05: Independence

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Quick slide reference

05a_chain

Generalized Chain Rule

Chain Rule

Definition of conditional probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

The Chain Rule:

$$P(EF) = P(E|F)P(F)$$

Generalized Chain Rule

$$P(E_1 E_2 E_3 \ldots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1 E_2) \ldots P(E_n|E_1 E_2 \ldots E_{n-1})$$

Quick check

$P(E_1 E_2 E_3 \ldots E_n) = P(E_1)P(E_2|E_1) \ldots P(E_n|E_1 E_2 \ldots E_{n-1})$ Chain Rule

You are going to a friend's Halloween party.

Let C = there is candy
M = there is music
W = you wear a costume
E = no one wears your costume

An awesome party means that all of these events must occur.

What is $P(\text{awesome party}) = P(CMWE)$?

A. $P(C)P(M|C)P(W|CM)P(E|CMW)$
B. $P(M)P(C|M)P(W|MC)P(E|MCW)$
C. $P(W)P(E|W)P(CM|EW)$
D. A, B, and C
E. None/other

Quick check

$P(E_1 E_2 E_3 \ldots E_n) =$
$P(E_1)P(E_2|E_1) \ldots P(E_n|E_1 E_2 \ldots E_{n-1})$ Chain Rule

You are going to a friend's Halloween party.

Let C = there is candy
M = there is music
E = no one wears your costume
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An awesome party means that all of these events must occur.

What is $P(\text{awesome party}) = P(CMEW)$?

A. $P(C)P(M|C)P(E|CM)P(W|CME)$
B. $P(M)P(C|M)P(E|MC)P(W|MCE)$
C. $P(W)P(E|W)P(CM|EW)$
D. A, B, and C
E. None/other

Chain Rule is a way of introducing "order" and "procedure" into probability.

Think of the children

Two parents both have an (A, a) gene pair.

- Each parent will pass on one of their genes (each gene equally likely) to their child.
- The probability of a single child having curly hair (recessive trait) is 0.25.
- There are three children.

What is the probability that all three children have curly hair?

Let E_1, E_2, E_3 be the events that child 1, 2, and 3 have curly hair, respectively.

$$P(E_1 E_2 E_3) = P(E_1)P(E_2|E_1)P(E_3|E_1 E_2)$$

Independence I

Independence

Two events E and F are defined as independent if:

$$P(EF) = P(E)P(F)$$

Otherwise E and F are called dependent events.

If E and F are independent, then:

$$P(E|F) = P(E)$$

Intuition through proof

Independent events E and F ⟺ $P(EF) = P(E)P(F)$

Statement:

If E and F are independent, then $P(E|F) = P(E)$.

Proof:

$$P(E|F) = \frac{P(EF)}{P(F)}$$ Definition of conditional probability

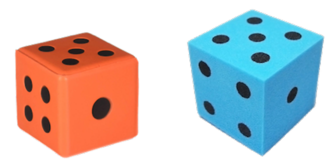
$$= \frac{P(E)P(F)}{P(F)}$$ Independence of E and F

$$= P(E)$$ Taking the bus to cancellation city

Knowing that F happened does not change our belief that E happened.

Dice, our misunderstood friends

Independent events E and F ⟺ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$



- Roll two 6-sided dice, yielding values D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 5$

$G = \{(1,4),(2,3),(3,2),(4,1)\}$

1. Are E and F independent?

   $P(E) = 1/6$
   $P(F) = 1/6$
   $P(EF) = 1/36$

   ✅ independent

2. Are E and G independent?

   $P(E) = 1/6$
   $P(G) = 4/36 = 1/9$
   $P(EG) = 1/36 \neq P(E)P(G)$

   ❌ dependent

Generalizing independence

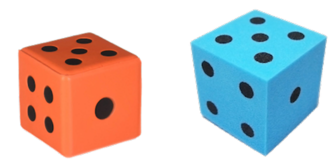
Three events E, F, and G are independent if:

$P(EFG) = P(E)P(F)P(G)$, and
$P(EF) = P(E)P(F)$, and
$P(EG) = P(E)P(G)$, and
$P(FG) = P(F)P(G)$

n events $E_1, E_2, \dots, E_n$ are independent if:

for $r = 1, \dots, n$:
for every subset $E_1, E_2, \dots, E_r$:
$$P(E_1, E_2, \dots, E_r) = P(E_1)P(E_2)\cdots P(E_r)$$

