02: Combinatorics

Jerry Cain
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Quick slide reference

Today's discussion thread: https://edstem.org/us/courses/5090/discussion/334922

02a_permutations

Permutations II

Summary of Combinatorics

Counting tasks on n objects

Sort objects (permutations)

Choose k objects (combinations)

Put objects in r buckets

Distinct (distinguishable)

Sort n distinct objects

# of permutations =

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable)
    - $n!$
  - Some distinct
- Choose k objects (combinations)
- Put objects in r buckets

Sort semi-distinct objects

Order n distinct objects	$n!$

All distinct

Ayesha Tim Irina Joey Waddie

Some indistinct

Coke Tim Coke Joey Waddie

Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

permutations of distinct objects = permutations considering some objects are indistinct × Permutations of just the indistinct objects

Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

$$\frac{\text{permutations of distinct objects}}{\text{Permutations of just the indistinct objects}} = \text{permutations considering some objects are indistinct}$$

General approach to counting permutations

When there are n objects such that

n_1 are the same (indistinguishable or indistinct), and

n_2 are the same, and

…

n_r are the same,

The number of unique orderings (permutations) is

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}.$$

For each group of indistinct objects,
Divide by the overcounted permutations.

Sort semi-distinct objects

Order n semi-distinct objects $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$

How many permutations?

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$
- Choose k objects (combinations)
- Put objects in r buckets

Strings

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many orderings of letters are possible for the following strings?

1. BOBA

2. MISSISSIPPI

Strings

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many orderings of letters are possible for the following strings?

1. BOBA $= \frac{4!}{2!} = 12$

2. MISSISSIPPI $= \frac{11!}{1!4!4!2!} = 34{,}650$

Unique 6-digit passcodes with six smudges

$$\text{Total} = 6!$$
$$= 720 \text{ passcodes}$$

Unique 6-digit passcodes with five smudges

02b_combinations_i

Combinations I

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$
- Choose k objects (combinations)
  - Distinct
- Put objects in r buckets

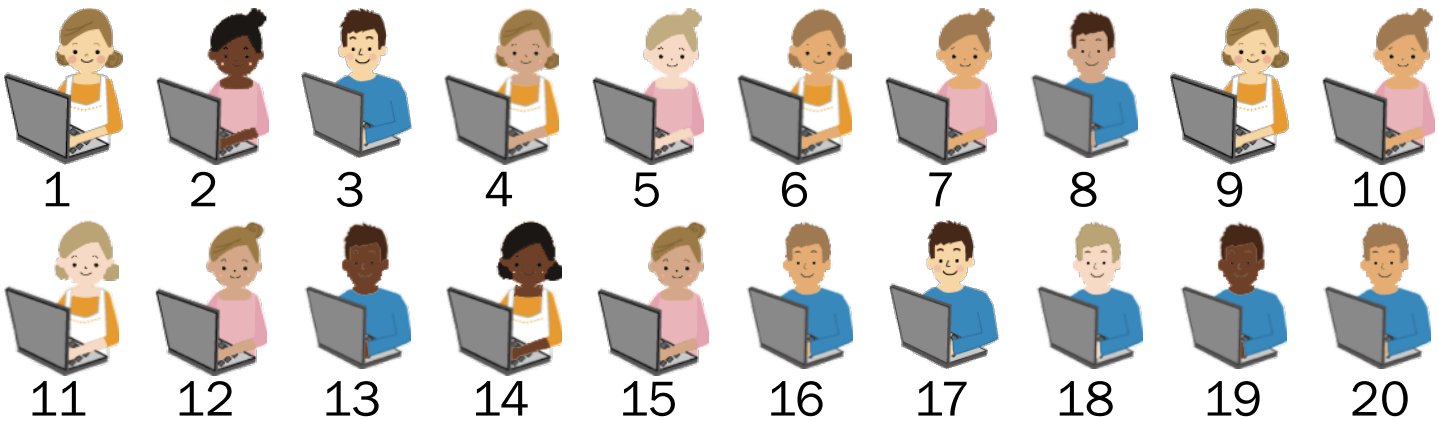
Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

