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CS 109

Quiz #1 Solution
April 26, 2021

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Take-Home Quiz information

Each quiz will be a 47-hour open-book, open-note exam. We have designed this quiz to approximate about 1-3 hours of active work (*before* typesetting).

- You can submit multiple times; we will only grade the last submission you submit before 10:00am (Pacific time) on Friday, April 23rd. No late submissions can be accepted. When uploading, please assign pages to each question.
- You should upload your submission as a PDF to Gradescope. We provide a LaTeX template if you find it useful, but we will accept any legible submission. You may also find the CS109 Probability LaTeX reference useful: `https://www.overleaf.com/project/5f650a577489e90001f065be`
- Course staff assistance will be limited to clarifying questions of the kind that might be allowed on a traditional, in-person exam. If you have questions during the exam, please ask them as private posts via our discussion forum. We will not have any office hours for answering quiz questions during the quiz, and we can't answer any questions about course material while the quiz is out.
- **For each problem, briefly explain/justify how you obtained your answer at a level such that a future CS109 student would be able to understand how to solve the problem. If it's not fully clear how you arrived at your answer, you will not receive full credit.** It is fine for your answers to be a well-defined mathematical expression including summations, products, factorials, exponents, and combinations, unless the question *specifically* asks for a numeric quantity or closed form. Where numeric answers are required, fractions are fine.

Honor Code Guidelines for Take-Home Quizzes

This exam must be completed individually. It is a violation of the Stanford Honor Code to communicate with any other humans about this exam (other than CS109 course staff), to solicit solutions to this exam, or to share your solutions with others.

The take-home exams are open-book: open lecture notes, handouts, textbooks, course lecture videos, and internet searches for conceptual information (e.g., Wikipedia). Consultation of other humans in any form or medium (e.g., communicating with classmates, asking questions on sites like Chegg or Stack Overflow) is prohibited. All work done with the assistance of any external material in any way (other than provided CS109 course materials) must include citation (e.g., "Referred to Wikipedia page on X for Question 2."). Copying solutions is unacceptable, even with citation. If by chance you encounter solutions to the problem, navigate away from that page before you feel tempted to copy.

If you become aware of any Honor Code violations by any student in the class, your commitments under the Stanford Honor Code obligate you to inform course staff. ***Please remember that there is no reason to violate your conscience to complete a take-home exam in CS109.***

I acknowledge and accept the letter and spirit of the Honor Code:

Name (typed or written): ______________________________

1 San Francisco Department of Health [30 points]

You've been hired by the San Francisco Department of Health to schedule a collection of safety and training meetings over the course of the 2022 calendar year. You need to schedule a total of 18 meetings, but otherwise very little has been decided about how they should be scheduled.

a. (7 points) Assume you decide that each meeting should cover the same safety material and provide the same training, but you're redundantly scheduling all 18 meetings over the course of the year so SFDH employees have many opportunities to attend. How many ways can the 18 meetings–each indistinguishable from all others—be scheduled across all 12 months?

Answer.

There are 18 objects are being distributed across 12 ordered buckets, where the objects are meetings and the buckets are months. This is classic divider method, where the number of way is given by $\binom{18+12-1}{12-1} = \binom{29}{11} = 34,597,290$.

In addition to providing an expression above,
please compute a numeric answer: 34,597,290

b. (8 points) How many ways can these same 18 meetings be scheduled as in part a., with the added constraint that there be at least one meeting scheduled per month?

Answer.

12 of the indistinguishable meetings are constrained to be distributed evenly across all 12 months, leaving a mere 6 meetings to be scheduled. We still use the divider method, but this time we use $n = 6$ instead of $n = 18$, since we only have flexibility with 6 meetings, not all 18. $\binom{6+12-1}{12-1} = \binom{17}{11} = 12,376$.

In addition to providing an expression above,
please compute a numeric answer: 12,376

c. (7 points) How many ways can these 18 meetings be scheduled such that there are at least one but at most three meetings per month?

Answer.

One approach is to take your answer to part b and then enact some form of the forbidden method to brute-force subtract the number of distributions that place too many meetings in one or more months. Since we have pre-allocated 1 meeting on each of the 12 months, we now assess how many ways we can allocate *at most two* on each month (instead of *at most three* which was the original problem statement).

