

# 14: Conditional Expectation

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Jerry Cain

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# Quick slide reference

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# Discrete conditional distributions

# Discrete conditional distributions

Recall the definition of the conditional probability of event  $E$  given event  $F$ :

$$P(E|F) = \frac{P(EF)}{P(F)}$$

For discrete random variables  $X$  and  $Y$ , the **conditional PMF** of  $X$  given  $Y$  is

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Different notation,  
same idea:

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

# Discrete probabilities of CS109

Each student responds with:

Year  $Y$

- 1: Frosh/Soph
- 2: Jr/Sr
- 3: Co-term/grad/NDO

Timezone  $T$  (12pm California time in my timezone is):

- -1: AM
- 0: noon
- 1: PM

	Joint PMF		
	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.06	.01	.01
$T = 0$	.29	.14	.09
$T = 1$	.30	.08	.02

$$P(Y = 3, T = 1)$$

Joint PMFs sum to 1.

# Discrete probabilities of CS109

The below are conditional probability tables for conditional PMFs

(A)  $P(Y = y|T = t)$  and (B)  $P(T = t|Y = y)$ .

- Which is which?
- What's the missing probability?

	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.09	.04	.08
$T = 0$	.45	.61	.75
$T = 1$	.46	.35	.17

	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.75	.125	?
$T = 0$	.56	.27	.17
$T = 1$	.75	.2	.05

	Joint PMF		
	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.06	.01	.01
$T = 0$	.29	.14	.09
$T = 1$	.30	.08	.02



# Discrete probabilities of CS109

The below are conditional probability tables for conditional PMFs

(A)  $P(Y = y|T = t)$  and (B)  $P(T = t|Y = y)$ .

1. Which is which?
2. What's the missing probability?

	Joint PMF		
	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.06	.01	.01
$T = 0$	.29	.14	.09
$T = 1$	.30	.08	.02

(B)  $P(T = t|Y = y)$

	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.09	.04	.08
$T = 0$	.45	.61	.75
$T = 1$	.46	.35	.17

$$.30 / (.06 + .29 + .30)$$

(A)  $P(Y = y|T = t)$

	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.75	.125	.125
$T = 0$	.56	.27	.17
$T = 1$	.75	.2	.05

$$1 - .75 - .125$$

Conditional PMFs also sum to 1 conditioned on different events!

# Extended to Amazon



Roll over image to zoom in

**FINEDINE**  
**Stainless Steel Mixing Bowls by Finedine (Set of 6) Polished Mirror Finish Nesting Bowl, 1/4 - 1.5-3 - 4-5 - 8 Quart - Cooking Supplies**

★★★★★ 2,566 customer reviews | 75 answered questions

Amazon's Choice for "stainless steel mixing bowls"

Price: **\$24.99 & FREE Shipping** on orders over \$25 shipped by Amazon. [Details](#)

Get \$40 off instantly: Pay \$0.00 upon approval for the Amazon.com Store Card.

✓prime | Try Fast, Free Shipping \*

- With graduating sizes of 1/4, 1.5, 3, 4, 5 and 8 quart, the bowl set allows users to be well equipped for serving fruit salads, marinating for the grill, and adding last ingredients for dessert.
- Stainless steel bowls with commercial grade metal that can be used as both baking mixing bowls and serving bowls. These metal bowls won't stain or absorb odors and resist rust for years of durability.
- An easy to grip rounded-lip on the stainless steel bowl set makes handling easier while a generous wide rim allows contents to flow evenly when pouring; flat base stabilizes the silver bowls making mixing all the easier.
- A space saving stackable design helps de-clutter kitchen cupboards while the attractive polished mirror finish on the large mixing bowls adds a luxurious aesthetic.
- This incredible stainless steel mixing bowl set is refrigerator, freezer, and dishwasher safe for quick and easy meal prep and clean up. They'd also make a great gift!

[Compare with similar items](#)

**Used & new** (7) from \$20.62 & FREE shipping on orders over \$25.00. [Details](#)

Report incorrect product information.









**Packaging may reveal contents.** Choose **Conceal Package** at checkout.

**KELIWA**  
*Easy home baking*  
[Shop now](#)

Kelima 12 Cup Silicone Muffin - Cupcake Baking Pan / Silicone Mold Non - Stick ...  
 ★★★★★ 1,803  
 \$9.99 ✓prime

[Ad feedback](#)

## Customers who

 ExcelSteel Stainless Steel Colanders, Set of 3 ★★★★★ 301 \$15.83 ✓prime	 EasyLife 18/8 Stainless Steel Measuring Spoons, Set of 6 for Measuring Dry and Liquid Ingredients ★★★★★ 1,854 #1 Best Seller in Measuring Spoons \$9.99 ✓prime	 New Star Foodservice 42917 Stainless Steel 4pcs Measuring Cups and Spoons Combo Set ★★★★★ 1,042 #1 Best Seller in Specialty Spoons \$9.95 ✓prime	 Rubbermaid Easy Find Lids Food Storage Containers, Racer Red, 42-Piece Set 1880801 ★★★★★ 10,319 \$19.99 ✓prime	 Miusco 5 Piece Silicone Cooking Utensil Set with Natural Acacia Hard Wood Handle ★★★★★ 461 \$20.99 ✓prime	 Bellemain Micro-perforated Stainless Steel 5-quart Colander-Dishwasher Safe ★★★★★ 2,797 #1 Best Seller in Colanders \$19.95 ✓prime	 AmazonBasics 6-Piece Nonstick Bakeware Set ★★★★★ 67 \$19.99 ✓prime	 HOMWE Kitchen Cutting Board (3-Piece Set)   Juice Grooves w/ Easy-Grip Handles   BPA-Free, ... ★★★★★ 240 \$14.97 ✓prime
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P(bought item X | bought item Y)



# Quick check

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Number or function?

