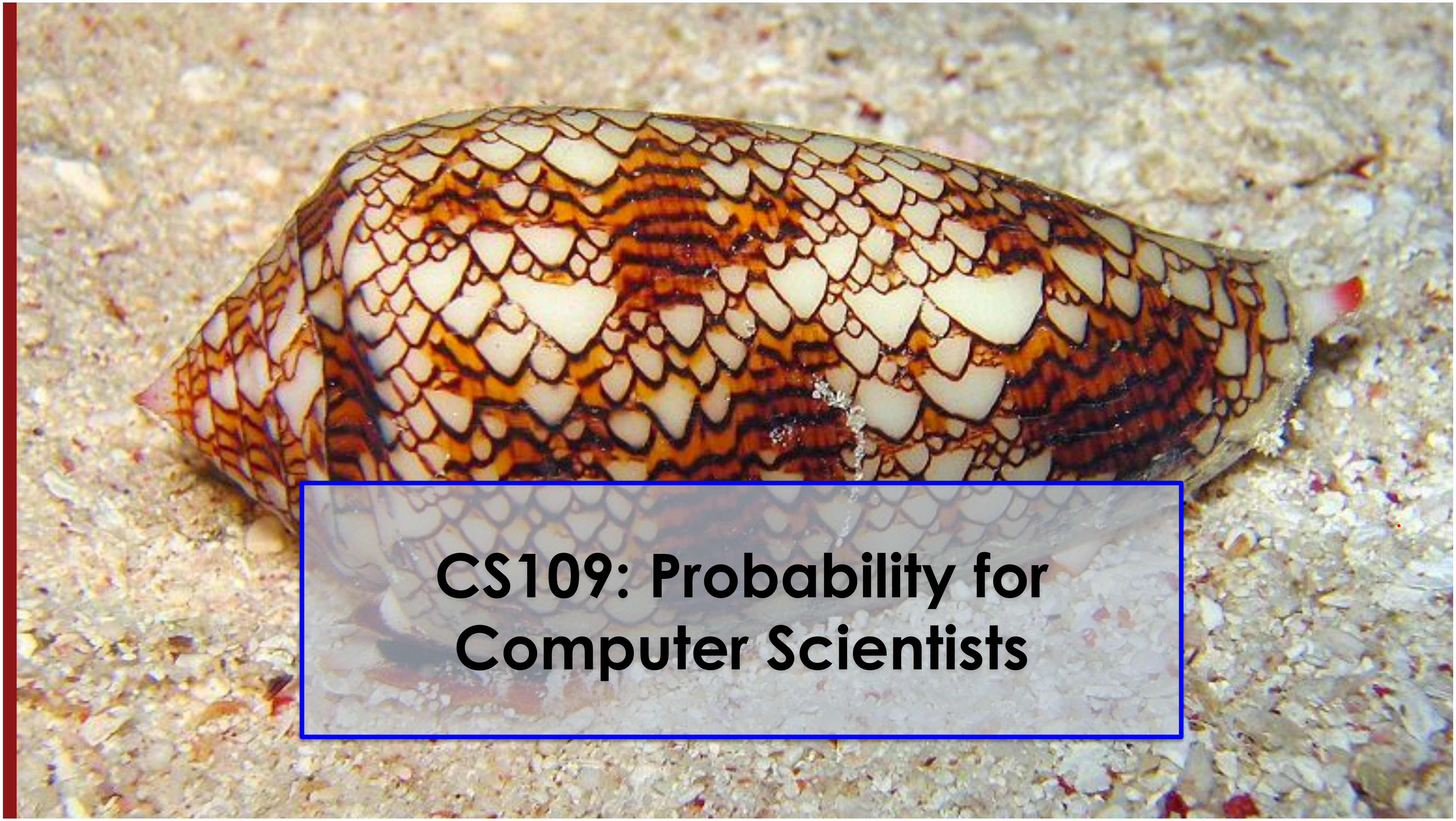
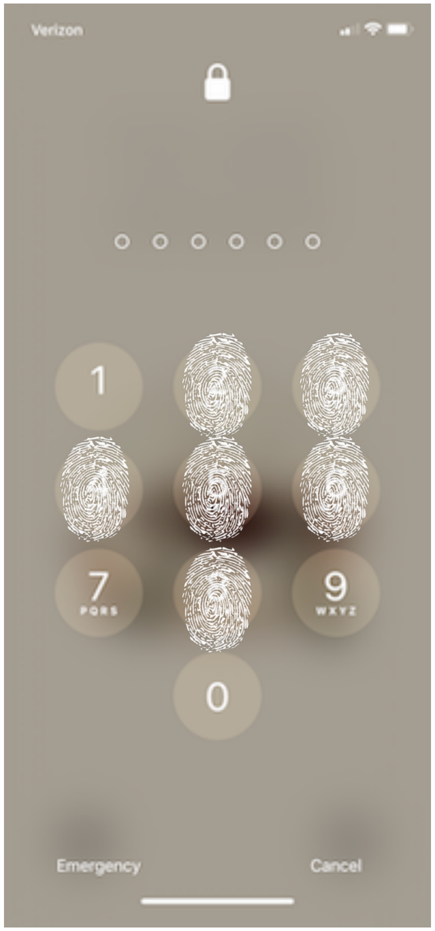


