

o8: Poisson and More

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April 14, 2021

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Poisson

Before we start

The natural exponent e :

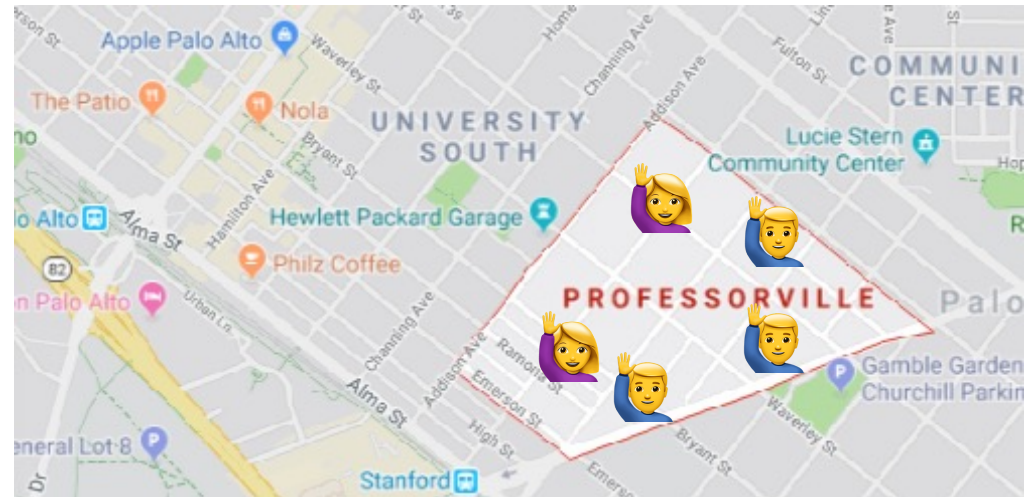
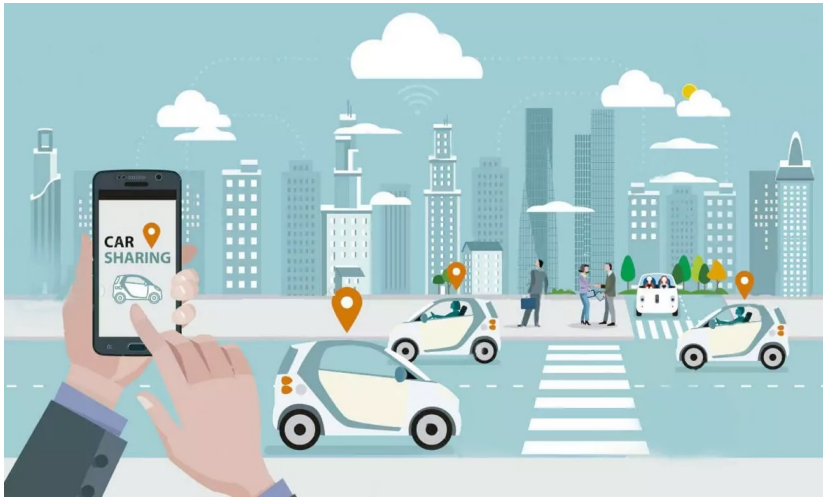
$$\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

[https://en.wikipedia.org/wiki/E_\(mathematical_constant\)](https://en.wikipedia.org/wiki/E_(mathematical_constant))

Jacob Bernoulli
while studying
compound interest
in 1683



Algorithmic ride sharing



Probability of k requests from this area in the next 1 minute?

Suppose we know:

On average, $\lambda = 5$ requests per minute

Algorithmic ride sharing, approximately

Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

Break a minute down into 60 seconds:



At each second:

- Independent trial
- You get a request (1) or you don't (0).

Let $X = \#$ of requests in minute.

$$E[X] = \lambda = 5$$

$$X \sim \text{Bin}(n = 60, p = 5/60)$$

$$P(X = k) = \binom{60}{k} \left(\frac{5}{60}\right)^k \left(1 - \frac{5}{60}\right)^{n-k}$$



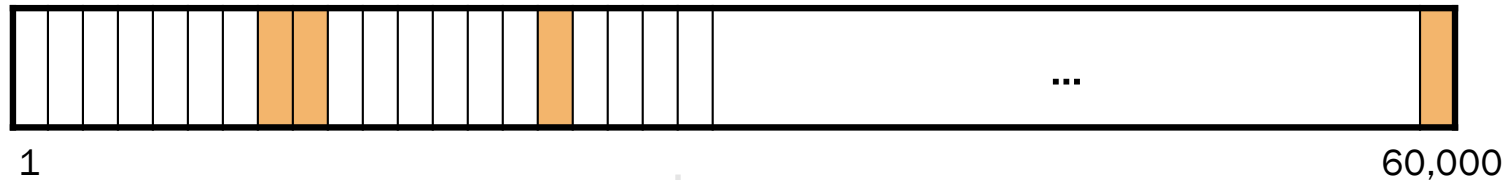
But what if there are *two* requests in the same second?

