09: Continuous RVs

Jerry Cain
April 16, 2021
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Continuous RVs
Not all values are discrete

```python
import numpy as np
def np.random.random()
```
People heights

You are volunteering at the local elementary school.

- To choose a t-shirt for your new buddy Jordan, you need to know their height.

1. What is the probability that your buddy is \textbf{54.0923857234} inches tall?

2. What is the probability that your buddy is between \textbf{52–56} inches tall?

Essentially 0
Integrals

Integral = area under a curve

Loving, not scary
Continuous RV definition

A random variable $X$ is **continuous** if there is a **probability density function** $f(x) \geq 0$ such that for $-\infty < x < \infty$:

$$P(a \leq X \leq b) = \int_{a}^{b} f(x) \, dx$$

Integrating a PDF must always yield valid probabilities, and therefore the PDF must also satisfy

$$\int_{-\infty}^{\infty} f(x) \, dx = P(-\infty < X < \infty) = 1$$

Also written as: $f_X(x)$
Today’s main takeaway, #1

Integrate $f(x)$ to get probabilities.

PDF Units: probability per units of $X$

$P(52 \leq X \leq 56) = \int_{52}^{56} f(x) \, dx$
## PMF vs PDF

**Discrete** random variable $X$

**Probability mass function (PMF):**

$$ p(x) $$

To get probability:

$$ P(X = x) = p(x) $$

Both are measures of how **likely** $X$ is to take on a value.

**Continuous** random variable $X$

**Probability density function (PDF):**

$$ f(x) $$

To get probability:

$$ P(a \leq X \leq b) = \int_a^b f(x) \, dx $$
Computing probability

Let $X$ be a continuous RV with PDF:

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

What is $P(X \geq 1)$?
Computing probability

Let $X$ be a continuous RV with PDF:

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

What is $P(X \geq 1)$?

**Strategy 1**: Integrate

$$P(1 \leq X < \infty) = \int_{1}^{\infty} f(x)dx = \int_{1}^{2} \frac{1}{2}x dx$$

$$= \frac{1}{2} \left( \frac{1}{2} x^2 \right) \bigg|_{1}^{2} = \frac{1}{2} \left[ 2 - \frac{1}{2} \right] = \frac{3}{4}$$

**Strategy 2**: Know triangles

$$1 - \frac{1}{2} \left( \frac{1}{2} \right) = \frac{3}{4}$$

Wait...is this even legal?

$$P(0 \leq X < 1) = \int_{0}^{1} f(x)dx \text{??}$$
Today’s main takeaway, #2

For a continuous random variable $X$ with PDF $f(x)$, 

$$P(X = c) = \int_c^c f(x) \, dx = 0.$$ 

Contrast with PMF in discrete case: $P(X = c) = p(c)$
PDF Properties

For a continuous RV $X$ with PDF $f$,

$$P(a \leq X \leq b) = \int_{a}^{b} f(x) \, dx$$

True/False:

1. $P(X = c) = 0$
2. $P(a \leq X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a < X \leq b)$
3. $f(x)$ is a probability
4. In the graphed PDF above, $P(x_1 \leq X \leq x_2) > P(x_2 \leq X \leq x_3)$
PDF Properties

For a continuous RV $X$ with PDF $f$,

$$P(a \leq X \leq b) = \int_a^b f(x) \, dx$$

True/False:

1. $P(X = c) = 0$

2. $P(a \leq X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a < X \leq b)$

3. $f(x)$ is a probability

4. In the graphed PDF above, $P(x_1 \leq X \leq x_2) > P(x_2 \leq X \leq x_3)$

Compare area under the curve $f$
Uniform RV
def An **Uniform** random variable $X$ is defined as follows:

$$X \sim \text{Uni}(\alpha, \beta)$$

**PDF**

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

**Support:** $[\alpha, \beta]$ (sometimes defined over $[\alpha, \beta]$)

**Expectation**

$$E[X] = \frac{\alpha + \beta}{2}$$

**Variance**

$$\text{Var}(X) = \frac{(\beta - \alpha)^2}{12}$$
Quick check

If \( X \sim \text{Uni}(\alpha, \beta) \), the PDF of \( X \) is:

\[
f(x) = \begin{cases} 
\frac{1}{\beta - \alpha} & \text{if } \alpha \leq x \leq \beta \\
0 & \text{otherwise}
\end{cases}
\]

What is \( \frac{1}{\beta - \alpha} \) if the following graphs are PDFs of Uniform RVs \( X \)?

1. \( f(x) \)
2. \( f(x) \)
3. \( f(x) \)
Quick check

If \( X \sim \text{Uni}(\alpha, \beta) \), the PDF of \( X \) is:

\[
 f(x) = \begin{cases} 
 \frac{1}{\beta - \alpha} & \text{if } \alpha \leq x \leq \beta \\
 0 & \text{otherwise}
\end{cases}
\]

What is \( \frac{1}{\beta - \alpha} \) if the following graphs are PDFs of Uniform RVs \( X \)?

