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<table>
<thead>
<tr>
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<tbody>
<tr>
<td>3</td>
<td>Permutations II</td>
<td>02a_permutations</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>Combinations I</td>
<td>02b_combinations_i</td>
<td></td>
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<tr>
<td>29</td>
<td>Combinations II</td>
<td>02c_combinations_ii</td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>Buckets and The Divider Method</td>
<td></td>
<td>LIVE</td>
</tr>
</tbody>
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Today’s discussion thread: [https://edstem.org/us/courses/5090/discussion/334922](https://edstem.org/us/courses/5090/discussion/334922)
Permutations II
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
- Choose $k$ objects (combinations)
- Put objects in $r$ buckets

Distinct (distinguishable)
Sort $n$ distinct objects

Ayesha  Tim  Irina  Joey  Waddie

# of permutations =
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
  - Distinct (distinguishable)
    - $n!$

- Choose $k$ objects (combinations)
  - Some distinct

- Put objects in $r$ buckets
Sort semi-distinct objects

All distinct

Ayesha  Tim  Irina  Joey  Waddie

Some indistinct

Coke  Tim  Coke  Joey  Waddie

Order $n$ distinct objects $n!$
Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

\[
\text{permutations of distinct objects} = \text{permutations considering some objects are indistinct} \times \text{Permutations of just the indistinct objects}
\]
Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

\[
\text{Permutations of distinct objects} = \text{Permutations of just the indistinct objects} \times \text{Permutations considering some objects are indistinct}
\]
General approach to counting permutations

When there are \( n \) objects such that
- \( n_1 \) are the same (indistinguishable or indistinct), and
- \( n_2 \) are the same, and
- ...
- \( n_r \) are the same,

The number of unique orderings (permutations) is

\[
\frac{n!}{n_1! n_2! \cdots n_r!}.
\]

For each group of indistinct objects, divide by the overcounted permutations.
Sort semi-distinct objects

How many permutations?

Coke    Coke0    Coke    Coke0    Coke0
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
  - Distinct (distinguishable)
  - $n!$

- Choose $k$ objects (combinations)
  - Some distinct
  - $\frac{n!}{n_1!n_2!\cdots n_r!}$

- Put objects in $r$ buckets

Lisa Yan, Chris Piech, Mehran Sahami, and Jerry Cain, CS109, Spring 2021
Strings

How many orderings of letters are possible for the following strings?

1. **BOBA**

2. **MISSISSIPPI**
Strings

How many orderings of letters are possible for the following strings?

1. BOBA

\[ = \frac{4!}{2!} = 12 \]

2. MISSISSIPPI

\[ = \frac{11!}{1!4!4!2!} = 34,650 \]
Unique 6-digit passcodes with six smudges

Total = 6!

= 720 passcodes
Unique 6-digit passcodes with five smudges
Combinations I
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
  - Distinct (distinguishable)
  - $n!$

- Choose $k$ objects (combinations)
  - Some distinct
  - Distinct
  - $\frac{n!}{n_1!n_2!\cdots n_r!}$

- Put objects in $r$ buckets
  - Distinct

Lisa Yan, Chris Piech, Mehran Sahami, and Jerry Cain, CS109, Spring 2021
Combinations with cake

There are \( n = 20 \) people. How many ways can we choose \( k = 5 \) people to get cake?

Consider the following generative process...
Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line

$n!$ ways
Combinations with cake

There are $n = 20$ people. How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line
   - $n!$ ways

2. Put first $k$ in cake room
   - 1 way
Combinations with cake

There are $n = 20$ people.

How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line $n!$ ways
2. Put first $k$ in cake room 1 way

Lisa Yan, Chris Piech, Mehran Sahami, and Jerry Cain, CS109, Spring 2021
Combinations with cake

There are \( n = 20 \) people.
How many ways can we choose \( k = 5 \) people to get cake?

1. \( n \) people get in line
   \( n! \) ways

2. Put first \( k \) in cake room
   1 way

3. Allow cake group to mingle
   \( k! \) different permutations lead to the same mingle
Combinations with cake

There are $n = 20$ people.

How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line
   - $n!$ ways

2. Put first $k$ in cake room
   - 1 way

3. Allow cake group to mingle
   - $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

Lisa Yan, Chris Piech, Mehran Sahami, and Jerry Cain, CS109, Spring 2021
Combinations with cake

There are \( n = 20 \) people. How many ways can we choose \( k = 5 \) people to get cake?

1. \( n \) people get in line
   - \( n! \) ways

2. Put first \( k \) in cake room
   - 1 way

3. Allow cake group to mingle
   - \( k! \) different permutations lead to the same mingle

4. Allow non-cake group to mingle
   - \((n - k)!\) different permutations lead to the same mingle
Combinations

A combination is an unordered selection of $k$ objects from a set of $n$ distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k! (n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!}$$

1. Order $n$ distinct objects
2. Take first $k$ as chosen
3. Overcounted: any ordering of chosen group is same choice
4. Overcounted: any ordering of unchosen group is same choice
Combinations

A combination is an unordered selection of \( k \) objects from a set of \( n \) distinct objects.

The number of ways of making this selection is

\[
\frac{n!}{k! \, (n - k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n - k)!} = \binom{n}{k}
\]

Binomial coefficient

Note: \( \binom{n}{n-k} = \binom{n}{k} \)
Probability textbooks

How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3! 3!} = 20 \text{ ways}$$
Combinations II
### Summary of Combinatorics

Counting tasks on $n$ objects

- **Sort objects (permutations)**
  - Distinct (distinguishable)
  - Distinct
  - Some distinct
  - $n!$

- **Choose $k$ objects (combinations)**
  - Distinct
  - 1 group
  - $\binom{n}{k}$

- **Put objects in $r$ buckets**
  - $r$ groups
  - $\frac{n!}{n_1! n_2! \cdots n_r!}$
General approach to combinations

The number of ways to choose $r$ groups of $n$ distinct objects such that
For all $i = 1, \ldots, r$, group $i$ has size $n_i$, and
$\sum_{i=1}^{r} n_i = n$ (all objects are assigned), is

$$\frac{n!}{n_1! n_2! \cdots n_r!} = \left(\begin{array}{c} n \\ n_1, n_2, \ldots, n_r \end{array}\right)$$

Multinomial coefficient
Datacenters

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

<table>
<thead>
<tr>
<th>Datacenter</th>
<th># machines</th>
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<tr>
<td>A</td>
<td>6</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
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A. \( \binom{13}{6,4,3} = 60,060 \)

B. \( \binom{13}{6} \binom{7}{4} \binom{3}{3} = 60,060 \)

C. \( 6 \cdot 1001 \cdot 10 = 60,060 \)

D. A and B

E. All of the above
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Strategy: Combinations into 3 groups

Group 1 (datacenter A): \( n_1 = 6 \)
Group 2 (datacenter B): \( n_2 = 4 \)
Group 3 (datacenter C): \( n_3 = 3 \)
Datacenters

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Strategy: Product rule with 3 steps

1. Choose 6 computers for A \( \binom{13}{6} \)
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Datacenters

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1. Choose 6 computers for A \( \binom{13}{6} \)
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3. Choose 3 computers for C \( \binom{3}{3} \)

Choose \( k \) of \( n \) distinct objects into \( r \) groups of size \( n_1, \ldots, n_r \) \( \binom{n}{n_1, n_2, \ldots, n_r} \)

Your approach will determine if you use binomial/multinomial coefficients or factorials.