CS109: Probability for Computer Scientists

Permutations I

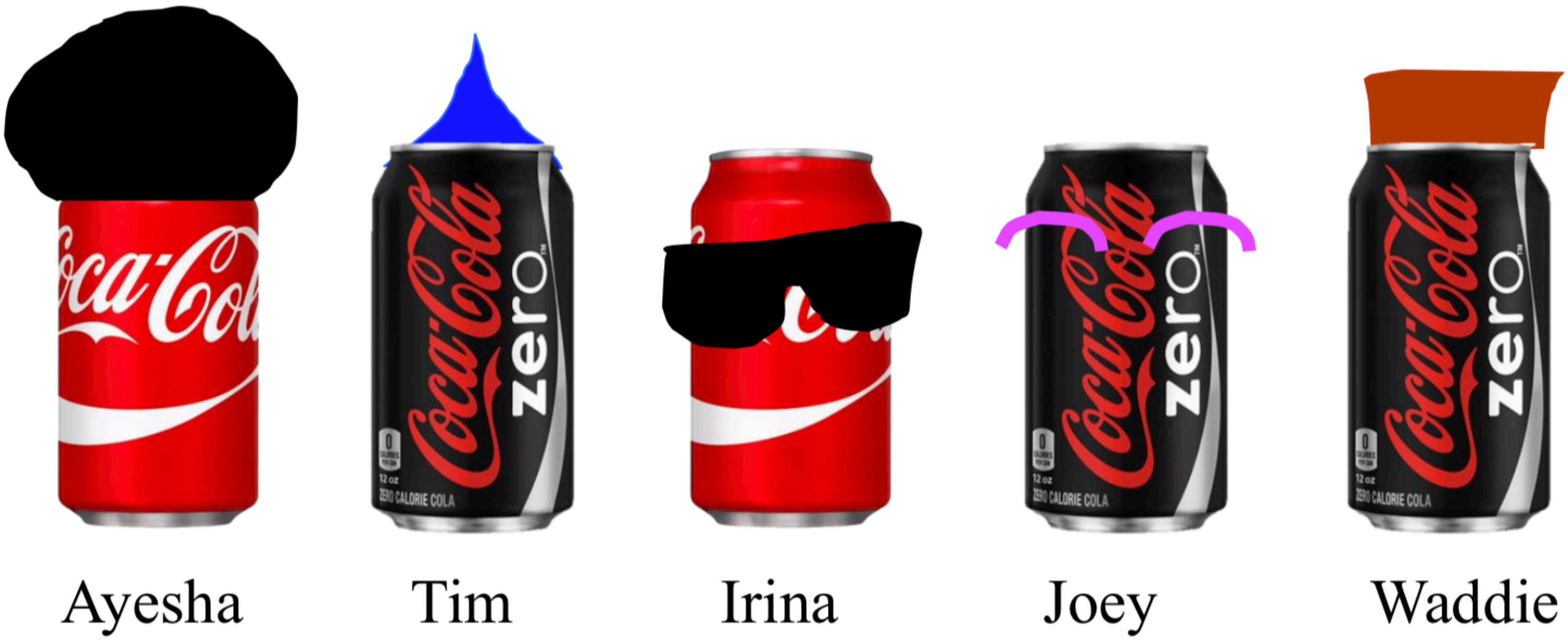
Unique 6-digit passcodes with six smudges



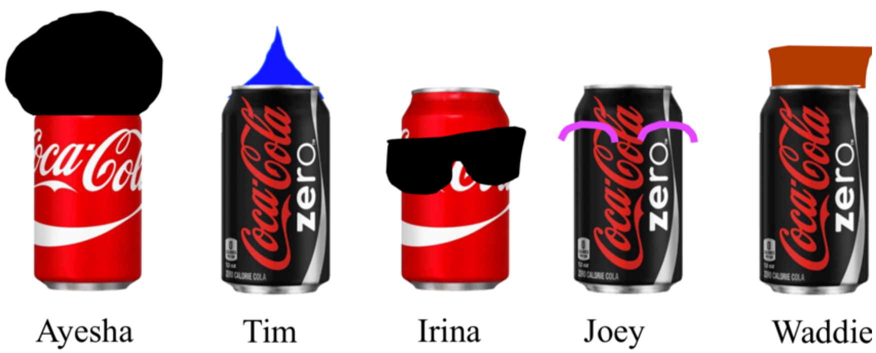
How many unique 6-digit passcodes are possible if a phone password uses each of **six** distinct numbers?

Sort n distinct objects

Sort n distinct objects



Sort n distinct objects



1^{st} 2^{nd} 3^{rd} 4^{th} 5^{th}

Steps:

1. Choose 1^{st} can 5 options
2. Choose 2^{nd} can 4 options

…

5. Choose 5^{th} can 1 option

$$\text{Total} = 5 \times 4 \times 3 \times 2 \times 1$$
$$= 120$$

Permutations

- A permutation is an ordered arrangement of objects.

- The number of unique orderings (permutations) of n distinct objects is

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

Unique 6-digit passcodes with six smudges

How many unique 6-digit passcodes are possible if a phone password uses each of **six** distinct numbers?

$$\text{Total} = 6! = 720 \text{ passcodes}$$

How many unique passcodes are possible if a phone password is some ordered subset of any 6 digits?

$$\begin{aligned}\text{Total} &= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \\ &= \frac{10!}{4!} = 151200 \text{ passcodes}\end{aligned}$$