Dice, increasingly misunderstood (still our friends)



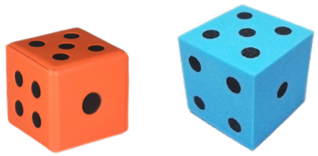
- Each roll of a 6-sided die is an independent trial.
- Two rolls: D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 7$

$G = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

1. Are E and F independent? ✅
2. Are E and G independent?
3. Are F and G independent?
4. Are E, F, G independent?

$P(EF) = 1/36$

Dice, increasingly misunderstood (still our friends)



- Each roll of a 6-sided die is an independent trial.
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$G = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

1. Are E and F ✅ independent?
2. Are E and G ✅ independent?
3. Are F and G ✅ independent?
4. Are E, F, G ❌ independent?

$P(EF) = 1/36$

Pairwise independence is not sufficient to prove independence of >2 events!

05b_independence_ii

Independence II

Independent trials

We often are interested in experiments consisting of n independent trials.

- n trials, each with the same set of possible outcomes
- n-way independence: an event in one subset of trials is independent of events in other subsets of trials

Examples:

- Flip a coin n times
- Roll a die n times
- Send a multiple choice survey to n people
- Send n web requests to k different servers

Think of the children as independent trials

Two parents both have an (A, a) gene pair.

- Each parent will pass on one of their genes (each gene equally likely) to their child.
- The probability of a single child having curly hair (recessive trait) is 0.25.
- There are three children. **Each child is an independent trial.**

What is the probability that all three children have curly hair?

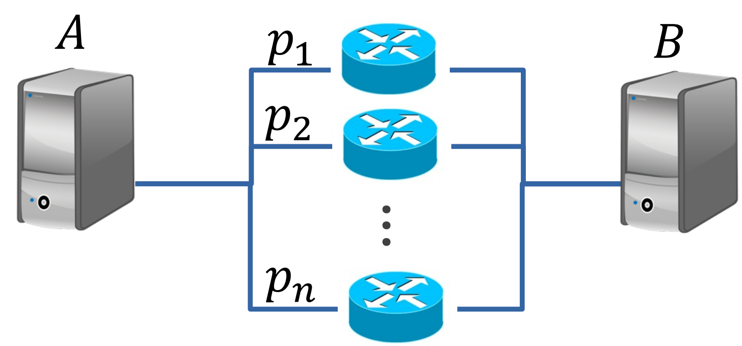
Let E_1, E_2, E_3 be the events that child 1, 2, and 3 have curly hair, respectively.

$$P(E_1 E_2 E_3) = P(E_1)P(E_2|E_1)P(E_3|E_1 E_2)$$

Network reliability

Consider the following parallel network:

- n independent routers, each with probability p_i of functioning (where $1 \leq i \leq n$)
- E = functional path from A to B exists.



What is $P(E)$?

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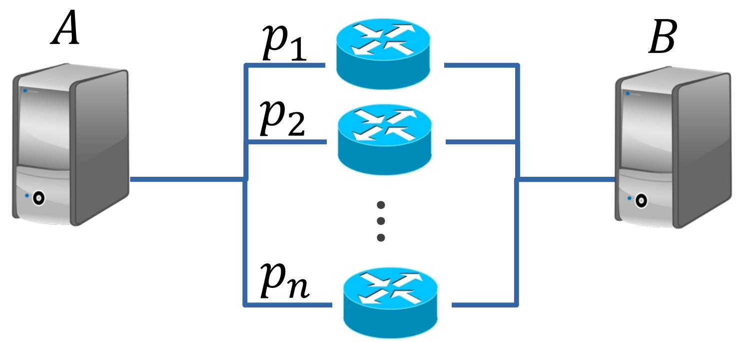

What is $P(E)$?

$$
\begin{aligned}
P(E) &= P(\geq 1 \text{ one router works}) \\
&= 1 - P(\text{all routers fail}) \\
&= 1 - (1-p_1)(1-p_2)\cdots(1-p_n) \\
&= 1 - \prod_{i=1}^{n}(1-p_i)
\end{aligned}
$$

≥ 1 with independent trials: take complement