- How many ways can you assign all six additional meetings to a single month? In precisely 12 ways, and all of them are verboten. Subtract them!
- How many ways can you assign five of the six additional meetings to a single month? In precisely 12 * 11 ways. How? Choose which of the twelve months gets five meetings,

and then choose which of the other 11 months gets the one remaining one, for a total of 132. Subtract them!!

- How many ways can you assign four of the six additional meetings to a single month? In precisely $12 \cdot \binom{2+11-1}{11-1} = \binom{12}{10} = 792$. How did we get that? Choose which of the 12 months gets precisely 4 meetings, and then distribute the two remaining meetings over the 11 other months using the divider method. Subtract them!!!
- How many ways can you assign three of the six additional meetings to a single month? In $12 * \binom{3+11-1}{11-1}$ ways. In this case, we must be careful to remove all the selections that we over counted where there were two groups of three on different months. This occurred $\binom{12}{2}$ times. Therefore, the result is $12 * \binom{3+11-1}{11-1} - \binom{12}{2} = 3,366$. Subtract them!!!!

Answer is then $12,376 - 12 - 132 - 792 - 3,366 = 8,074$.

In addition to providing an expression above, please compute a numeric answer: 8,074

d. (8 points) Assuming there are 92 SFDH employees, and each of the 92 is required to attend exactly one of the 18 distinct offerings, how many ways can all 92 employees be assigned to the 18 meetings, the only constraint being that each meeting is required to have 5 or 6 people? Your answer should be simplified as much as possible, but you needn't reduce it to a single number as you did for earlier parts of this question.

Answer.

Here, both the employees and the meetings are taken to be distinct. Because all meetings are constrained to have at least 5 people, then 90 of the 92 must be distributed across the 18 meetings. That means that two of the 18 meetings have rosters of 6 people, and the remaining 16 only have five. We need to choose which two meetings of the eighteen get six, and for each choice, we need to figure out all the ways to distribute 92 employees across 16 groups of size 5 and two groups of size 6. So, the number of ways to do this is:

$$\binom{18}{2}\binom{92}{6,6,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5} = \binom{18}{2} \cdot \frac{92!}{6!^2 5!^{16}}$$

Of course, this is easily computed in one's head to be

1, 823, 985, 452, 085, 893, 664, 213, 932, 031, 176, 765, 464, 579, 802, 973, 726, 523, 800, 609, 177, 027, 983, 340, 112, 978, 661, 218, 859, 959, 729, 818, 420, 284, 347, 034, 097, 035, 879, 437, 528, 379, 873, 120, 602, 775, 038, 722, 048, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000

2 Art Curation [15 points]

Jerry and Doris have abandoned their respective careers in education and puppy paw modeling and have taken up new work as art curators. As luck would have it, the San Francisco Museum of Modern Art has hired them both to independently appraise modern paintings. Doris has a keen eye for art and can spot a forgery—that is, identifies fake painting as fake—with probability 0.91, whereas Jerry is more easily fooled and only spots a fake painting as fake with probability 0.82. Doris and Jerry also do an excellent good job at certifying authentic art as authentic. When Doris sees an authentic modern art painting, she certifies it as authentic with probability 0.99. Jerry does the same thing, but with probability 0.84.

Those working at SFMoMA are adamant that, based on past experience, the probability that each painting they consider is authentic with probability 0.6. Assuming a single painting is fake, Jerry and Doris independently identify that painting as fake. Similarly, Jerry and Doris independently identify authentic paintings as authentic.

a. (5 points) Assuming a painting is authentic, what is the probability that neither Jerry nor Doris believe it to be authentic?

Answer.

Let's define some events for the problem. Let A be the event that a painting is authentic, let D be the event that Doris says a painting is authentic, and let J be the event that Jerry says a painting is authentic. We have:

$$P(A) = 0.6, \text{ so that } P(A^C) = 0.4$$
$$P(D|A) = 0.99 \text{ and } P(D^C|A) = 0.01$$
$$P(D^C|A^C) = 0.91 \text{ and } P(D|A^C) = 0.09$$
$$P(J|A) = 0.84, \text{ and } P(J^C|A) = 0.16$$
$$P(J^C|A^C) = 0.82, \text{ so } P(J|A^C) = 0.18$$

The problem is asking for $P(D^C J^C|A)$, which is $P(D^C|A)P(J^C|A)$, or $0.01 \cdot 0.16 = 0.0016$.