Algorithmic ride sharing, approximately

Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

Break a minute down into 60,000 **milliseconds**:



At each **millisecond**:

- Independent trial
- You get a request (1) or you don't (0).

Let $X = \#$ of requests in minute.

$$E[X] = \lambda = 5$$

$$X \sim \text{Bin}(n = 60000, p = \lambda/n)$$

$$P(X = k) = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$



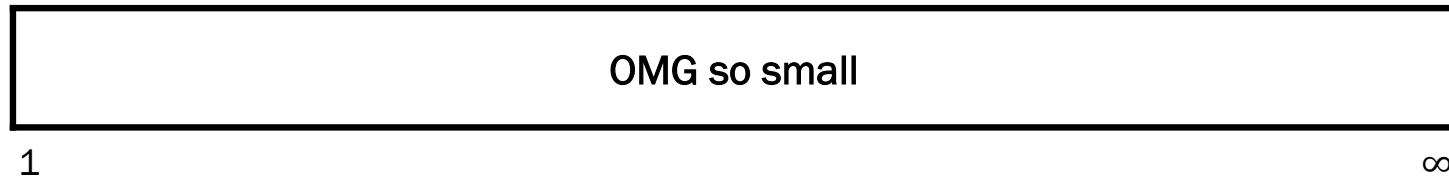
But what if there are *two* requests in the same **millisecond**?

Algorithmic ride sharing, approximately

Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

Break a minute down into **infinitely small** buckets:



For each time bucket:

- Independent trial
- You get a request (1) or you don't (0).

Let $X = \#$ of requests in minute.

$$E[X] = \lambda = 5$$

$$X \sim \text{Bin}(n, p = \lambda/n)$$

$$P(X = k) = \lim_{n \rightarrow \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

Who wants to see some cool math?

Binomial in the limit

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

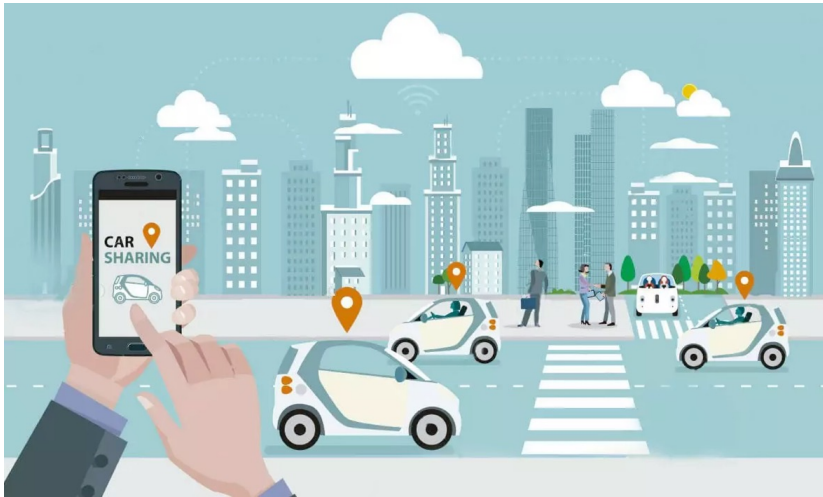
$$P(X = k) = \lim_{n \rightarrow \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \stackrel{\text{Expand}}{=} \lim_{n \rightarrow \infty} \frac{n!}{k!(n-k)!} \frac{\lambda^k}{n^k} \frac{\left(1 - \frac{\lambda}{n}\right)^n}{\left(1 - \frac{\lambda}{n}\right)^k}$$

$$\stackrel{\text{Rearrange}}{=} \lim_{n \rightarrow \infty} \frac{n!}{n^k(n-k)!} \frac{\lambda^k}{k!} \frac{\left(1 - \frac{\lambda}{n}\right)^n}{\left(1 - \frac{\lambda}{n}\right)^k} \stackrel{\text{Def natural exponent}}{=} \lim_{n \rightarrow \infty} \frac{n!}{n^k(n-k)!} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{\left(1 - \frac{\lambda}{n}\right)^k}$$

$$\stackrel{\text{Expand}}{=} \lim_{n \rightarrow \infty} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{(n-k)!}{(n-k)!} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{\left(1 - \frac{\lambda}{n}\right)^k}$$

$$\stackrel{\text{Limit analysis + cancel}}{=} \lim_{n \rightarrow \infty} \frac{n^k}{n^k} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{1} \stackrel{\text{Simplify}}{=} \frac{\lambda^k}{k!} e^{-\lambda}$$

Algorithmic ride sharing



Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

**Poisson
distribution**

Poisson, continued

Poisson Random Variable

Consider an experiment that lasts a fixed interval of time.

def A **Poisson** random variable X is the number of successes over the experiment duration, assuming **the time that each success occurs is independent** and the average # of requests over time is constant.



