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CS 109

Problem Set #4
April 30, 2021

Problem Set #4
Due: 10:00am on Monday, May 10th

With problems by Mehran Sahami, Chris Piech, Lisa Yan, and Jerry Cain.

- Submit on Gradescope by 10:00am Pacific on Monday, May 10th, for a small, "on-time" bonus.
- All students have a pre-approved extension, or "grace period" that extends until Wednesday 10:00am Pacific, when they can submit with no penalty. **The grace period expires on 10:00am Pacific on Wednesday, May 12th**, after which we cannot accept further late submissions.
- **Collaboration policy:** You are encouraged to discuss problem-solving strategies with each other as well as the course staff, but you must write up your own solutions and submit individual work. Please cite any collaboration at the top of your submission.
- **Tagging written problems**: When you submit your written PDF on Gradescope you must tag your PDF, meaning that you must assign pages of your PDF as answers to particular questions so that we can properly grade your submission. For problem sets, we are deducting **2 points** for any submissions that do not have all questions tagged.
- **For each problem, briefly explain/justify how you obtained your answer.** Brief explanations of your answer are necessary to get full credit for a problem even if you have the correct numerical answer. The explanations help us determine your understanding of the problem whether or not you got the correct answer. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide. It is fine for your answers to include summations, products, factorials, exponentials, or combinations; you don't need to calculate those all out to get a single numeric answer.
- If you handwrite your solutions, you are responsible for making sure that you can produce **clearly legible** scans of them for submission. You may also use any word processing software you like for writing up your solutions. On the CS109 webpage we provide a template file and tutorial for the LaTeX system, if you'd like to use it.

Written Problems

Submit your solutions to these written problems as a single pdf file on Gradescope.

1. Say that of all the students who will attend Stanford, each will buy at most one laptop computer when they first arrive at school. 40% of students will purchase a PC, 30% will purchase a Mac, 10% will purchase a Linux machine and the remaining 20% will not buy any laptop at all. If 15 students are asked which, if any, laptop they purchased, what is the probability that exactly 6 students will have purchased a PC, 4 will have purchased a Mac, 2 will have purchased a Linux machine, and the remaining 3 students will have not purchased any laptop?

Continued on next page. . .

2. Recall the example of zero sum games for teams with ELO scores S_1 (Team 1) and S_2 (Team 2). When a game is played between the two teams, they each sample an ability (A_1 and A_2 for Teams 1 and 2, respectively) from a normal distribution with mean equal to the team's ELO score and constant variance. The variance is different for different types of games; for this problem, we will use the GO rating variance of $\sigma^2 = (2000/7)^2$. In lecture, we talked about how to calculate the probability that a team wins via sampling. In this problem, we will work out a closed form calculation.
    a. What is the probability distribution for the difference between A_1 and A_2, defined as $A_1 - A_2$?
    b. A team wins if their sampled ability on game day is larger. Come up with a closed form expression for the probability that Team 1 wins.
    c. The best human GO player in the world is Ke Jie, with an ELO score of 3670. Alpha GO is a computer with an ELO score of 5200. How many independent games would they have to play before the expected number of games that Ke wins is ≥ 1?

3. An artificial intelligence algorithm is being used to make a binary prediction (G for guess) for whether a person will repay a microloan. An important question to ask: is the algorithm fair with respect to a binary demographic (D for demographic)? To answer this question we are going to analyze the historical predictions of the algorithm and compare the predictions to the true outcome (T for truth) and the growing field of algorithmic fairness. Consider the following joint probability table from the history of the algorithm's predictions:

	D = 0		**D = 1**	
	G = 0	G = 1	G = 0	G = 1
T = 0	0.21	0.32	0.01	0.01
T = 1	0.07	0.28	0.02	0.08

D: is the demographic of an individual (binary)
G: is the prediction made by the algorithm (binary)
T: is the true result (binary)

Recall that cell in the joint table where $(A = i, C = j, Y = k)$ is the probability, $P(A = i, C = j, Y = k)$. For all questions, justify your answer. You may leave your answers with terms that could be input into a calculator.