Consider the following generative process…

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

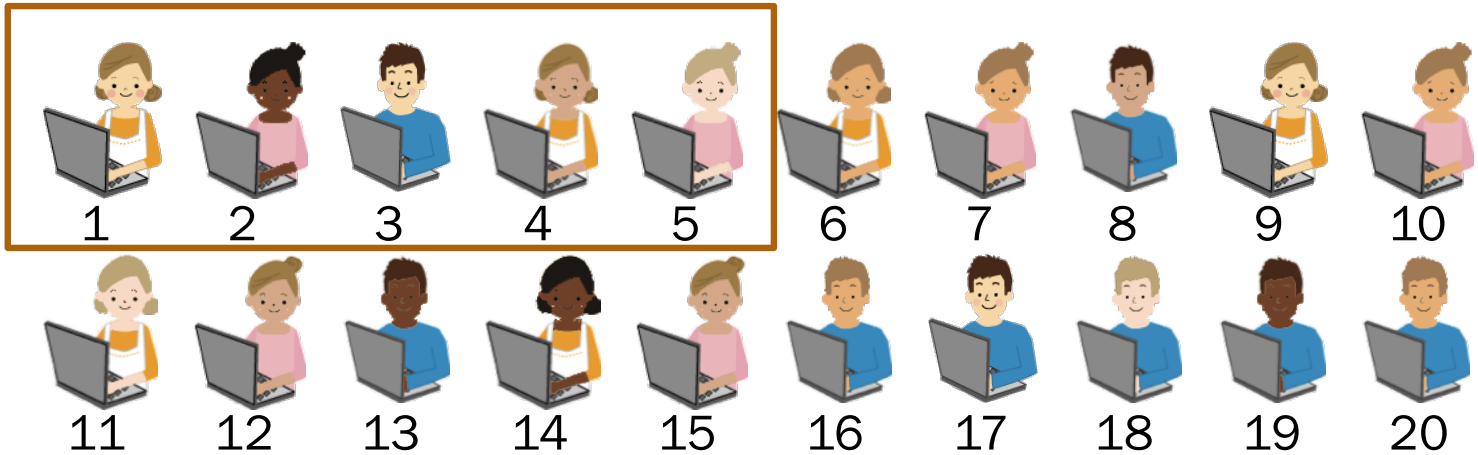


1. n people get in line

   $n!$ ways

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

$n!$ ways

2. Put first k in cake room

1 way

Combinations with cake

There are $n = 20$ people.
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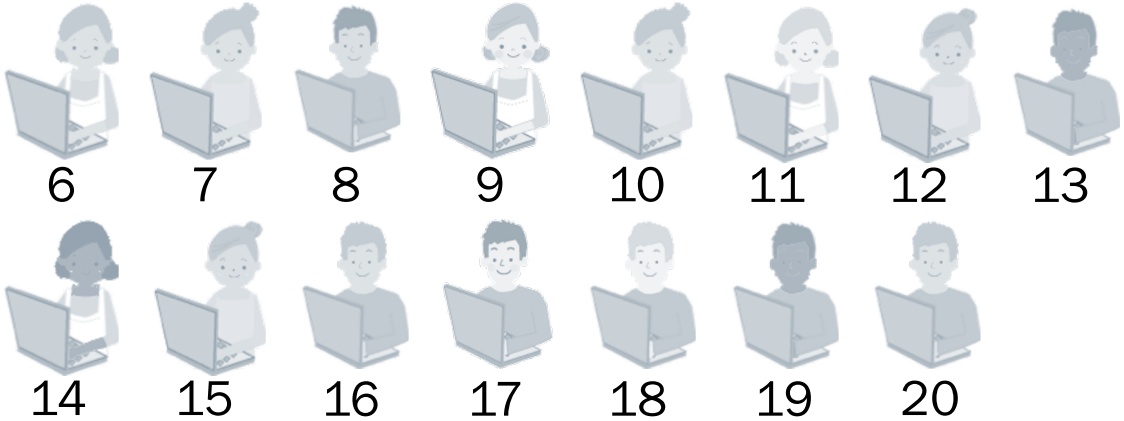

