Jerry Cain
CS 109

Section 5
May 4, 2021

Section 5

Based on the work of many CS109 instructors and course staff members.

1 Problems

1.1 Hat-Check Again??

Recall the hat-check problem from an earlier discussion section: n people go to a party and drop off their hats to a hat-check person. When the party is over, a different hat-check person is on duty, and returns the n hats randomly back to each person. Let X be the random variable representing the number of people who get their own hat back. We showed last time that $E[X] = 1$ for any n.

What is $Var(X)$?

1.2 Conditional Expectation

Let $X \sim Geo(p)$. Use the Law of Total Expectation to prove that $E[X] = 1/p$, by conditioning on whether the first flip is heads or tails.

1.3 Random Number of Random Variables

Let N be a non-negative integer-valued random variable—that is, a random variable that takes on values in $\{0, 1, 2, \ldots\}$. Let $X_1, X_2, X_3, \ldots$ be an infinite sequence of independent and identically distributed random variables (independent of N), each with mean μ, and $X = \sum_{i=1}^{N} X_i$ be the sum of the first N of them.

Before doing any work, what do you *think* $E[X]$ will turn out to be? Then show it mathematically to see if your intuition is correct.

1.4 Binary Trees

Consider the following function for constructing binary trees:

```
struct Node {
    Node *left;
    Node *right;
};

Node *randomTree(float p) {
    if (randomBool(p)) {  // returns true with probability p
        Node *newNode = new Node;
        newNode->left = randomTree(p);
        newNode->right = randomTree(p);
        return newNode;
    } else {
        return nullptr;
    }
}
```

The `if` branch is taken with probability p (and the `else` branch with probability $1 - p$). A tree with no nodes is represented by `nullptr`; so a tree node with no left child has `nullptr` for the `left` field (and the same for the right child).

Let X be the number of nodes in a tree returned by `randomTree`. You can assume $0 < p < 0.5$. What is $E[X]$, in terms of p?