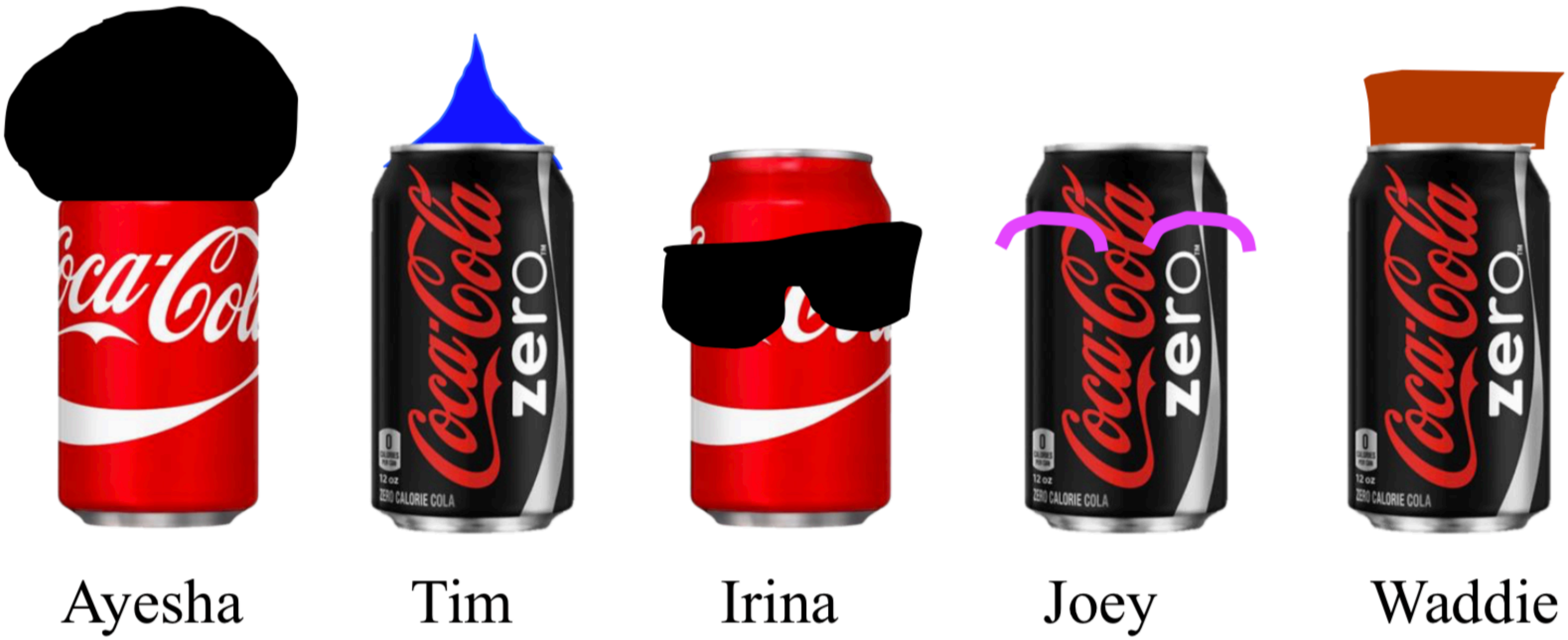
Permutations II

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable)
- Choose k objects (combinations)
- Put objects in r buckets

Sort n distinct objects



# of permutations =

Summary of Combinatorics

Counting tasks on n objects

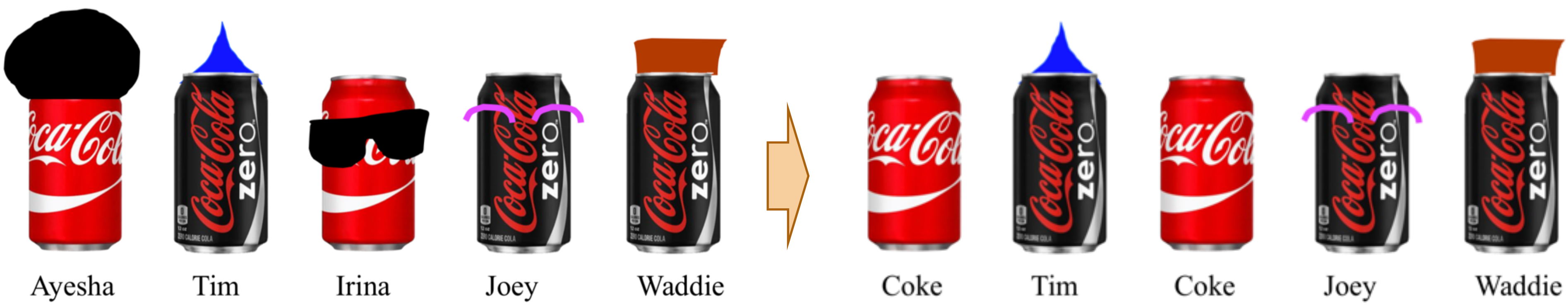
- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct
- Choose k objects (combinations)
- Put objects in r buckets

Sort semi-distinct objects

Order n distinct objects	$n!$

All distinct

Some indistinct



Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

permutations of distinct objects $=$ permutations considering some objects are indistinct $\times$ Permutations of just the indistinct objects

Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

$$\frac{\text{permutations of distinct objects}}{\text{Permutations of just the indistinct objects}} = \text{permutations considering some objects are indistinct}$$

General approach to counting permutations

When there are n objects such that

n_1 are the same (indistinguishable or indistinct), and

n_2 are the same, and

…

n_r are the same,

The number of unique orderings (permutations) is

$$\frac{n!}{n_1!\, n_2! \cdots n_r!}.$$

For each group of indistinct objects,
Divide by the overcounted permutations.

Sort semi-distinct objects

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many permutations?

$$\frac{5!}{2!\,3!}$$

Coke Coke0 Coke Coke0 Coke0

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $$\frac{n!}{n_1!\, n_2! \cdots n_r!}$$
- Choose k objects (combinations)
- Put objects in r buckets

Strings

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many letter orderings are possible for the following strings?

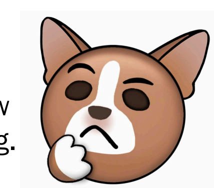
$$\frac{10!}{3!\,1!\,1!\,1!}$$

1. $\overbrace{\text{CHRISDADDY}}^{10}$

$$\frac{11!}{1!\,4!\,4!\,2!}$$

2. MISSISSIPPI

   1 4 4 2



This is Jerry's dog, Doris. She puts her little Doris paw up to her chin when she's thinking.

Strings

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many letter orderings are possible for the following strings?

