03: Intro to Probability

Jerry Cain
Arpil 2, 2021
Quick slide reference

3  Defining Probability                          03a_definitions
13 Axioms of Probability                       03b_axioms
20 Equally likely outcomes                     03c_elo
30 Exercises                                   LIVE
37 Corollaries                                 LIVE

Today’s discussion thread: https://edstem.org/us/courses/5090/discussion/334923
Defining Probability

Gradescope quiz, blank slide deck, etc.
http://cs109.stanford.edu/
Key definitions

An experiment in probability:

Sample Space, $S$: The set of all possible outcomes of an experiment.
Event, $E$: Some subset of $S$ ($E \subseteq S$).
Key definitions

Sample Space, $S$

- Coin flip
  $S = \{\text{Heads, Tails}\}$

- Flipping two coins
  $S = \{(H,H), (H,T), (T,H), (T,T)\}$

- Roll of 6-sided die
  $S = \{1, 2, 3, 4, 5, 6\}$

- # emails in a day
  $S = \{x \mid x \in \mathbb{Z}, x \geq 0\}$

- TikTok hours in a day
  $S = \{x \mid x \in \mathbb{R}, 0 \leq x \leq 24\}$

Event, $E$

- Flip lands heads
  $E = \{\text{Heads}\}$

- $\geq 1$ head on 2 coin flips
  $E = \{(H,H), (H,T), (T,H)\}$

- Roll is 3 or less:
  $E = \{1, 2, 3\}$

- Low email day ($\leq 20$ emails)
  $E = \{x \mid x \in \mathbb{Z}, 0 \leq x \leq 20\}$

- Wasted day ($\geq 5$ TT hours):
  $E = \{x \mid x \in \mathbb{R}, 5 \leq x \leq 24\}$
What is a probability?

A number between 0 and 1 to which we ascribe meaning.*

*our belief that an event $E$ occurs.
What is a probability?

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

$n = \# \text{ of total trials}$

$n(E) = \# \text{ trials where } E \text{ occurs}$

Let $E = \text{ the set of outcomes where you hit the target.}$
What is a probability?

Let $E = \text{the set of outcomes where you hit the target.}$

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

$n = \# \text{ of total trials}$

$n(E) = \# \text{ trials where } E \text{ occurs}$

$P(E) \approx 0.00$
What is a probability?

\[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

Let \( E \) = the set of outcomes where you hit the target.

\[ n = \# \text{ of total trials} \]
\[ n(E) = \# \text{ trials where } E \text{ occurs} \]

\[ P(E) \approx 0.500 \]
What is a probability?

\[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

\( n = \# \) of total trials
\( n(E) = \# \) trials where \( E \) occurs

Let \( E \) = the set of outcomes where you hit the target.

\[ P(E) \approx 0.667 \]
What is a probability?

\[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

- \( n \) = # of total trials
- \( n(E) \) = # trials where \( E \) occurs

Let \( E \) = the set of outcomes where you hit the target.

\[ P(E) \approx 0.458 \]
Not just yet...
Axioms of Probability
Quick review of sets

$E$ and $F$ are events in $S$.

Experiment:
- Die roll
- $S = \{1, 2, 3, 4, 5, 6\}$
- Let $E = \{1, 2\}$, and $F = \{2, 3\}$
Quick review of sets

Review of Sets

$E$ and $F$ are events in $S$.

Experiment:

Die roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2, 3\}$

**Def** Union of events, $E \cup F$

The event containing all outcomes in $E$ or $F$.

$E \cup F = \{1, 2, 3\}$
Quick review of sets

$E$ and $F$ are events in $S$.

Experiment:

Die roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2, 3\}$

**def** Intersection of events, $E \cap F$

The event containing all outcomes in $E$ **and** $F$.

**def** Mutually exclusive events $F$ and $G$ means that $F \cap G = \emptyset$

$E \cap F = EF = \{2\}$
Quick review of sets

$E$ and $F$ are events in $S$.