1. \( f(x) \)
   \[
   \frac{1}{25}
   \]

2. \( f(x) \)
   \[
   2
   \]

3. \( f(x) \)
   \[
   \frac{1}{10}
   \]
### Expectation and Variance

**Discrete RV** $X$

\[
E[X] = \sum_x x \cdot p(x)
\]

\[
E[g(X)] = \sum_x g(x) \cdot p(x)
\]

**Continuous RV** $X$

\[
E[X] = \int_{-\infty}^{\infty} x \cdot f(x) \, dx
\]

\[
E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x) \, dx
\]

Both continuous and discrete RVs

\[
E[aX + b] = aE[X] + b
\]

\[
\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2
\]

\[
\text{Var}(aX + b) = a^2 \text{Var}(X)
\]

**Linearity of Expectation**

**Properties of variance**

**TL;DR:** $\sum_{x=a}^{b} \Rightarrow \int_{a}^{b}$
Uniform RV expectation

\[ E[X] = \int_{-\infty}^{\infty} x \cdot f(x) \, dx \]

\[ = \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta - \alpha} \, dx \]

\[ = \frac{1}{\beta - \alpha} \cdot \frac{1}{2} x^2 \bigg|_{\alpha}^{\beta} \]

\[ = \frac{1}{\beta - \alpha} \cdot \frac{1}{2} (\beta^2 - \alpha^2) \]

\[ = \frac{1}{2} \cdot \frac{(\beta + \alpha)(\beta - \alpha)}{\beta - \alpha} = \frac{\alpha + \beta}{2} \]

Interpretation:
Average the start & end
**Uniform Random Variable**

**def** An **Uniform** random variable $X$ is defined as follows:

$$f(x) = \begin{cases} 
\frac{1}{\beta - \alpha} & \text{if } \alpha \leq x \leq \beta \\
0 & \text{otherwise}
\end{cases}$$

$X \sim \text{Uni}(\alpha, \beta)$

- **Support:** $[\alpha, \beta]$ (sometimes defined over $(\alpha, \beta)$)
- **PDF**
- **Expectation**
  $$E[X] = \frac{\alpha + \beta}{2}$$
- **Variance**
  $$\text{Var}(X) = \frac{(\beta - \alpha)^2}{12}$$

On your own time

Just now

Lisa Yan, Chris Piech, Mehran Sahami, and Jerry Cain, CS109, Spring 2021
Exponential RV
Grid of random variables

Number of successes

<table>
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<tr>
<th>One trial</th>
<th>Several trials</th>
<th>Interval of time</th>
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<tr>
<td>Ber($p$)</td>
<td>Bin($n, p$)</td>
<td>Poi($\lambda$)</td>
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<tr>
<td>One success</td>
</tr>
<tr>
<td>NegBin($r, p$)</td>
</tr>
<tr>
<td>Exp($\lambda$)</td>
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- $n = 1$
- $r = 1$
Consider an experiment that lasts a duration of time until success occurs. **Def** An **Exponential** random variable $X$ is the amount of time until success.

**Exponential Random Variable**

\[ X \sim \text{Exp}(\lambda) \]

- **PDF**
  \[ f(x) = \begin{cases} 
  \lambda e^{-\lambda x} & \text{if } x \geq 0 \\
  0 & \text{otherwise}
  \end{cases} \]

- **Support:** $(0, \infty)$

- **Expectation**
  \[ E[X] = \frac{1}{\lambda} \]

- **Variance**
  \[ \text{Var}(X) = \frac{1}{\lambda^2} \]

**Examples:**
- Time until next earthquake
- Time for request to reach web server
- Time until end of cell phone contract
Interpreting Exp(\(\lambda\))

**def** An **Exponential** random variable \(X\) is the amount of time until success.

\[
X \sim \text{Exp}(\lambda) \quad \text{Expectation} \quad E[X] = \frac{1}{\lambda}
\]

Based on the expectation \(E[X]\), what are the units of \(\lambda\)?
Interpreting Exp(\(\lambda\))

**def** An **Exponential** random variable \(X\) is the amount of time until success.

\[
X \sim \text{Exp}(\lambda) \\
\text{Expectation} \quad E[X] = \frac{1}{\lambda}
\]

Based on the expectation \(E[X]\), what are the units of \(\lambda\)?

- e.g., average # of successes per second

For both Poisson and Exponential RVs,
\(\lambda = \# \text{ successes/time}\).
Earthquakes

1906 Earthquake
Magnitude 7.8
Earthquakes

Major earthquakes (magnitude 8.0+) occur once every 500 years.*

1. What is the probability of a major earthquake in the next 30 years?

We know on average:

\[
\begin{align*}
\text{500 years} & \quad \text{earthquake} \\
0.002 \quad \text{earthquakes} & \quad \text{year} \\
1 \quad \text{earthquakes} & \quad \text{500 years}
\end{align*}
\]

*In California, according to historical data from USGS, 2015
Earthquakes

Major earthquakes (magnitude 8.0+) occur once every 500 years.*

1. What is the probability of a major earthquake in the next 30 years?

Define events/ RVs & state goal

Solve

\[
X \sim \text{Exp}(\lambda) \quad E[X] = \frac{1}{\lambda} \\
 f(x) = \lambda e^{-\lambda x} \quad \text{if } x \geq 0
\]

\[
X \sim \text{Exp}(\lambda = 0.002) \\
\lambda: \text{year}^{-1} = \frac{1}{500}
\]

Want: \( P(X < 30) \)

*In California, according to historical data from USGS, 2015
Earthquakes

Major earthquakes (magnitude 8.0+) occur once every 500 years.*

1. What is the probability of a major earthquake in the next 30 years?
2. What is the standard deviation of years until the next earthquake?

Define events/ RVs & state goal

\[ X: \text{when next earthquake happens} \]
\[ X \sim \text{Exp}(\lambda = 0.002) \]
\[ \lambda: \text{year}^{-1} \]

Want: \( P(X < 30) \)

*In California, according to historical data from USGS, 2015