    a. What is $P(D = 1)$?
    b. What is $P(G = 1|D = 1)$?
    c. Fairness definition #1: Parity. An algorithm satisfies "parity" if the probability that the algorithm makes a *positive prediction* ($G = 1$) is the same regardless of the demographic variable. Does this algorithm satisfy parity?
    d. Fairness definition #2: Calibration. An algorithm satisfies "calibration" if the probability that the algorithm is *correct* ($G = T$) is the same regardless of demographics. Does this algorithm satisfy calibration?
    e. Fairness definition #3: Equality of odds. An algorithm satisfies "equality of odds" if the probability that the algorithm *predicts a positive outcome* ($G = 1$) is the same regardless

of demographics *given* that the outcome will occur ($T = 1$). Does this algorithm satisfy equality of odds?

4. The joint probability density function of continuous random variables X and Y is given by:

$$f_{X,Y}(x, y) = c\left(x + \frac{xy}{2}\right) \qquad \text{where } 0 < x < 1, 0 < y < 1$$

   a. What is the value of c in order for $f_{X,Y}(x, y)$ to be a valid probability density function?
   b. Are X and Y independent? Explain why or why not.
   c. What is the marginal density function of X?
   d. What is $P(X > Y)$?
   e. What is $P(Y > 0.5 \mid X > 0.5)$?
   f. What is $E[X]$?

5. Our ability to fight contagious diseases depends on our ability to model them. One person is exposed to llama-flu. The method below models the number of individuals who will get infected.

```
from scipy import stats
"""
Return number of people infected by one individual.
"""
def num_infected():
  # most people are immune to llama flu.
  # stats.bernoulli(p).rvs() returns 1 w.p. p (0 otherwise)
  immune = stats.bernoulli(p = 0.99).rvs()
  if immune: return 0

  # people who are not immune spread the disease far by
  # making contact with k people (up to 100).
  spread = 0
  # returns random # of successes in n trials w.p. p of success
  k = stats.binom(n = 100, p = 0.25).rvs()
  for i in range(k):
    spread += num_infected()

  # total infections will include this individual
  return spread + 1
```

   What is the expected return value of `numInfected()`?

6. Let X and Y be independent random variables such that both X and $Y \sim \text{Exp}(1)$).

   a. Let's define $L = min(X, Y)$. What is $P(L \leq 4)$?
   b. Now let's define $H = max(X, Y)$. What is $P(H \leq 4)$?
   c. What is $\rho(L, H)$? Intuitively explain why the correlation is positive.

7. We know that the climate is warming. Even small changes to the mean global temperature (such a 1.5° C increase from pre-industrial levels) will change global weather patterns,

increasing human suffering and premature death. Given this, and given the uncertainty about how much technologies for removing CO_2 from the air may improve, how much should we reduce emissions now?

Imagine that you are currently choosing between the following options:

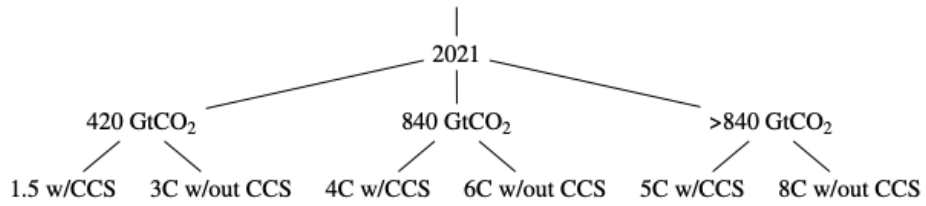
A. Cut emissions significantly, aiming for a carbon budget of 420 $GtCO_2$
B. Cut emissions moderately, aiming for a carbon budget of 840 $GtCO_2$
C. Continue with business as usual, expecting that emissions will significantly exceed 840 $GtCO_2$

In the future, technologies for removing CO_2 from the air such as carbon capture and storage (CCS) may improve such that they will be able to decrease the impact of past emissions on the climate – or they may not. Betting on this technology is a risk. We do not yet know to what extent these technologies can mitigate global warming.