Examples:

- # earthquakes per year
- # server hits per second
- # of emails per day

Poisson Random Variable

Consider an experiment that lasts a fixed interval of time.

def A **Poisson** random variable X is the number of successes over the experiment duration, assuming the time that each success occurs is independent and the average # of requests over time is constant.

$$X \sim \text{Poi}(\lambda)$$

Support: $\{0, 1, 2, \dots\}$

PMF

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Expectation $E[X] = \lambda$

Variance $\text{Var}(X) = \lambda$

Examples:

- # earthquakes per year
- # server hits per second
- # of emails per day

Yes, expectation == variance for Poisson RV! More later.

Simeon-Denis Poisson



French mathematician (1781 – 1840)

- Published his first paper at age 18
- Professor at age 21
- Published over 300 papers

“Life is only good for two things: doing mathematics and teaching it.”

Earthquakes

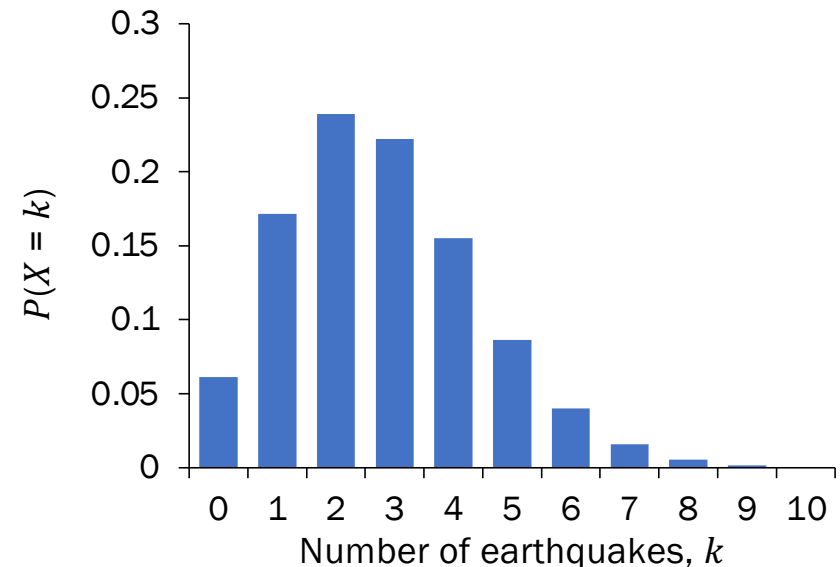
$$X \sim \text{Poi}(\lambda) \quad p(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$
$$E[X] = \lambda$$

There are an average of 2.79 major earthquakes in the world each year, and major earthquakes occur independently.

What is the probability of 3 major earthquakes happening next year?

1. Define RVs

2. Solve



Are earthquakes really Poissonian?

Bulletin of the Seismological Society of America

Vol. 64

October 1974

No. 5

IS THE SEQUENCE OF EARTHQUAKES IN SOUTHERN CALIFORNIA,
WITH AFTERSHOCKS REMOVED, POISSONIAN?