1.  $P(X = 2|Y = 5)$
2.  $P(X = x|Y = 5)$
3.  $P(X = 2|Y = y)$
4.  $P(X = x|Y = y)$

True or false?

5.  $\sum_x P(X = x|Y = 5) = 1$
6.  $\sum_y P(X = 2|Y = y) = 1$
7.  $\sum_x \sum_y P(X = x|Y = y) = 1$
8.  $\sum_x \left( \sum_y P(X = x|Y = y)P(Y = y) \right) = 1$



# Quick check

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Number or function?

1.  $P(X = 2|Y = 5)$   
number
2.  $P(X = x|Y = 5)$   
1-D function
3.  $P(X = 2|Y = y)$   
1-D function
4.  $P(X = x|Y = y)$   
2-D function

True or false?

5.  $\sum_x P(X = x|Y = 5) = 1$  true
6.  $\sum_y P(X = 2|Y = y) = 1$  false
7.  $\sum_x \sum_y P(X = x|Y = y) = 1$  false
8.  $\sum_x \left( \sum_y P(X = x|Y = y)P(Y = y) \right) = 1$  true

14b\_cond\_expectation

# Conditional Expectation

# Conditional expectation

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Recall the the conditional PMF of  $X$  given  $Y = y$ :

$$p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

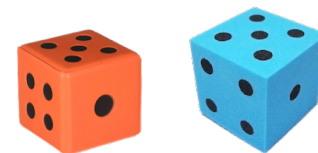
The **conditional expectation** of  $X$  given  $Y = y$  is

$$E[X|Y = y] = \sum_x xP(X = x|Y = y) = \sum_x xp_{X|Y}(x|y)$$

# It's been so long, our dice friends

$$E[X|Y = y] = \sum_x xp_{X|Y}(x|y)$$

- Roll two 6-sided dice.
- Let roll 1 be  $D_1$ , roll 2 be  $D_2$ .
- Let  $S =$  value of  $D_1 + D_2$ .



1. What is  $E[S|D_2 = 6]$ ?  $E[S|D_2 = 6] = \sum_{x=7}^{12} xP(S = x|D_2 = 6)$

$$= \left(\frac{1}{6}\right) (7 + 8 + 9 + 10 + 11 + 12)$$
$$= \frac{57}{6} = 9.5$$

Intuitively:  $6 + E[D_1] = 6 + 3.5 = 9.5$

Let's prove this!

# Properties of conditional expectation

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## 1. LOTUS:

$$E[g(X)|Y = y] = \sum_x g(x)p_{X|Y}(x|y)$$

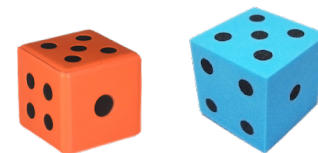
## 2. Linearity of conditional expectation:

$$E\left[\sum_{i=1}^n X_i | Y = y\right] = \sum_{i=1}^n E[X_i | Y = y]$$

## 3. Law of total expectation (next time)

# It's been so long, our dice friends

$$E[X|Y = y] = \sum_x xp_{X|Y}(x|y)$$

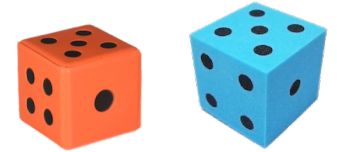


- Roll two 6-sided dice.
  - Let roll 1 be  $D_1$ , roll 2 be  $D_2$ .
  - Let  $S = \text{value of } D_1 + D_2$ .
1. What is  $E[S|D_2 = 6]$ ?  $\frac{57}{6} = 9.5$
  2. What is  $E[S|D_2]$ ?
    - A. A function of  $S$
    - B. A function of  $D_2$
    - C. A number
  3. Give an expression for  $E[S|D_2]$ .



# It's been so long, our dice friends

$$E[X|Y = y] = \sum_x x p_{X|Y}(x|y)$$



- Roll two 6-sided dice.
- Let roll 1 be  $D_1$ , roll 2 be  $D_2$ .
- Let  $S = \text{value of } D_1 + D_2$ .

1. What is  $E[S|D_2 = 6]$ ?

$$\frac{57}{6} = 9.5$$

2. What is  $E[S|D_2]$ ?

- A. A function of  $S$
- B.** A function of  $D_2$
- C. A number

3. Give an expression for  $E[S|D_2]$ .

$$E[S|D_2 = d_2] = E[D_1 + d_2|D_2 = d_2]$$

$$= \sum_{d_1} (d_1 + d_2) P(D_1 = d_1 | D_2 = d_2)$$

$$= \sum_{d_1} d_1 P(D_1 = d_1) + d_2 \sum_{d_1} P(D_1 = d_1)$$

( $D_1 = d_1, D_2 = d_2$   
independent  
events)

$$= E[D_1] + d_2 = 3.5 + d_2$$

$$E[S|D_2] = 3.5 + D_2$$