1. CHRISDADDY $= \frac{10!}{3!} = 604{,}800$

2. MISSISSIPPI $= \frac{11!}{1!4!4!2!} = 34{,}650$

Unique 6-digit passcodes with six smudges

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many unique 6-digit passcodes are possible if a phone password uses each of **six** distinct numbers?

$$\text{Total} = 6!$$
$$= 720 \text{ passcodes}$$

Unique 6-digit passcodes with five smudges

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many unique 6-digit passcodes are possible if a phone password uses each of **five** distinct numbers?

Steps:

1. Choose digit to repeat — 5 outcomes
2. Create passcode — (sort 6 digits: 4 distinct, 2 indistinct)

Total $= 5 \times \frac{6!}{2!}$

$= 1{,}800$ passcodes

Combinations I

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$
- Choose k objects (combinations)
  - Distinct
- Put objects in r buckets

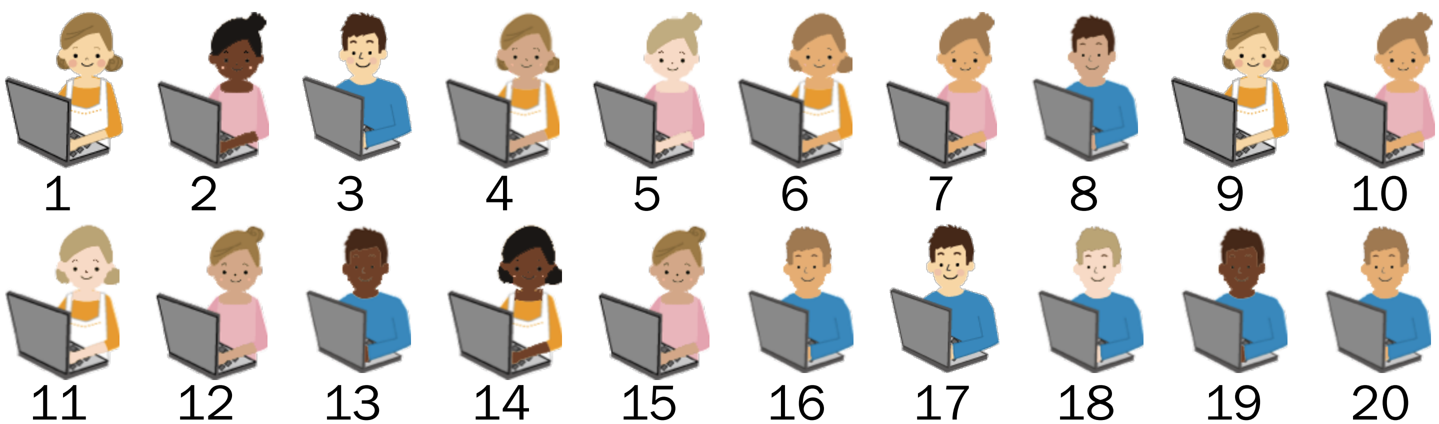
Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