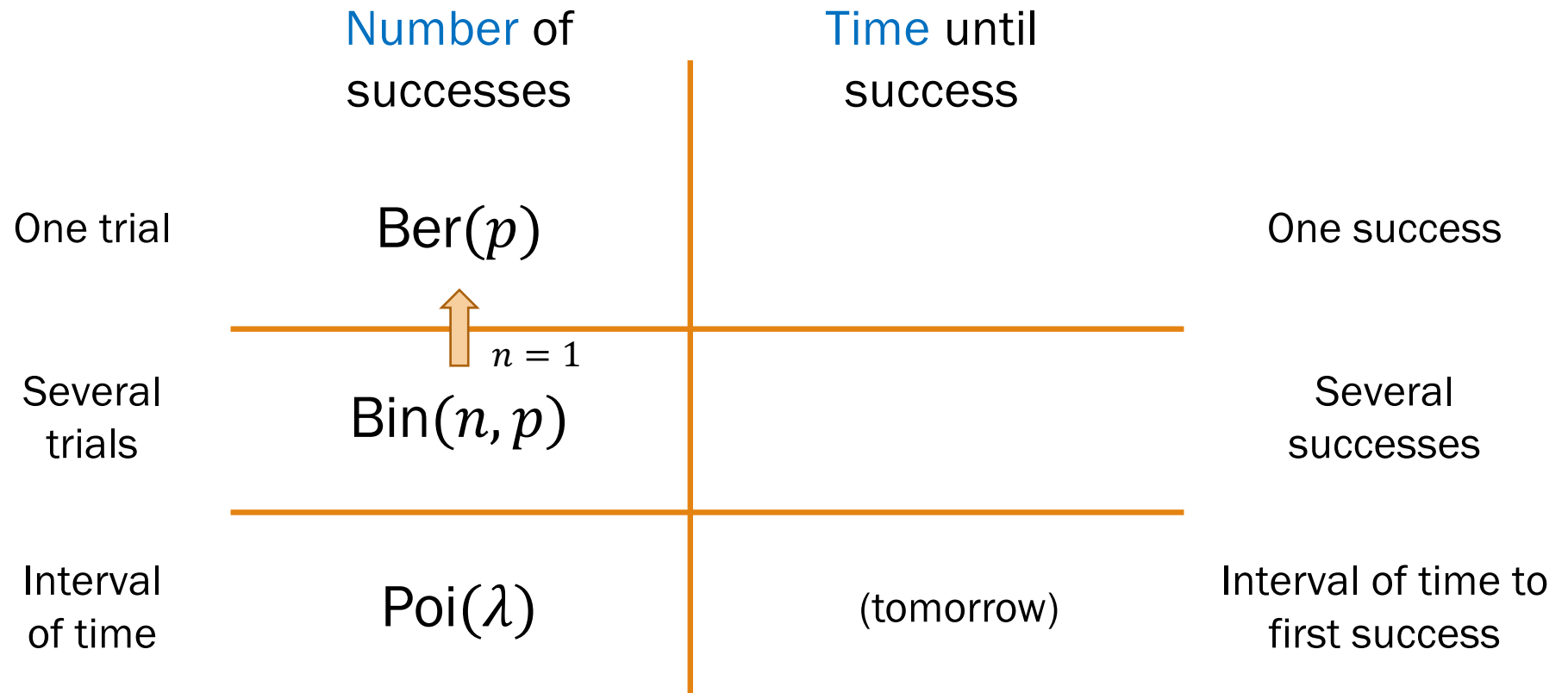
BY J. K. GARDNER and L. KNOPOFF

ABSTRACT

Yes.

Other Discrete RVs

Grid of random variables



Focus on understanding how and when to use RVs, not on memorizing PMFs.

Geometric RV

Consider an experiment: independent trials of $\text{Ber}(p)$ random variables.

def A **Geometric** random variable X is the # of trials until the first success.

$$X \sim \text{Geo}(p)$$

Support: $\{1, 2, \dots\}$

PMF

$$P(X = k) = (1 - p)^{k-1} p$$

Expectation

$$E[X] = \frac{1}{p}$$

Variance

$$\text{Var}(X) = \frac{1-p}{p^2}$$

Examples:

- Flipping a coin ($P(\text{heads}) = p$) until first heads appears
- Generate bits with $P(\text{bit} = 1) = p$ until first 1 generated

Negative Binomial RV

Consider an experiment: independent trials of $\text{Ber}(p)$ random variables.

def A **Negative Binomial** random variable X is the # of trials until r successes.

$X \sim \text{NegBin}(r, p)$	PMF	$P(X = k) = \binom{k-1}{r-1} (1-p)^{k-r} p^r$
	Expectation	$E[X] = \frac{r}{p}$
	Variance	$\text{Var}(X) = \frac{r(1-p)}{p^2}$