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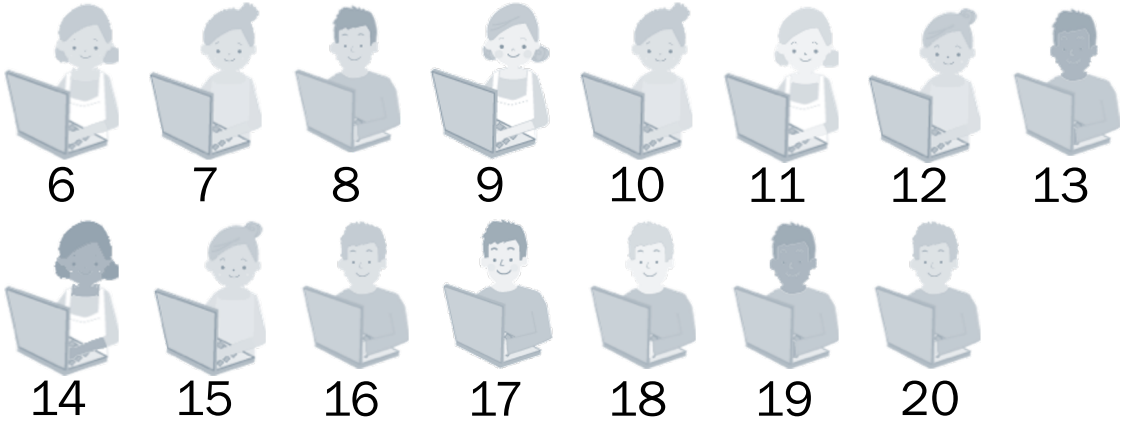

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

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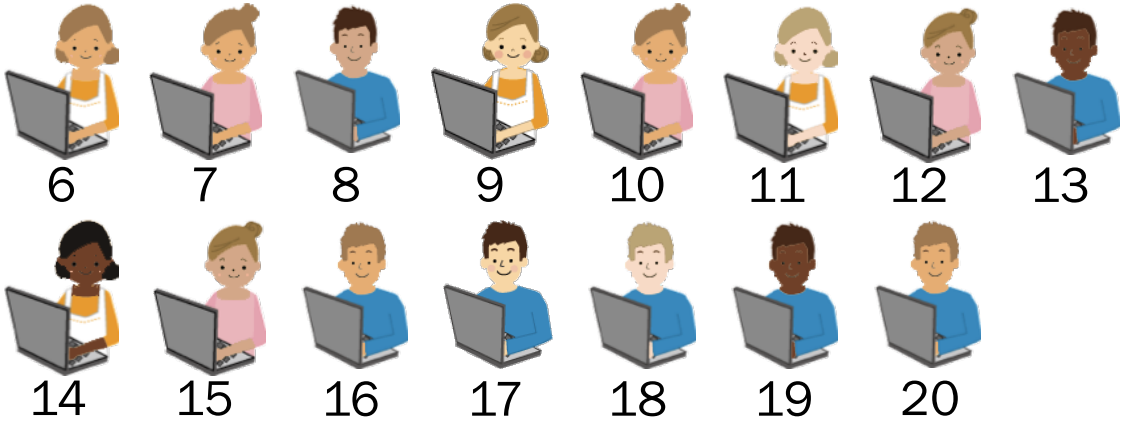

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

   $(n-k)!$ different permutations lead to the same mingle

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!}$$

1. Order n distinct objects
2. Take first k as chosen
3. Overcounted: any ordering of chosen group is same choice
4. Overcounted: any ordering of unchosen group is same choice

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!} = \binom{n}{k}$$

Binomial coefficient

Note: $\binom{n}{n-k} = \binom{n}{k}$

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

02c_combinations_ii

Combinations II

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\,n_2!\cdots n_r!}$
- Choose k objects (combinations)
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups
- Put objects in r buckets

General approach to combinations

The number of ways to choose r groups of n distinct objects such that

For all $i = 1, \ldots, r$, group i has size n_i, and

$\sum_{i=1}^{r} n_i = n$ (all objects are assigned), is

$$\frac{n!}{n_1!\, n_2! \cdots n_r!} = \binom{n}{n_1, n_2, \cdots, n_r}$$

Multinomial coefficient

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	6
B	4
C	3

A. $\binom{13}{6,4,3} = 60{,}060$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

C. $6 \cdot 1001 \cdot 10 = 60{,}060$

D. A and B

E. All of the above

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Strategy: Combinations into 3 groups

Group 1 (datacenter A): $n_1 = 6$

Group 2 (datacenter B): $n_2 = 4$

Group 3 (datacenter C): $n_3 = 3$

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

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Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

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Strategy: Product rule with 3 steps

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2. Choose 4 computers for B $\binom{7}{4}$
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Your approach will determine if you use binomial/multinomial coefficients or factorials.