Consider the three principles presented in class as guides to choosing between actions that may have significant future effects: the Precautionary Principle, the Future Risk-Avoidance Principle, and the Maximin Principle.

a. Which of these three principles is most useful for making choices about the future? (You are welcome to introduce a different decision principle if you clearly define it.) Justify your answer in 2-4 sentences.
b. What would your chosen principle say about whether we should cut emissions significantly, moderately, or not at all (options A, B, or C)? Assume that the results of each choice are summarized in the decision tree below; that you have control over the choice to reduce or not reduce emissions now; and that you have no control over how effective or ineffective CCS turns out to be in the future. Explain your answer in 2-4 sentences.

Note: As always, we are more interested in your process than in your final answer. We are looking for (1) demonstrated understanding of your chosen principle and (2) thoughtful reflection on the question, no matter your final verdict.

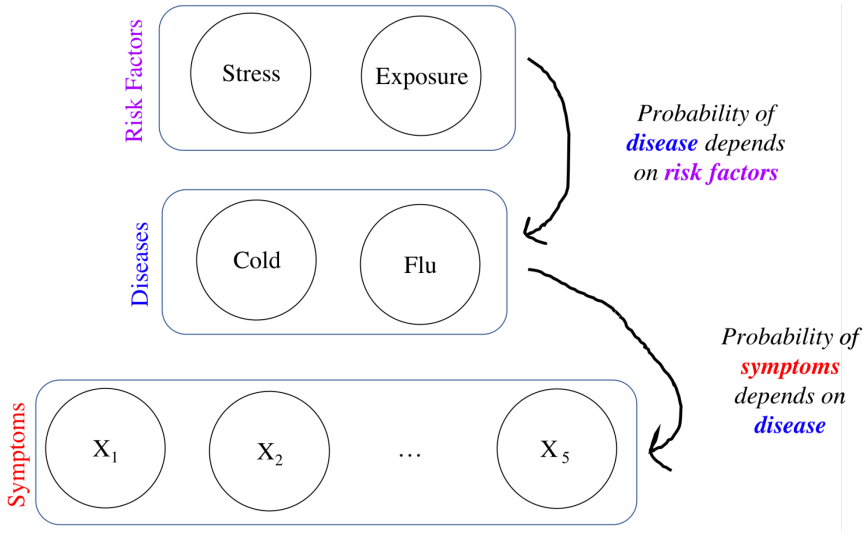


Coding Problems

Download the starter code and implement the functions as described. You MUST use the starter code. Do not add import statements to the starter code. When you are ready to submit, upload only and `webmd.py` and `federalist.py` directly to Gradescope under "PSet4 - Coding" (not a zip file). Do not modify the file name or any function names. Do not submit any other files, such as `util.py` or `probabilities.py`; if you submit extra files, they'll simply be ignored during grading.

The Gradescope autograder will report your problem score; there are no hidden tests. You can submit an unlimited number of times. We will only use the score from your most recent submission. No partial credit will be given if you do not pass the autograder.

8. We are writing a WebMD program that is slightly larger than the one we worked through in class. In this program we predict whether a user has a flu ($F = 1$) or cold ($C = 1$) based on knowing any subset of 5 potential binary symptoms (e.g., headache, sniffles, fatigue, cough, fever) and a subset of binary risk factors (exposure, stress).



The file `probabilities.py` implements the below functions. You **should not** modify these functions as we may use different probabilities when grading.

- The functions `probStress()` and `probExposure()` return the prior probabilities for stress and exposure, respectively.
- The functions `probCold(s, e)` and `probFlu(s, e)` return the probability that a patient has a cold or flu, given the state of the risk factors stress (`s`) and exposure (`e`).
- The function `probSymptom(i, f, c)` returns the probability that the `i`th symptom (X_i) takes on value 1, given the state of cold (`c`) and flu (`f`): $P(X_i = 1 | F = f, C = c)$, for $i = 1, \ldots, 5$.