Experiment:
- Die roll
  - $S = \{1, 2, 3, 4, 5, 6\}$
  - Let $E = \{1, 2\}$, and $F = \{2, 3\}$

**def Complement** of event $E$, $E^C$

The event containing all outcomes in that are **not** in $E$.

$E^C = \{3, 4, 5, 6\}$
3 Axioms of Probability

Definition of probability: \( P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \)

Axiom 1: \( 0 \leq P(E) \leq 1 \)

Axiom 2: \( P(S) = 1 \)

Axiom 3: If \( E \) and \( F \) are mutually exclusive \( (E \cap F = \emptyset) \), then \( P(E \cup F) = P(E) + P(F) \)
Axiom 3 is the (analytically) useful Axiom

Axiom 3: If \( E \) and \( F \) are mutually exclusive (\( E \cap F = \emptyset \)), then
\[
P(E \cup F) = P(E) + P(F)
\]

More generally, for any sequence of mutually exclusive events \( E_1, E_2, \ldots \):
\[
P \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} P(E_i)
\]

(like the Sum Rule of Counting, but for probabilities)
Equally Likely Outcomes
Equally Likely Outcomes

Some sample spaces have equally likely outcomes.

• Coin flip: $S = \{\text{Head, Tails}\}$
• Flipping two coins: $S = \{(H, H), (H, T), (T, H), (T, T)\}$
• Roll of 6-sided die: $S = \{1, 2, 3, 4, 5, 6\}$

If we have equally likely outcomes, then $P(\text{Each outcome}) = \frac{1}{|S|}$

Therefore $P(E) = \frac{\# \text{ outcomes in } E}{\# \text{ outcomes in } S} = \frac{|E|}{|S|}$ (by Axiom 3)
Roll two dice

Roll two 6-sided fair dice. What is $P(\text{sum} = 7)$?

$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

$E =$

$P(E) = \frac{|E|}{|S|}$

Equally likely outcomes
Target revisited
Target revisited

Let $E =$ the set of outcomes where you hit the target.

Screen size = $800 \times 800$

Radius of target: 200

The dart is equally likely to land anywhere on the screen. What is $P(E)$, the probability of hitting the target?

$$|S| = 800^2$$
$$|E| \approx \pi \cdot 200^2$$

$$P(E) = \frac{|E|}{|S|} \approx \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$
Target revisited

Let $E$ = the set of outcomes where you hit the target.

Screen size = $800 \times 800$

Radius of target: 200

The dart is equally likely to land anywhere on the screen. What is $P(E)$, the probability of hitting the target?

$$|S| = 800^2 \quad |E| \approx \pi \cdot 200^2$$

$$P(E) = \frac{|E|}{|S|} \approx \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$
Not equally likely outcomes

Play the lottery.
What is $P(\text{win})$?

$S = \{\text{Lose, Win}\}$    
$E = \{\text{Win}\}$

$P(E) = \frac{|E|}{|S|} = \frac{1}{2} = 50\%$?

The hard part: defining outcomes consistently across sample space and events
Cats and sharks

4 cats and 3 sharks in a bag. 3 drawn. What is \( P(1 \text{ cat and } 2 \text{ sharks drawn}) \)?

**Note:** Do indistinct objects give you an equally likely sample space?

(No)
**Cats and sharks (ordered solution)**

4 cats and 3 sharks in a bag. 3 drawn. What is \( P(1 \text{ cat and 2 sharks drawn}) \)?

**Define**

- \( S = \) Pick 3 distinct items
- \( E = 1 \) distinct cat, 2 distinct sharks

\[
P(E) = \frac{|E|}{|S|} \quad \text{Equally likely outcomes}
\]

Make indistinct items distinct to get equally likely outcomes.
Cats and sharks (unordered solution)

4 cats and 3 sharks in a bag. 3 drawn. What is $P(1$ cat and 2 sharks drawn$)$?

Define

- $S =$ Pick 3 distinct items
- $E =$ 1 distinct cat, 2 distinct sharks

Equally likely outcomes

$$P(E) = \frac{|E|}{|S|}$$

Make indistinct items distinct to get equally likely outcomes.