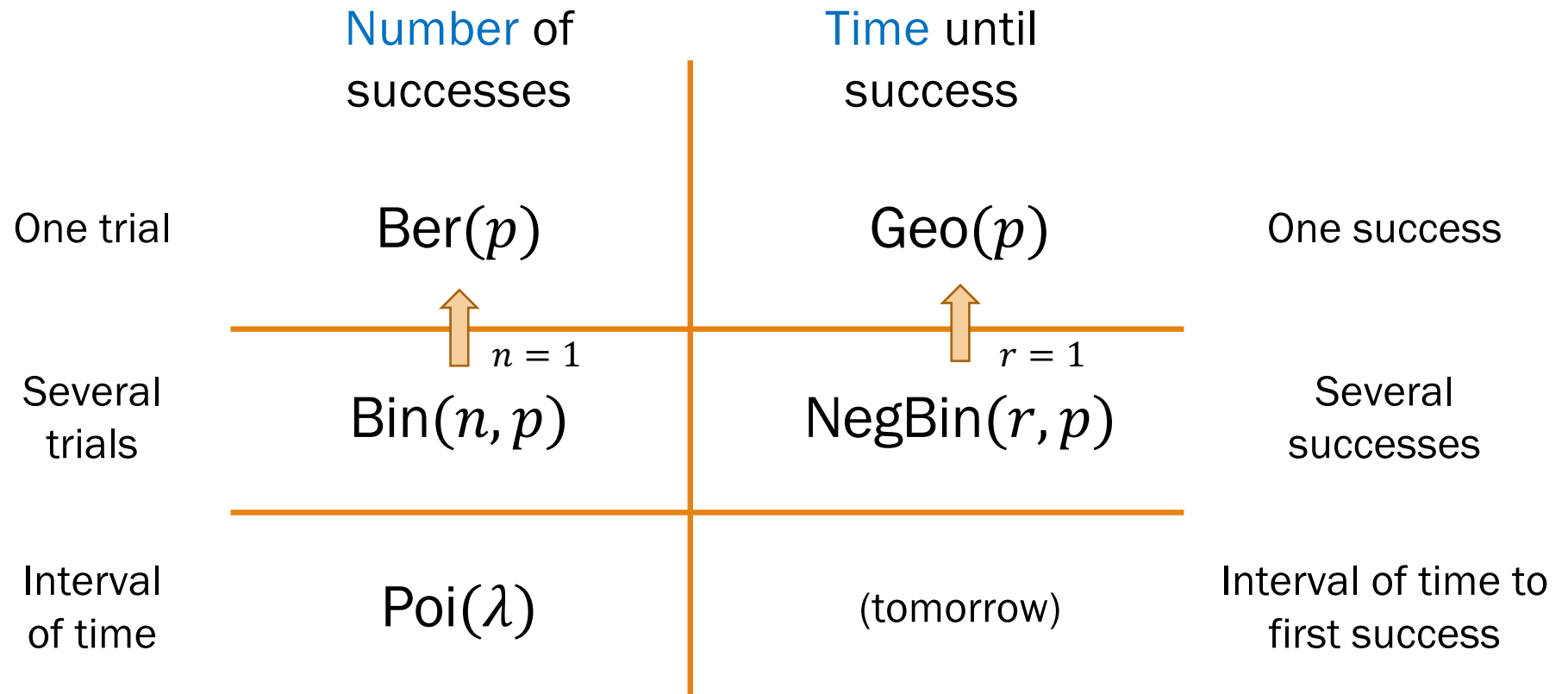
Support: $\{r, r+1, \dots\}$

Examples:

- Flipping a coin until r^{th} heads appears
- # of strings to hash into table until bucket 1 has r entries

$$\text{Geo}(p) = \text{NegBin}(1, p)$$

Grid of random variables



Catching Pokemon

Wild Pokemon are captured by throwing Pokeballs at them.

- Each ball has probability $p = 0.1$ of capturing the Pokemon.
- Each ball is an independent trial.

What is the probability that you catch the Pokemon on the 5th try?

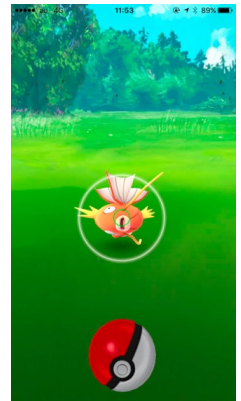
1. Define events/
RVs & state goal

$X \sim$ some distribution

Want: $P(X = 5)$

2. Solve

- A. $X \sim \text{Bin}(5, 0.1)$
- B. $X \sim \text{Poi}(0.5)$
- C. $X \sim \text{NegBin}(5, 0.1)$
- D. $X \sim \text{NegBin}(1, 0.1)$
- E. $X \sim \text{Geo}(0.1)$
- F. None/other

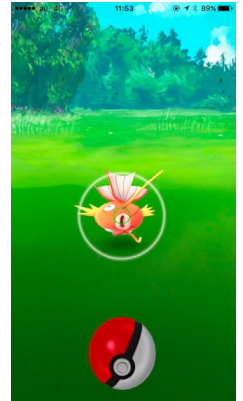


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Catching Pokemon

$$X \sim \text{Geo}(p) \quad p(k) = (1 - p)^{k-1}p$$

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$$X \sim \text{Geo}(0.1)$$

$$\text{Want: } P(X = 5)$$