We would like to write code that computes the probability $P(\text{Flu} = 1 | \text{Exposure} = 1, X_2 = 1)$: the probability of flu *conditioned on observing* that the patient has had exposure to a sick friend and that they are experiencing Symptom 2 (sniffles):

Implement the function `inferProbFlu()` in `webmd.py` that computes the probability $P(\text{Flu} = 1 | \text{Exposure} = 1, X_2 = 1)$ using **Rejection Sampling**. This function takes one parameter `ntrials`, which is the number of observations to generate:

```
def inferProbFlu(ntrials=1000000) # P(Flu = 1 | Exposure = 1 and X2 = 1)
```

The Federalist Papers

The Federalist Papers is a body of essays written between 1787 and 1788 by Alexander Hamilton, James Madison, and John Jay under the collective pseudonym "Publius." Suppose that James Madison and Alexander Hamilton both claim to have written Federalist Paper No. 53 (we do not

consider John Jay as a possible author in this problem). Over the next few problems, we will tackle the problem of deciding who wrote this paper probabilistically, given the contents of the document and historical evidence of the two authors' penmanship. It's a big task, but you all are ready :-)

Chapter 1: Mathematical Model

Before we get to the code, we need to first understand the model, which we discuss in Lecture Notes 11 as follows: Let there be n total words in the unknown document. There are m unique words in the document, where c_i is the number of times word i appears, and $\sum_{i=1}^{m} c_i = n$. If Hamilton wrote the document, then each of the n words has probability p_i of being unique word i, where $i = 1, \ldots, m$. If Madison wrote the document, then each of the n words has probability q_i of being unique word i.

9. **[Written]** We set up the problem as comparing the probabilities that each author wrote the document: If the probability that Hamilton is the author (given the document's contents) is greater than the probability that Madison is the author (given the document's contents), then we predict that Hamilton is the author (Equation (1) in Lecture Notes 11, Section 3). Otherwise, we predict Madison.

   We make two assumptions: first, prior to knowing the document's contents, Hamilton and Madison are equally likely to be the author; second, given the author, the document's contents can be modeled using a Multinomial random variable, where the probability of producing word i is known (p_i for Hamilton, q_i for Madison). Use Bayes' Theorem to show that Equation (1) is equivalent to the below expression (Equation (2) in Lecture Notes 11):

   $$\frac{\prod_{i=1}^{m} p_i^{c_i}}{\prod_{i=1}^{m} q_i^{c_i}} > 1$$

   If the above inequality holds, then we predict Hamilton as the author. Otherwise, we predict Madison.

Chapter 2: Data processing

It's time to code! While the Gradescope autograder tests each function you write, we **urge** you to develop incrementally and **test locally**. We suggest that you verify that you pass all Gradescope tests on previous parts *before* moving to next parts, though it may help to read ahead.

To run and test your code locally, run `python3 federalist.py` (if OS X; if Windows, run `python federalist.py`) in your Python environment. This will run the `main` method, which runs each of the functions you are asked to write. We won't be grading anything in `main`, but the provided code shows how the functions might be used; you can modify `main` as you like.

10. We collect each author's word frequency data from existing documents, which involves recording word counts (which we've implemented for you) and computing word probabilities (your task).

    In `util.py`, the provided function `make_word_count_map` takes a *word list* (i.e., a Python list of strings) and returns a dictionary with key:value pairs (word: number of times the key appeared in the word list). Do not modify this function. For example:

```
>>> word_list = ['you', 'are', 'a', 'friend', 'and', 'a', 'role', 'model']
>>> word_counts = make_word_count_map(word_list))
>>> print(word_counts)
{'you': 1, 'are': 1, 'a': 2, 'friend': 1, 'and': 1, 'role': 1, 'model': 1}
```

a. **[Coding]** In `federalist.py`, implement the function `make_word_prob_map`, which takes a word list and returns a dictionary with key:value pairs (word: probability that this word appears in the word list). Approximate the probability as the frequency of the word in the word list, i.e. the probability that the word is drawn if all words in the word list are equally likely. (Hint: It might be useful to use the provided `make_word_count_map` function.)

b. **[Written]** The provided function `get_probability` in `util.py` takes two parameters—a map of word probabilities (i.e., the return value of `make_word_prob_map`) and a single word—and returns the probability of the word `word` appearing. Notice that it returns a small `EPSILON` value if the word does not exist in the `word_map` dictionary. Why is this `EPSILON` term necessary? Briefly explain in 1-3 sentences. (Do not modify this function; just read the implementation.)