Consider the following generative process…

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

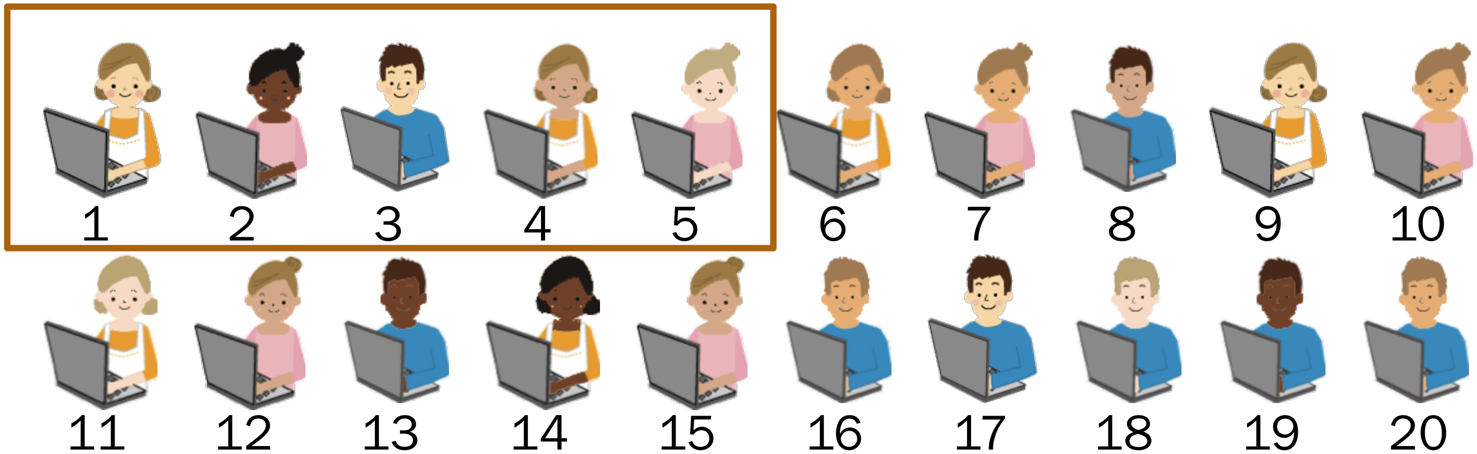


1. n people get in line

$n!$ ways

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

Combinations with cake

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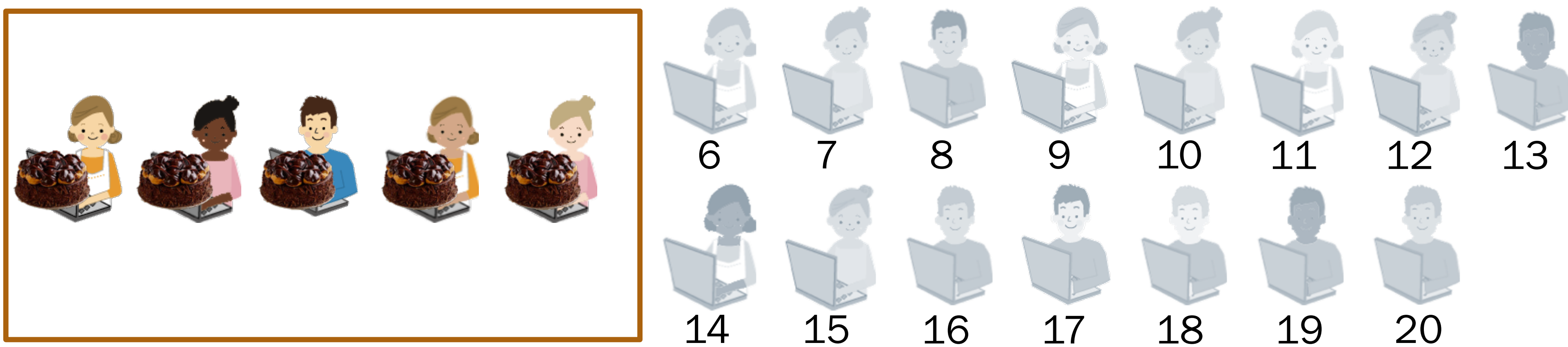

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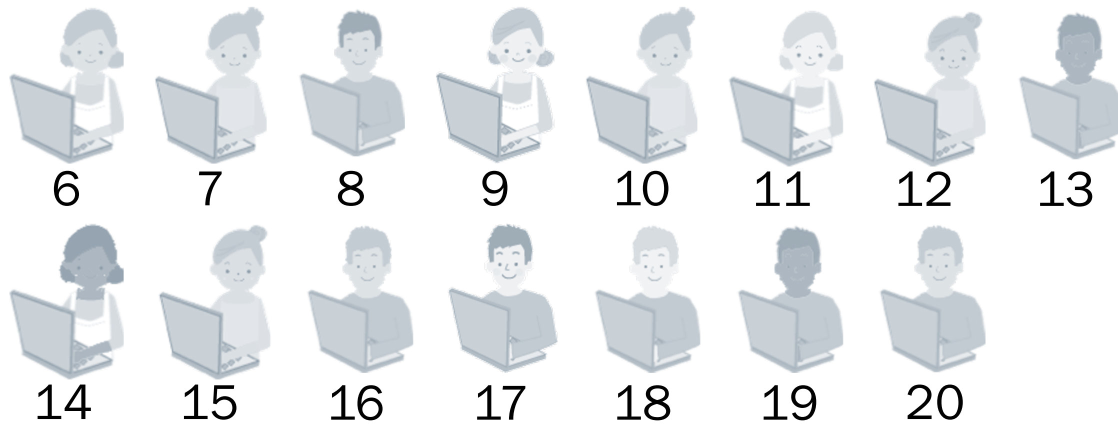

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

Combinations with cake

There are $n = 20$ people.
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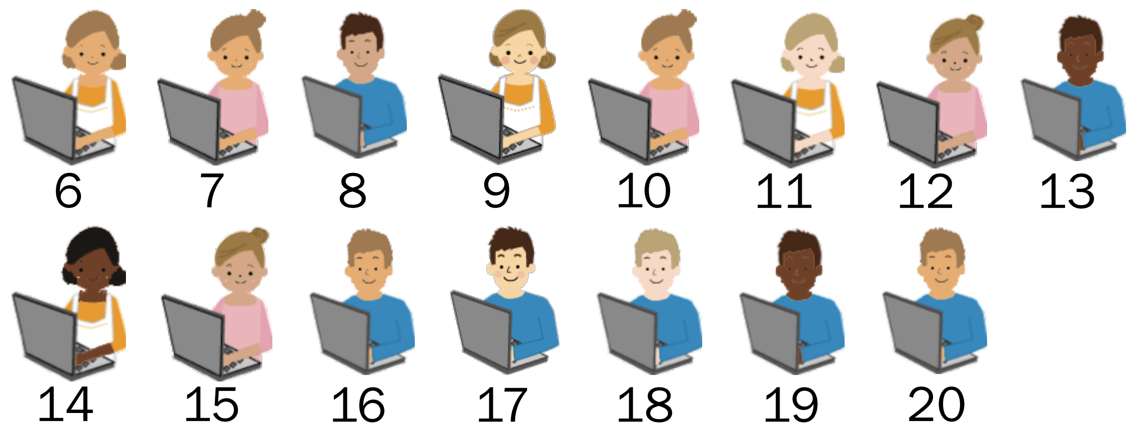


1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

   $(n-k)!$ different permutations lead to the same mingle

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k!(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!}$$

1. Order n distinct objects
2. Take first k as chosen
3. Overcounted: any ordering of chosen group is same choice
4. Overcounted: any ordering of unchosen group is same choice

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!} = \binom{n}{k}$$

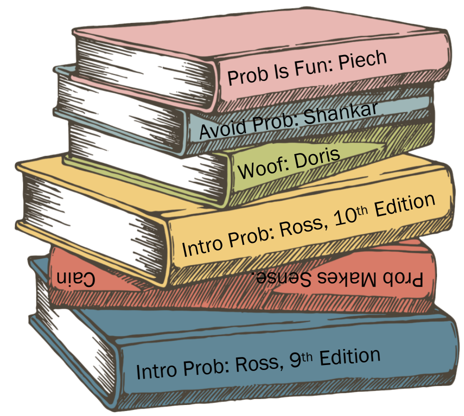
Binomial coefficient

Fun Fact: $\boxed{\binom{n}{k} = \binom{n}{n-k}}$

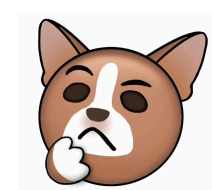
Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

How many ways are there to choose 3 books from a set of 6 distinct books?