In addition to providing an expression above,
please compute a numeric answer: 0.0016

b. (5 points) Assuming a painting is a forgery, what is the probability that exactly one of Doris and Jerry identify it as a forgery?

Answer.

The approach is similar to that for part a, except that there are more mutually exclusive conditional events that need to be union-ed. In particular, we need to compute $P(DJ^C|A^C) + P(D^C J|A^C) = P(D|A^C)P(J^C|A^C) + P(D^C|A^C)P(J|A^C)$. Numerically, this is $0.09 \cdot 0.82 + 0.91 \cdot 0.18 = 0.2375$.

In addition to providing an expression above,
please compute a numeric answer: 0.2376

c. (5 points) Given that both Doris and Jerry certify a painting as authentic, what is the probability the painting is really a forgery?

Answer.

Here, we're asking for $P(A^C|DJ)$. The best approach is the employ Bayes' Theorem of, in my presentation below, to effectively derive it from scratch.

$$\begin{aligned}
P(A^C|DJ) &= \frac{P(A^C DJ)}{P(DJ)} \\
&= \frac{P(DJ|A^C)P(A^C)}{P(DJ)} \\
&= \frac{P(D|A^C)P(J|A^C)P(A^C)}{P(DJ|A)P(A) + P(DJ|A^C)P(A^C)} \\
&= \frac{P(D|A^C)P(J|A^C)P(A^C)}{P(D|A)P(J|A)P(A) + P(D|A^C)P(J|A^C)P(A^C)} \\
&= \frac{0.09 \cdot 0.18 \cdot 0.4}{0.99 \cdot 0.84 \cdot 0.6 + 0.09 \cdot 0.18 \cdot 0.4} = 0.01282
\end{aligned}$$

In addition to providing an expression above,
please compute a numeric answer: 0.01282

3 Listening to Music [15 points]

Tim has 30 albums of 10 songs each on his iPhone, and he listens to a total of 12 songs on shuffle every morning before classes begin. Each of the 300 songs is equally likely to be chosen, although a song is never selected more than once.

Oh, and one of the 30 albums, Neil Diamond's The Jazz Singer, is Tim's absolute favorite.

a. (5 points) Compute $p_X(x)$, which defines the number of songs x that just so happen to be from his favorite album.

Answer.

Here, X is a random variable with support of $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. For any given value of $X = x$, the number of ways one can choose x songs from Tim's favorite album is $\binom{10}{x}$, and for each choice, the number of ways to independently choose the remaining $12 - x$ songs from the other 290 songs is $\binom{290}{12-x}$. The total number of ways to choose any 12 songs from a set of 300, without concern for favorite albums, is $\binom{300}{12}$. That means that:

$$p_X(x) = \frac{\binom{10}{x}\binom{290}{12-x}}{\binom{300}{12}}.$$

b. (5 points) What is the probability two or more songs from the same album (not necessarily his favorite) are among the 12 selected?

Answer.

In this case, it's easier to compute the probability that no two songs are drawn from the same album (we'll just call it p), and then subtracting that probability from 1. To compute p, we need to count the number of ways which 12 of the 30 albums will contribute exactly one song, and for each chosen album we count the number of ways one of its 10 songs can be selected for the shuffle. Once we do that, our answer is $1 - p$.

$$\begin{aligned} p &= \frac{\binom{30}{12} \cdot 10^{12}}{\binom{300}{12}} \\ &= 0.0974 \\ 1 - p &= 0.9026 \end{aligned}$$

In addition to providing an expression above,
please compute a numeric answer: 0.9026

c. (5 points) If Tim hears one particular song (The First Time Ever I Saw Your Face, Roberta Flack) in the shuffle, it makes him so sad that he stops listening immediately, even if fewer than 12 songs have played. Given this new piece of information, what is the expected number of songs Tim listens to?

Answer.

This song that makes Tim so sad is equally likely to occupy any one of 300 positions in an arbitrary permutation of all songs. That means the probability that Tim listens to any number of songs between 1 and 11 inclusive is $\frac{1}{300}$, since those are the probability we hear some Roberta Flack in each of those positions of the shuffle. The probability that Tim listens to 12 songs is $\frac{289}{300}$, which is the complement probability of the sum of the first 11. Therefore, the expected number of songs Tim listens to is $\sum_{k=1}^{11} k \cdot \frac{1}{300} + 12 \cdot \frac{289}{300} = 11.78$.

In addition to providing an expression above,
please compute a numeric answer: 11.78