Chapter 3: A Small Lesson in Underflow

A good practice in engineering is to first test our implementation on a small task before running it on the final task. The next part of this problem will show that while our prediction model is mathematically correct, it is numerically unstable and will not work on large documents.

We aren't using `unknown.txt` in this chapter—we're testing our model on a smaller file. All the information you need for this part is in the data table below, but if you're interested, the smaller file is also included in the starter code as `underflow.txt`.

11. **[Written]** Before writing code, you'll test the theory out by hand. Try to compute the ratio $(\prod_{i=1}^{m} p_i^{c_i})/(\prod_{i=1}^{m} q_i^{c_i})$ for a smaller document whose p_i, q_i and c_i values are provided in the table below (taken from `hamilton.txt`, `madison.txt`, and `underflow.txt`, respectively). You may assume that the below words are the only words that appear in this smaller document.

i	word	p_i	q_i	c_i
1	"the"	0.08	0.094	193
2	"of"	0.059	0.056	128
3	"to"	0.037	0.031	73
4	"and"	0.032	0.031	62

Feel free to use a calculator, Python, WolframAlpha, or whatever computing tool you like. What you will most likely encounter is an issue related to *arithmetic underflow*: The smallest representable number in Python is approximately 5×10^{-324}, beyond which Python rounds down to zero (Wikipedia link for those interested: `https://en.wikipedia.org/wiki/Double-precision_floating-point_format`)

Using the information above, briefly explain why arithmetic underflow will occur in this (very plausible) situation. We aren't grading code for this question, so please include your work (e.g., reasoning, numeric justification, or code screenshot) only on the PDF that you upload to Gradescope.

12. **[Written]** The ratio of probabilities we used above leads to arithmetic underflow. In practice, we can use the properties of logarithms to produce numerically stable results.

    Use the properties of logarithms to show that the inequality in Problem 9 is equivalent to the below expression (Equation (3) on Lecture Notes 11). If the below inequality holds, then we predict Hamilton as the author (otherwise, we predict Madison):

$$\sum_{i=1}^{m} c_i \log p_i - \sum_{i=1}^{m} c_i \log q_i > 0$$

Chapter 4: The Final Stretch

It's time to bring all of our insights together to design a computationally tractable text analyzer!

13. **[Coding]** Let's use the log expression you derived in Problem 12 to determine the author of `unknown.txt`.

    a. In `federalist.py`, implement `calculate_doc_log_prob`. This function takes two parameters:

       - a word probability map (i.e. the return value from `make_word_prob_map`) for a known author, either Madison or Hamilton, and
       - a map of word counts (i.e. the return value from `make_word_count_map`) for a mystery document whose author we're trying to predict.

       This function should return $\sum_{i=1}^{m} c_i \log f_i$, where f_i is the probability of word i, drawn from the first parameter to the function. This quantity is one of the terms in the expression you derived in Problem 12. (In practice, f_i will be either p_i or q_i, depending on the arguments that are passed to the function.)

       While you could theoretically use any log function, the Gradescope autograder expects that you will use the natural logarithm function in NumPy, which we've already imported for you (e.g., `log(10)` will return approximately `2.302`).

       If you're having trouble with this function, we recommend tracing out the logic on Problem 11's small example.

    b. In `federalist.py`, implement `determine_author`, which takes four parameters and is called from the `main` function as below:

       - `word_list1`: a list of words that appear in `hamilton.txt`
       - `author1`: the string `"Alexander Hamilton"`
       - `word_list2`: a list of words that appear in `madison.txt`
       - `author2`: the string `"James Madison"`
       - `unknown_word_list`: a list of words that appear in `unknown.txt`

       This function should return your prediction of the author: either `author1`, i.e., `"Alexander Hamilton"`, or `author2`, i.e., `"James Madison"`. (Hint: you'll want to use both the provided functions in `util.py` and the functions you wrote in previous parts.)

You've just written a program that predicts the author of an unknown document. Give yourself a pat on the back! Exciting!