$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$



Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books? $\binom{6}{3} = \frac{6!}{3!\,3!} = 20$ ways

2. What if we do not want to read both the 9th and 10th edition of Ross?

A. $\binom{6}{3} - \binom{6}{2} = 5$ ways

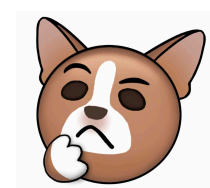
B. $\frac{6!}{3!3!2!} = 10$

C. $2 \cdot \binom{4}{2} + \binom{4}{3} = 16$

D. $\binom{6}{3} - \binom{4}{1} = 16$

E. Both C and D

F. Something else



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2. What if we do not want to read both the 9th and 10th edition of Ross?

Strategy 1: Sum Rule

neither: $\binom{4}{3}$

9th: $1 \cdot \binom{4}{2}$

10th: $1 \cdot \binom{4}{2}$

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books? $\binom{6}{3} = \frac{6!}{3!\,3!} = 20$ ways

2. What if we do not want to read both the 9th and 10th edition of Ross?

Strategy 2: "Forbidden method" (unofficial name)

$$\binom{6}{3} - \underbrace{\binom{4}{1}}_{\text{forbidden}} = 16$$

Forbidden method: It is sometimes easier to exclude invalid cases than to include cases.

Combinations II

Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\, n_2! \cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups
- **Put objects in r buckets**

General approach to combinations

The number of ways to choose r groups of n distinct objects such that

For all $i = 1, \ldots, r$, group i has size n_i, and

$\sum_{i=1}^{r} n_i = n$ (all objects are assigned), is

$$\frac{n!}{n_1!\, n_2! \cdots n_r!} = \binom{n}{n_1, n_2, \cdots, n_r}$$

Multinomial coefficient

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	6
B	4
C	3

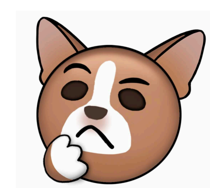
A. $\binom{13}{6,4,3} = 60{,}060$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

C. $6 \cdot 1001 \cdot 10 = 60{,}060$

D. A and B

E. All of the above



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How many different divisions are possible?

A. $\binom{13}{6,4,3} = 60{,}060 = \frac{13!}{6!\,4!\,3!}$

Strategy: Combinations into 3 groups

Group 1 (datacenter A): $n_1 = 6$

Group 2 (datacenter B): $n_2 = 4$

Group 3 (datacenter C): $n_3 = 3$

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

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Group 3 (datacenter C): $n_3 = 3$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

$\frac{3!}{3!\,0!}$

Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \dots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

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Group 1 (datacenter A): $n_1 = 6$

Group 2 (datacenter B): $n_2 = 4$

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B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

Your approach will determine if you use binomial/multinomial coefficients or factorials.

Buckets and The Divider Method

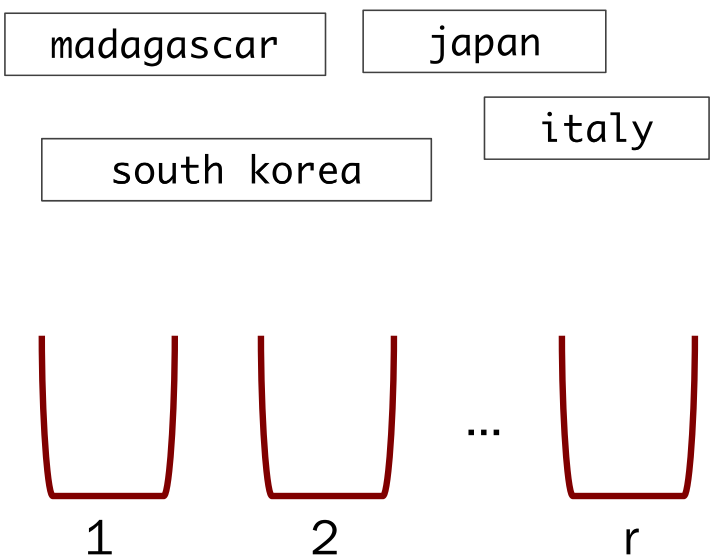
Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\, n_2! \cdots n_r!}$
- Choose k objects (combinations)
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- Put objects in r buckets
  - Distinct
  - Indistinct

~~Balls and urns~~ Hash tables and distinct strings

How many ways are there to hash n distinct strings to r buckets?



Steps:

1. Bucket 1st string r
2. Bucket 2nd string r

...

n. Bucket n^{th} string r

n

r^n outcomes

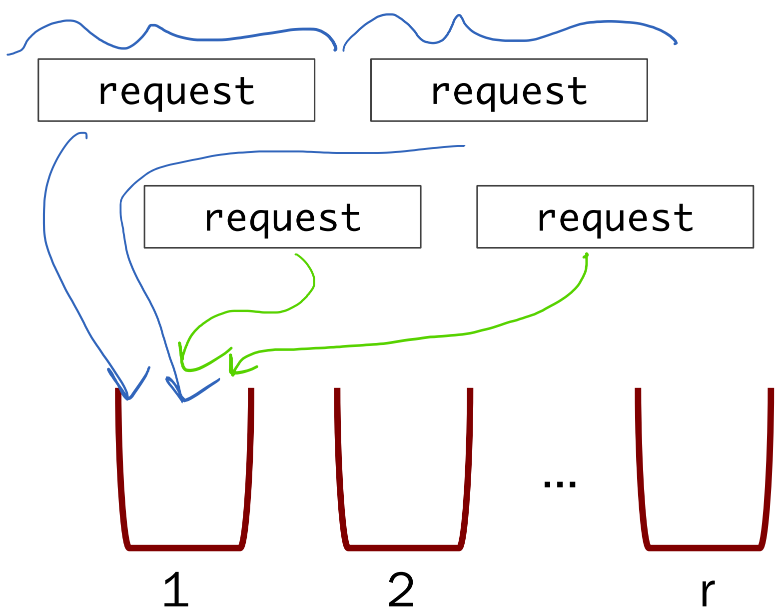
Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\, n_2! \cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct: r^n
  - Indistinct

Servers and indistinct requests

How many ways are there to distribute n indistinct web requests to r servers?



