Jerry Cain
CS109

Section 01: Analytic Probability

Problems by Tim Gianitsos, Alex Tsun, and Jerry Cain

Overview of Section Materials

The warmup questions provided will help students review concepts introduced in lectures. The section problems are meant to apply these concepts in more complex scenarios similar to what you will see in problem sets and quizzes.

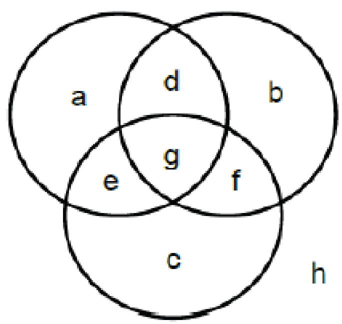
1 Warmups

1.1 Lecture 1: Counting

The Inclusion Exclusion Principle for three sets is:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Explain why in terms of a venn-diagram.



1.2 Lecture 2 Warmup: Permutations and Combinations

Suppose there are 7 blue fish, 4 red fish, and 8 green fish in a large fishing tank. You drop a net into it and end up with 6 fish. What is the probability you get 2 of each color?

1.3 Lecture 3 Warmup: Axioms of Probability

Decide whether each of the three statements below is true or false.

$$P(A) + P(A^C) = 1, \quad P(A \cap B) + P(A \cap B^C) = 1, \quad P(A) = 0.4 \wedge P(B) = 0.6 \implies A = B^C$$

2 Problems

2.1 Lecture 1 Generative Processes: The Birthday Problem

When solving a counting problem, it can often be useful to come up with a generative process, a series of steps that “generates" examples. A correct generative process to count the elements of set A will (1) *generate every element of A* and (2) *not generate any element of A more than once*. If our process has the added property that (3) *any given step always has the same number of possible outcomes*, then we can use the product rule of counting.

Example: Say we want to count the number of ways to roll two (distinct) dice where one die is even and one die is odd. Our process could be: (1) roll the first die and note if the value is even or odd, then (2) count the number of ways the second die can be rolled for a value of the opposite parity. Since the first step has 6 options and the second step has 3 options regardless of the outcome of the first step, the number of possibilities is $6 * 3 = 18$.

Problem: Assume that birthdays happen on any of the 365 days of the year with equal likelihood (we’ll ignore leap years).

a. What is the probability that of the n people in class, at least two people share the same birthday?

b. What is the probability that this class contains exactly one pair of people who share a birthday?

2.2 Lecture 2: Flipping Coins

One thing that students often find tricky when learning combinatorics is how to figure out when a problem involves permutations and when it involves combinations. Naturally, we will look at a problem that can be solved with both approaches. Pay attention to what parts of your solution represent distinct objects and what parts represent indistinct objects.

Problem: We flip a fair coin n times, hoping (for some reason) to get k heads.

a. How many ways are there to get exactly k heads? Characterize your answer as a *permutation* of H’s and T’s.

b. For what x and y is your answer to part a equal to $\binom{x}{y}$? Why does this *combination* make sense as an answer?

c. What is the probability that we get exactly k heads?

2.3 Lecture 2: Combinatorial Proofs

Example: Prove why $\binom{n}{k} = \binom{n}{n-k}$.

A fully algebraic proof is possible by simply showing the left hand side is equivalent to the right, as with:

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!((n-n)+k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$$

An equally compelling proof is a combinatorial one, which relies on our ability to describe the counting problem in two, equivalent ways. In this case, we can simply say the above is true because choosing k items from a set of n items is equivalent to choosing the $n-k$ items to be excluded. These types of proofs are called **combinatorial proofs**, or **story proofs**.

a. Present a combinatorial proof arguing that $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$.

b. Present a combinatorial proof arguing that $\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$ for all integer values of k, assuming r, s, m, and n are integer constants.