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CS109

Section #2
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Section #2: Random Variables

Overview of Section Materials

The warmup questions provided will help students practice concepts introduced in lectures. The section problems are meant to apply these concepts in more complex scenarios similar to what you will see in problem sets and quizzes.

1 Warmups

1.1 Independence

1. Definitions: Cite Bayes' Theorem.
2. True or False. Note that true means true for ALL cases.
   (a) In general, $P(AB|C) = P(B|C)P(A|BC)$
   (b) If A and B are independent, so are A and B^C.

1. Bayes' Theorem: $P(E|F) = \frac{P(F|E)P(E)}{P(F)}$
2. (a) True
   (b) True

1.2 Random Variables and Expectation

1. Definitions:
   (a) If we let X be a random variable, then what is $E[X]$? What is $E[g(X)]$?
   (b) For random variables $X_1, \ldots, X_n$, what is $E[\sum_{i=1}^{n} X_i]$?
2. True or False: For any random variable X, $E[X^2] = E[X]^2$.

1. Definitions:
   (a) $E[X] = \sum_x x p_X(x)$ and $E[g(X)] = \sum_x g(x) p_X(x)$.
   (b) $E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$
2. False

2 Problems

2.1 Taking Expectation: Breaking Vegas

Preamble: When a random variable fits neatly into a family we've seen before (e.g. Binomial), we get its expectation for free. When it does not, we have to use the definition of expectation.

Problem: If you bet on "Red" in Roulette, there is $p = 18/38$ that you with win \$Y and a $(1 - p)$ probability that you lose \$Y. Consider this algorithm for a series of bets:

1. Let Y = \$1.
2. Bet Y.
3. If you win, then stop.
4. If you lose, then set Y to be 2Y and goto step (2).

What are your expected winnings when you stop? It will help to recall that the sum of a geometric series $a^0 + a^1 + a^2 + \cdots = \frac{1}{1-a}$ if $0 < a < 1$. Vegas breaks you: Why doesn't everyone do this?

Let X be the number of dollars that your earn.
The possible values of x are from the outcomes of: winning on your first bet, winning on your second bet, and so on.

$$\begin{aligned}
E[X] &= \frac{18}{38} + \frac{20}{38}\frac{18}{38}(2-1) + \left(\frac{20}{38}\right)^2\frac{18}{38}(4-2-1) + \ldots \\
&= \sum_{i=0}^{\infty}\left(\frac{20}{38}\right)^i\left(\frac{18}{38}\right)\left(2^i - \sum_{j=0}^{i-1}2^j\right) \\
&= \left(\frac{18}{38}\right)\sum_{i=0}^{\infty}\left(\frac{20}{38}\right)^i \\
&= \left(\frac{18}{38}\right)\frac{1}{1-\frac{20}{38}} = 1
\end{aligned}$$

Real games have maximum bet amounts. You have finite money and casinos can kick you out. But, if you had no betting limits and infinite money, then go for it! (and tell me which planet you are living on).

2.2 Linearity of Expectation: Hat-Check

Preamble: Typically, it is easier to use linearity of expectation for sums of random variables than it is to manually compute a PMF and apply the definition.

Problem: n people go to a party and drop off their hats to a hat-check person. When the party is over, a different hat-check person is on duty, and returns the n hats randomly back to each person. Let X be the random variable representing the number of people who get their own hat back.

a. For $n = 3$, find $E[X]$ by first computing the probability mass function p_X, and then applying the definition of expectation.

b. Find a general formula for $E[X]$, for any positive integer n.

a. The number of people X who could get their hat back is in $\{0, 1, 3\}$ (why not 2?). One can enumerate the possibilities:

$$123 \rightarrow 3$$
$$132 \rightarrow 1$$
$$213 \rightarrow 1$$
$$231 \rightarrow 0$$
$$312 \rightarrow 0$$
$$321 \rightarrow 1$$

Hence, $p_X(0) = P(X = 0) = 1/3$, $p_X(1) = P(X = 1) = 1/2$, and $p_X(3) = P(X = 3) = 1/6$. We then have that

$$E[X] = \sum_x x p_X(x) = 0 \cdot (1/3) + 1 \cdot (1/2) + 3 \cdot (1/2) = 1.$$

b. For $i = 1, \ldots, n$, let X_i be the indicator variable of whether person i gets their hat back. That is, $X_i = 1$ if person i gets their hat back, and $X_i = 0$ otherwise. Then, $X = \sum_{i=1}^n X_i$. For a particular person i, the probability they get their hat back is exactly $1/n$ (why?), and so $E[X_i] = 1 \cdot (1/n) + 0 \cdot (1 - 1/n) = 1/n$.

By linearity of expectation,

$$E[X] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] = \sum_{i=1}^n \frac{1}{n} = n \cdot (1/n) = 1.$$

Now imagine finding the PMF for this random variable with n people/hats. There was no nice catch-all formula in part a) for $n = 3$, and so it would be extremely difficult/impossible to come up with one for general $P(X = k)$. Even if you could, evaluating the sum might be difficult. This is the power of linearity of expectation - though we don't know the PMF, we can still compute it easily by breaking it down into smaller pieces. Notice that people getting their hat backs are not independent events either!

2.3 Sending Bits to Space

Preamble: When sending binary data to satellites (or really over any noisy channel), the bits can be flipped with high probability. In 1947, Richard Hamming developed a system to more reliably send data. By using Error Correcting Hamming Codes, you can send a stream of 4 bits along with 3 redundant bits. If zero or one of the seven bits are corrupted, using error correcting codes, a receiver can identify the original 4 bits.

Problem: Lets consider the case of sending a signal to a satellite where each bit is independently flipped with probability $p = 0.1$.

a. If you send 4 bits, what is the probability that the correct message was received (i.e. none of the bits are flipped).

b. If you send 4 bits, with 3 Hamming error correcting bits, what is the probability that a correctable message was received?

c. Instead of using Hamming codes, you decide to send 100 copies of each of the four bits. If for every single bit, more than 50 of the copies are not flipped, the signal will be correctable. What is the probability that a correctable message was received?

Hamming codes are super interesting. It's worth looking up if you haven't seen them before!

a. Let Y be the number of 4 bits corrupted. Then $P(Y = k)$ is given as:

$$P(Y = 0) = \binom{4}{0}(0.1)^0(0.9)^4 = 0.656$$

b. Let Z be the number of 7 bits corrupted. A correctable message is received if Z equals 0 or 1:

$$\begin{aligned} P(\text{correctable}) &= P(Z = 0) + P(Z = 1) \\ &= \binom{7}{0}(0.1)^0(0.9)^7 + \binom{7}{1}(0.1)^1(0.9)^6 = 0.850 \end{aligned}$$

That is a 30% improvement!

c. Let X_i be the number of copies of bit i which are not corrupted. We can represent each as a random variable as we did in parts a and b.

$$\begin{aligned} P(\text{correctable}) &= \prod_{i=1}^{4} P(X_i > 50) \\ &= \prod_{i=1}^{4} \sum_{j=51}^{100} P(X_i = j) \\ &= \prod_{i=1}^{4} \sum_{j=51}^{100} \binom{100}{j}(0.9)^j(0.1)^{100-j} \\ &= \left(\sum_{j=51}^{100} \binom{100}{j}(0.9)^j(0.1)^{100-j} \right)^4 > 0.999 \end{aligned}$$

But now you need to send 400 bits, instead of the 7 required by hamming codes :-).