Goal

Server 1 has x_1 requests,
Server 2 has x_2 requests,
…
Server r has x_r requests

constraint: $\sum_{i=1}^{r} x_i = n$

Simple example: $n = 3$ requests and $r = 2$ servers

RRR|

RR|R

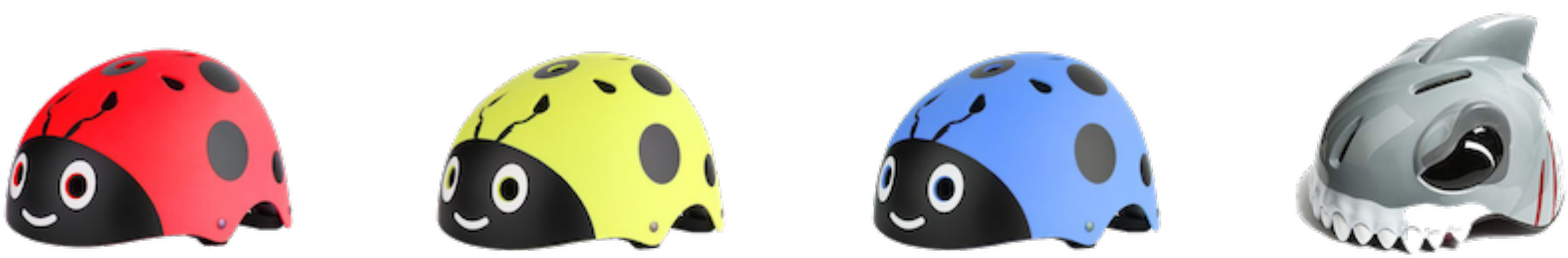
R|RR

|RRR

4 ways

Bicycle helmet sales

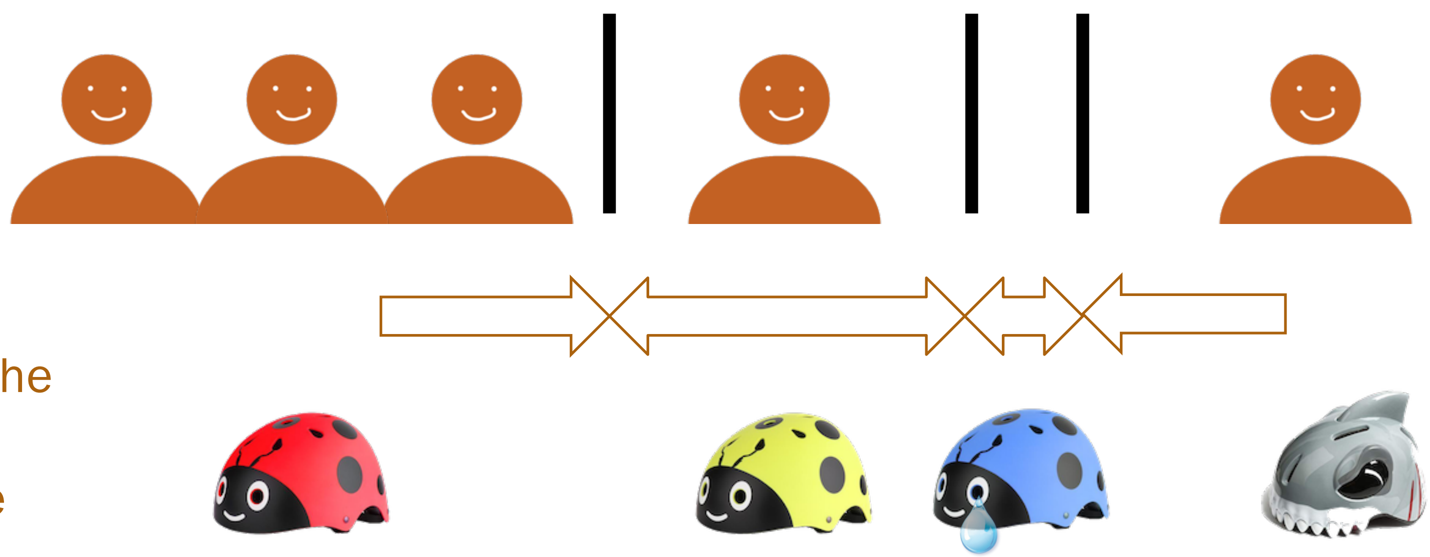
How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?



Bicycle helmet sales

1 possible assignment outcome:

Goal Order n indistinct objects and $r-1$ indistinct dividers.



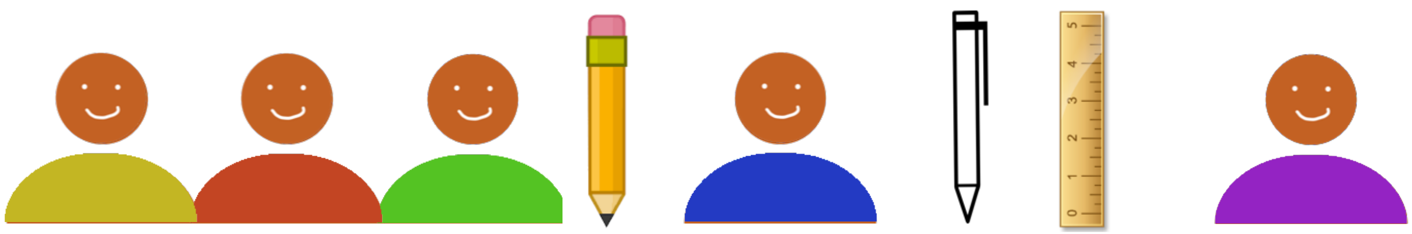
Consider the following generative process…

The divider method: A generative proof

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

Goal Order n indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct



The divider method: A generative proof

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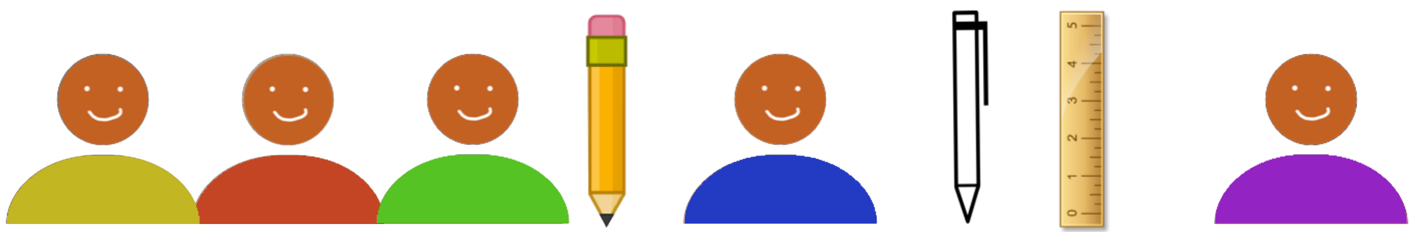

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

The divider method: A generative proof

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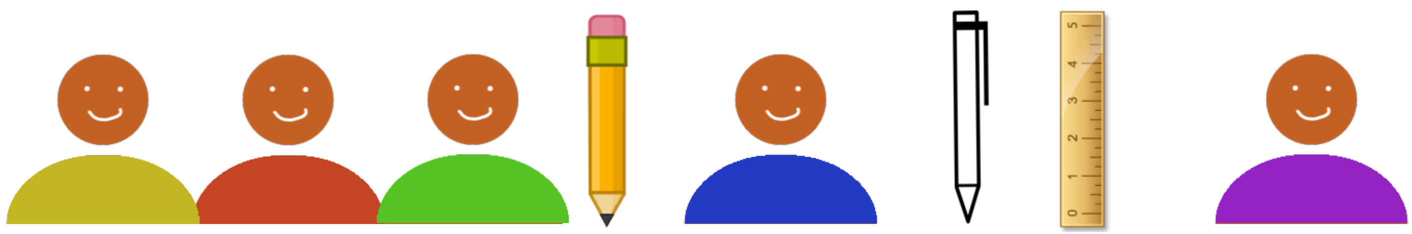

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

The divider method: A generative proof

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

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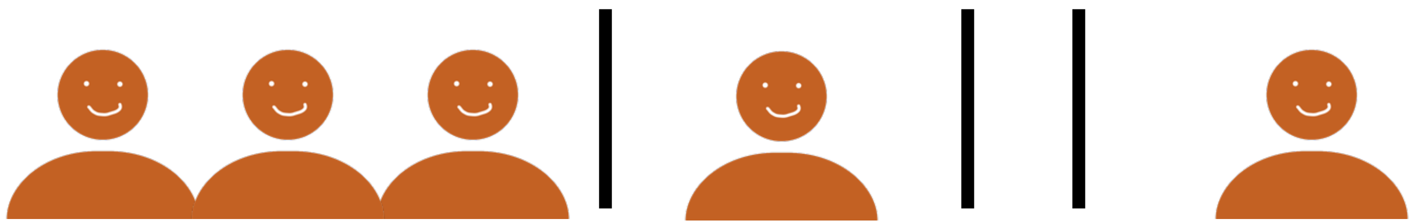


1. Order n distinct objects and $r-1$ distinct dividers

$$(n+r-1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

3. Make $r-1$ dividers indistinct

$$\frac{1}{(r-1)!}$$

The divider method

The number of ways to distribute n indistinct objects into r buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that
n are indistinct objects, and
$r - 1$ are indistinct dividers:

$$\text{Total} = \underbrace{(n + r - 1)!}\times \frac{1}{n!} \times \frac{1}{(r-1)!}$$

$$= \binom{n + r - 1}{r - 1} \text{ outcomes}$$

Integer solutions to equations

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

How many integer solutions are there to the following equation:

$$x_1 + x_2 + \cdots + x_r = n,$$

where for all i, x_i is an integer such that $0 \leq x_i \leq n$?

$$\binom{n+r-1}{r-1}$$

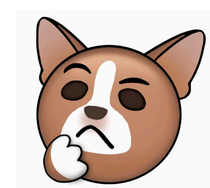
Positive integer equations can be solved with the divider method.

Venture capitalists

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have $10 million to invest in 4 companies (in $1 million increments).

1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1?
3. What if you don't have to invest all your money?



Venture capitalists. #1

Divider method (
n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?

$\binom{10+4-1}{4-1}$

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

$$x_i \geq 0$$

Solve

C_1	C_2	C_3	C_4
10	0	0	0
5	5	0	0
3	4	2	1

Venture capitalists. #2

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

$$x_1 \geq 3, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$$

$x_1 + x_2 + x_3 + x_4 = 7$

Solve

$\binom{7+4-1}{4-1}$

$= \binom{10}{3}$

Venture capitalists. #3

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?
3. What if you don't have to invest all your money?

Set up

$$x_1 + x_2 + x_3 + x_4 \leq 10$$

x_i: amount invested in company i

$$x_i \geq 0$$

Solve

$$\binom{10+5-1}{5-1}$$

$$\underbrace{x_1 + x_2 + x_3 + x_4}_{} + x_5 = 10$$

Summary of Combinatorics

Counting tasks on n objects

Sort objects (permutations)

Distinct (distinguishable)

Some distinct

Choose k objects (combinations)

Distinct

1 group

r groups

Put objects in r buckets

Distinct

Indistinct

- Determine if objects are distinct
- Use Product Rule if several steps
- Use Inclusion-Exclusion if different cases